Graph Theoretic Concepts for Highly Available Underlay Aware P2P Networks

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Abstract

In our previous work we have demonstrated that underlay awareness is necessary in P2P overlays for the availability of overlay paths and proved that the problem of formation of overlay networks with guaranteed availability is NP complete. Despite this complexity, underlay aware overlay networks, which use knowledge of the underlay to provide guaranteed levels of availability can be efficiently formed and maintained, under a specified set of constraints. In this technical report, we present the graph theoretic concepts developed, leading to the development of efficient algorithms for availability guaranteed overlay construction and maintenance.

1 Introduction

In our previous works[1, 2], we demonstrate how knowledge of the underlay topology is a necessary condition for creating truly available overlays and propose a strong model of availability that we term Manifest Availability and proved that the problem of creating manifest available overlays is NP Complete. This motivated the need for a weaker availability model that we term Latent Availability which was also NP complete but could be solved with a set of constraints.

In this work we present the concept of reduced overlays and develop the graph theoretic concepts necessary for developing polynomial time algorithms for forming latent availability overlays of a general degree of availability[3].

The rest of this Technical report is laid out as follows. The Latent Availability Model used in our work is stated in Section 2. Theoretical concepts developed, leading to the design of algorithms for latent availability overlay construction are presented in Section 3.

2 Latent Availability

The concepts illustrated here are based on the formalization of underlay aware overlays and availability[4]

The latent availability overlay network of degree k is a general overlay network which has the property that there are at least k distinct vertex disjoint paths in the underlay network between every pair of underlay nodes corresponding to the nodes in the overlay. Formally, a latent overlay graph $G_o = < V_o, E_o >$ of availability degree k on an underlay graph $G_u = < V_u, E_u >$ is a graph such that

C1. There exists a mapping function $\text{VERTEXMAP}$ which maps every node in the overlay graph to a distinct underlay node. This can be formally represented as:

$$\exists \text{VERTEXMAP} : (\forall p \forall q ((p \in V_o \land (q \in V_o) \land (p \neq q)) \Rightarrow (\exists a \exists b : ((a \in V_u) \land (b \in V_u) \land (a \neq b) \land (a = \text{VERTEXMAP}(p)) \land (b = \text{VERTEXMAP}(q))))$$

which means that if p and q are distinct overlay nodes, then their corresponding nodes in the underlay, mapped by vertexmap, say a and b, are also distinct nodes in the underlay.

C2. Between every pair of overlay nodes there are k node disjoint paths in the corresponding underlay. This can be formally represented as:

$$\forall p \forall q ((p, q \in V_o) \Rightarrow (\exists_{i=1}^k \text{path}_i : ((\text{path}_i \in PATHS_u) \land (\text{END}\_\text{NODES(path}_i) =$$
\{\text{VERTEXEXMAP}(p), \text{VERTEXEXMAP}(q)\})
\land (\forall j \neq l \implies (\text{INT\_NODES}(\text{path}_j) \cap 
\text{INT\_NODES}(\text{path}_l) = \Phi)))

which means that for every pair of overlay nodes, for example \(p\) and \(q\), there are \(k\) paths between them in the underlay such that they are pairwise node disjoint. Their end points are the vertexmaps of \(p\) and \(q\), but their internal node sets are disjoint.

We hereby refer to the problem of finding latent availability overlays on a given underlay as the latent availability problem. It has been proved that the latent availability problem is NP-Complete[4].

In a practical scenario on the internet, or a similar large network, every node in the network cannot be a potential overlay node or run the application software. It is decided by administrative policies, node capabilities and application requirements. Hence instead of considering the issue of whether a particular availability model can be supported by the given underlay, we can consider the mapping of the overlay nodes to underlay nodes, i.e., the VERTEXMAP, as fixed, and analyze whether the availability model can be supported by the set of overlay nodes on the given underlay. We name this as the spatial invariability condition.

We proved that under the spatial invariability condition, formation of latent availability overlays is polynomial time. The temporal variability criterion states that the size of the overlay graph is not constant over time. Nodes may join or leave at any point of time. Whenever a new overlay node is to be added to the overlay, the overlay has to be reconstructed and its availability requirements have to be assured. The impact this has on the latent availability model is analyzed below.

In the latent availability model, the temporal variability criterion says that the new node to join the network should also have \(k\) node disjoint paths to each existing overlay node. The problem of latent availability overlay can be now stated formally as: Given an underlay graph \(G_u\), and an overlay graph vertex set \(V_o\), an overlay node \(x\), and VERTEXMAP such that every overlay node is mapped to a distinct underlay node and for every node pair in the underlay which is mapped to from an overlay node by VERTEXMAP, there are at least \(k\) pairwise node disjoint paths between the node pairs in the underlay, determine \(G_o\), such that \(V_o' = V_o \cup \{v_x\}\), VERTEXMAP = VERTEXMAP \cup \{v_x, x\} and every node pair \(p, q \in V_o'\) is such that there are \(k\) node disjoint paths in \(G_o\) between VERTEXMAP\((p)\) and VERTEXMAP\((q)\).

