**Theorem 1** *A graph $G$ is chordal iff it has a perfect elimination ordering.*

**Proof:** The easy part is to show that if $G$ has a perfect elimination ordering, then it is chordal. Suppose, for contradiction, that this is false. Let $G$ be a graph with a perfect elimination ordering and suppose there is a chordless cycle $v_1, v_2, \ldots, v_l$ of length $l \geq 4$ in $G$. Let $v_i$ be the vertex in the cycle that occurs first in the perfect elimination ordering. Then $v_{i-1}$ and $v_{i+1}$ are neighbors of $v_i$ in $G$ that occur later in the ordering. Since the ordering is perfect, there must be an edge between $v_{i-1}$ and $v_{i+1}$, but this contradicts the assumption that the cycle is chordless.

Now, show the converse, that if $G$ is chordal then it has a perfect elimination ordering. Note that if $G$ is chordal, any induced subgraph of $G$ is also chordal. Therefore, it is sufficient to show that $G$ has a simplicial vertex, that is a vertex whose neighbors induce a complete graph. We put this vertex first in the elimination ordering, and by induction, find a perfect elimination ordering of the chordal graph obtained by deleting this vertex. To show this, we prove the following stronger claim by induction.

**Claim:** If $G$ is chordal then either $G$ is complete or it contains two non-adjacent simplicial vertices.

The proof is by induction on the number of vertices. If $|G| = 1$, then $G$ is complete and there is nothing to prove. Suppose $|G| > 1$ and $G$ is not complete. Then there exists a minimal subset $S$ of vertices such that $G - S$ has at least two connected components.

We claim that $S$ induces a complete subgraph of $G$. Suppose there are vertices $x$ and $y$ in $S$ such that $x$ is not adjacent to $y$. Let $C_1$ and $C_2$ be any two components of $G - S$. Then $x$ must have a neighbor in $C_1$ and also in $C_2$, otherwise $G - (S - x)$ also has at least two connected components, contradicting the minimality of $S$. Similarly, $y$ must have a neighbor in $C_1$ and $C_2$. Therefore there is a path $P_1$ between $x$ and $y$ in $G$ going only through vertices in $C_1$ and a path $P_2$ between $x$ and $y$ going only through vertices in $C_2$. Let $P_1$ and $P_2$ be the shortest possible such paths. Then $P_1 \cup P_2$ is a chordless cycle in $G$ of length at least 4, contradicting the assumption that $G$ is chordal. Therefore $S$ must induce a complete subgraph.

Let $G_1$ be the subgraph of $G$ induced by vertices in $S$ and $C_1$. $G_1$ must be chordal and has number of vertices less than $|G|$. Therefore, by induction, $G_1$ is either complete or has two non-adjacent simplicial vertices. In either case, $G_1$ has a simplicial vertex contained in $C_1$, and this is also a simplicial vertex in $G$. Similarly, by considering the graph $G_2$ induced by vertices in $S$ and $C_2$, and applying induction, we can find a simplicial vertex in $C_2$. This gives two non-adjacent simplicial vertices in $G$ and completes the proof.

**Theorem 2** *A graph $G$ is chordal iff it is the intersection graph of some collection of subtrees of some tree $T$.*

**Proof:** Let $T$ be any tree and $\{T_1, T_2, \ldots, T_n\}$ an arbitrary collection of subtrees of $T$. We show that their intersection graph is chordal. It is sufficient to show that it has a perfect elimination ordering, and to do that, it is sufficient to show there exists a simplicial vertex. The complete ordering then follows by induction.

Root the tree $T$ at an arbitrary vertex $r$. For each tree $T_i$, let $r_i$ be the vertex in $T_i$ that is closest to the root $r$ in $T$. Call $r_i$ the root of $T_i$. Let $T_1$ be the subtree such that the distance of $r_1$ from $r$ in $T$ is maximum, amongst all the $r_i$. We claim $T_1$ is a simplicial vertex in the intersection graph. Let $T_i$ be a neighbor of $T_1$ in the intersection graph. We claim $T_i$ must include $r_1$. If not, $T_i$ must contain some descendant of $r_1$, since it intersects $T_1$. Since $T_i$ is a subtree, all vertices in $T_i$ must be descendants of $r_1$. This implies that $r_i$ is further from $r$ than $r_1$, a contradiction. Thus all neighbors of $T_1$ contain $r_1$ and thus induce a complete subgraph in the intersection graph.

Conversely, let $G$ be a chordal graph and let $v_1, v_2, \ldots, v_n$ be a perfect elimination ordering of $G$. We show that there exists a tree $T$ and a collection $\{T_1, T_2, \ldots, T_n\}$ of subtrees of $T$ such that $v_i$ is adjacent to $v_j$ in $G$ iff $T_i$ and $T_j$ have non-empty intersection. The construction is by induction on $|G|$. If $|G| = 1$, let $T = T_1$ be the trivial tree with one vertex. Suppose, by induction, we have constructed a tree $T'$ and subtrees $\{T'_2, \ldots, T'_n\}$ of $T'$ such that $T'_i$ intersects $T'_j$ iff $v_i$ is adjacent to $v_j$ for $2 \leq i < j \leq n$.

Let $S$ be the neighbors of $v_1$ in $G$. Then $S$ induces a complete subgraph of $G$. Let $T'_S$ be the collection of subtrees of $T'$ corresponding to vertices in $S$. The subtrees in $T'_S$ are pairwise intersecting. We claim that if a collection of subtrees is pairwise intersecting, then there exists a vertex that is contained in all of them. This argument is the same as earlier. Root $T'$ at some vertex and take the deepest root of a subtree in $T'_S$. This must belong to all subtrees in $T'_S$. Let this vertex be $u$. Let $T$ be obtained from $T'$ by adding a leaf vertex $v$ adjacent to $u$. Let $T_1 = \{v\}$, $T_i = T'_i \cup \{v\}$ for $i \in S$ and $T_i = T'_i$ for $i \notin S$. This gives the required collection of subtrees whose intersection graph is the given chordal graph $G$.