This problem is polynomial time solvable as we prove in [4].

We have developed an underlay awareness model [4] and introduced the concept of reduced underlays. The work presented is this report is based on the referred underlay awareness model.

3 Graph theoretical concepts for underlay aware overlays

3.1 Reduced Underlay Graph

We define a reduced underlay graph of an underlay graph \(G_u\) as a multigraph \(G_{ru} =< V_{ru}, E_{ru} >\), where \(V_{ru} = V_u - \{x\} | (\text{degree}(x) = 1) \lor (\neg (\exists u(\text{VERTEXEXMAP}(u) = x)) \land (\text{degree}(x) \leq 2))\)
and \(E_{ru} = \{(x, y) | (x, y) \in V_{ru} \land (x, y) \in E_u) \lor \exists \text{path in PATHS}_{G}(\text{END\_NODES}(\text{path}) = (x, y) \land (\text{INT\_NODES}(\text{path}) \cap V_{ru} = \phi))\}

Which means that the reduced underlay graph consists of all nodes in the underlay except pendant\((\text{degree}=1)\) nodes and connector nodes (which have degree 2 and are not overlay nodes). The edges in the reduced underlay are either edges in the underlay, or simple paths in the underlay consisting of only connector nodes as internal nodes.

**Lemma I:** An overlay graph \(G_o\) forms a latent availability overlay of degree \(k\) on an underlay graph \(G_u\) if and only if \(G_o\) forms a latent availability overlay of degree \(k\) on the reduced underlay graph \(G_{ru}\)

**Proof:** 1. If \(G_o\) forms a latent availability overlay of degree \(k\) on the reduced underlay graph \(G_{ru}\) then \(G_o\) forms a latent availability overlay of degree \(k\) on an underlay graph \(G_u\).

We claim that any two paths in \(G_{ru}\) that are internally node disjoint in \(G_{ru}\) have mappings to distinct node disjoint paths in \(G_u\). Let \((x, p_1, p_2, .., p_n, y)\) and \((x, q_1, q_2, .., q_m, y)\) be the two node disjoint paths in \(G_{ru}\). Then \(p_1, p_2, .., p_n\) and \(q_1, q_2, .., q_m\) are all expander nodes or overlay nodes. Consider any edge \((p_i, p_{i+1})\) or \((q_i, q_{i+1})\) in the path. It implies that either there was an edge in \(G_u\) between the nodes or that there was a path between them in \(G_u\) consisting of only connector nodes (degree 2). Hence the paths \((x, p_1, p_2, .., p_n, y)\) and \((x, q_1, q_2, .., q_m, y)\) have corresponding paths \((x, p_0, p_1, p_2, .., p_{n-1}, p_n, p_{n+1} .., y)\) and \((x, q_0, q_1, .., q_{m-1}, q_m, q_{m+1} .., y)\) in \(G_u\), where all the nodes between \(p_i\) and \(p_{i+1}\) or \(q_i\) and \(q_{i+1}\) are connector nodes. It can be proved that the same node \(x\) cannot be \(p_{i+1}\) as well as \(q_{i+1}\) which means that the
connector nodes in the first path are distinct from the connector nodes in the second path. If there was such a node \( x \), it is connected to only two nodes as its degree is two. Hence the adjacent nodes would also have to be common in both paths. Following this line of reasoning for all the succeeding two degree nodes, the expander/overlay nodes \( p_i \) and \( q \) will have to be the same, thus implying that the paths were not internally node disjoint in \( G_{ru} \).

Thus it has been proved by contradiction that there exists no such node \( x \), thereby implying that the paths are node disjoint in \( G_u \).

2. If \( G_o \) forms a latent availability overlay of degree \( k \) on an underlay graph \( G_u \) then \( G_o \) forms a latent availability overlay of degree \( k \) on the reduced underlay graph \( G_{ru} \).

We claim that any two paths in \( G_u \) that are internally node disjoint in \( G_u \) have mappings to distinct node disjoint paths in \( G_{ru} \). Let \( (x, p_1, p_2, \ldots, p_n, y) \) and \( (x, q_1, q_2, \ldots, q_m, y) \) be the two node disjoint paths in \( G_u \). Each node in this is either a connector node or an expander/overlay node. Consider two nodes \( a \) and \( b \) in such a path that are expander/overlay nodes. Then \( a \) and \( b \) have an edge between them in \( G_{ru} \). Thus there is path in \( G_{ru} \) between \( x \) and \( y \) consisting of expander/overlay nodes in the path in \( G_u \). As the paths are internally node disjoint in \( G_u \), the set of internal nodes in the path in \( G_{ru} \) being a subset of the nodes in \( G_{ru} \), the paths in \( G_{ru} \) are also internally node disjoint.

Hence, we further restrict our problem to that of finding latent availability overlays for the given set of overlay nodes on the reduced underlay graph \( V_{ru} \). This reduces the size of the underlay graph, and thus further reduces the time complexity of the latent availability problem. We hereafter refer to the reduced underlay as the underlay graph in this paper, and use the notation \( G_u \) for the reduced underlay graph.

Whitney’s theorem - expansion lemma [5] states that if a node \( x \) is added to a \( k \) connected graph with \( k \) edges to \( k \) distinct nodes, then the resulting graph is also \( k \)-connected. This lemma forms the basis for the algorithms we develop for available overlay formation of degree two. But this theorem is not directly applicable as the availability constraints are different from connectivity in a graph. The underlay network cannot practically be \( k \) connected for \( k > 2 \) because that would imply that every node should have a degree at least as large as \( k \), which is not true for an underlay graph for a practical large scale network. The basis for the correctness of our algorithms for broker addition for latent availability overlays of degree two is built from Whitney’s expansion lemma.

The applicability of Whitney’s theorem shows the difference in underlying principles of the formation of latent degree availability \( 2(k = 2) \) [3] and greater than 2. In this report, we develop the graph theoretical concepts required for developing algorithms for latent availability algorithms of degree \( k \), where \( k \) is more than 2.

3.2 Formation of latent availability overlays of degree three or more

In case of latent availability overlays of degree \( k \) greater than 2, a construction technique based on Whitney’s theorem is not applicable, as every latent availability of degree three need not consist of only three-connected reduced underlay subgraphs, as can be observed from Figure 1.

Hence, \( k \) degree available overlay formation requires a different approach.

We take a deeper look at the problem of expanding existing overlays to include a new overlay node.

**Lemma II:** Given \( G_o \) a latent availability overlay of degree \( k \), on a reduced underlay graph \( G_u \), with \textsc{VertexMap}. If an expander node \( d \) in the reduced underlay network has \( t = \min(k, \text{degree}(d)) \) node disjoint paths to \( t \) nodes \( v_1 \ldots v_t \) in \( V_u \) which are all overlay nodes, then \( d \) has \( t \) node disjoint paths to every overlay node \( w \), in the underlay.

**Proof:** We give the proof by contradiction. Suppose there is an overlay node \( v_i \in V_o \) such that \( d \) has only \( t-s \) underlay node disjoint paths to \( v_i \), where \( 1 < s \leq t \). Then the removal of \( (t-s) \) nodes, one from each path...
can disconnect the node \( d \) from \( v_i \). Assume that \((t-s)\) such nodes fail, and \( d \) is disconnected from \( v_i \). As \( d \) has \( t \) node disjoint paths to different nodes \( v_1 \) to \( v_i \), at the most \((t-s)\) of these paths can fail due to the failure of \( t-s \) paths, even if all these failed nodes lie on different paths. Thus there will be connections from \( d \) to at least \( s \) nodes from \( v_1 \) to \( v_i \). As each of these overlay nodes has \( t \) node disjoint paths to \( v_i \), at the most only \( t-s \) of them can fail due to the failure of \((t-s)\) nodes, hence each of these nodes will still have at least \( s \) node disjoint paths to \( v_i \). This implies that \( d \) is not disconnected from \( v_i \). Hence \( d \) must have had \( t \) node disjoint paths to \( v_i \). Thus we have proved that \( d \) has \( t \) node disjoint paths to every node in the underlay which was chosen as an overlay node.

**Corollary:** Given \( G_o \), a latent availability overlay of degree \( k \), on a reduced underlay graph \( G_u \), and an expander node \( e \in V_u \) which has a degree \( d \geq k \) and \( e \) has \( k \) underlay node disjoint paths connecting it to \( k \) overlay nodes \( v_1, ... v_k \). Then \( e \) has \( k \) underlay node disjoint paths to each overlay node \( G_o \).

**Proof:** This follows from previous lemma with \( t = k \).

**Lemma III:** Let \( V_o \) forms a latent availability overlay of degree \( k \) on an underlay \( G_u \). Let \( B \) be the set of overlay nodes. Let \( M \subset B \) such that \(|M| = k\). If there exists an expander node \( e \) in \( V_u - B \), such that it has pairwise internally node disjoint paths to every overlay node in the subset \( M \), then \( V_o \cup N \) is a latent availability overlay of degree \( k \) on \( G_u \), where \( \text{VERTEXMAP}(N) = e \).

**Proof:** From the corollary of Lemma II, \( e \) has \( k \) node disjoint paths to every existing overlay node, hence by definition of latent overlay of degree \( k \), \( V_o \cup N \) is a latent availability overlay of degree \( k \) on \( G_o \) with new \( \text{VERTEXMAP} = \text{VERTEXMAP} \cup \{< N, e >\} \).

This lemma, with \( k=3 \), forms the basis of the Tri-marg algorithm[1].

4 Conclusion

High availability in the presence of the runtime failures is an important concern in modern distributed event based applications. Availability of the overlay networks is a necessary condition for guaranteed data delivery. We have developed the graph theoretical concepts which form the base for a practical solution for the problem of forming and maintaining underlay aware overlays with availability guarantees.

References


