

Solutions to Homework 1

1. This statement is actually incorrect as some of you have pointed out. A counterexample is obtained by taking a four cycle v_1, v_2, v_3, v_4 , adding four vertices u_1, u_2, u_3, u_4 and edges $u_i v_i$ for $1 \leq i \leq 4$. In this graph the only minimal separators are the singleton sets $\{v_i\}$, but the graph is not chordal. The correct statement should have been: A graph is chordal if and only if it is either complete or for every non-adjacent pair of vertices u, v , any minimal subset of vertices that separates u and v induces a complete subgraph. In one direction the proof is essentially same as was done in the class. Suppose G is chordal and not complete and there exists a pair of non-adjacent vertices u, v such that a minimal separator S separating u and v does not induce a complete subgraph. Then there are two non-adjacent vertices x, y in S , and we can find a chordless cycle in the graph as done in class, giving a contradiction. To prove the converse, if the graph is complete it is trivially chordal. So suppose that for every pair of non-adjacent vertices u, v , any minimal separator of u, v induces a complete subgraph. Suppose there is a chordless cycle $v_1, v_2, v_3, \dots, v_l$ for some $l \geq 4$. Consider a minimal separator of v_1 and v_3 (they are not adjacent since the cycle is chordless). It must include v_2 and also some vertex from the path $v_1, v_l, v_{l-1}, \dots, v_4, v_3$ other than v_1 or v_3 . Since any such separator induces a complete subgraph, v_2 must be adjacent to some vertex in $\{v_l, v_{l-1}, \dots, v_4\}$ contradicting the fact that the cycle is chordless.

2. The proof is by induction on the number of vertices. Let x be a simplicial vertex in G and C the neighbors of x . Suppose C is a maximal clique in $G - x$. Then the number of maximal cliques in G and $G - x$ is the same, with C replaced by $C \cup \{x\}$. The intersection between any two maximal cliques is the same in G and $G - x$. Therefore the maximum weight tree in the intersection graph of maximal cliques is the same in both, and the cliques containing any vertex other than x form a subtree of the tree, by induction. Since x is contained in only one maximal clique the statement is true for G as well.

Suppose C is not a maximal clique in $G - x$. Then the number of maximal cliques in G is one more than in $G - x$ with $C' = C \cup \{x\}$ the additional clique. Let T be a maximum weight spanning tree in the clique intersection graph of G . Then T must contain an edge joining C' to some clique C_1 such that $C \subseteq C_1$. Note that the weight of this edge is $|C|$ and every edge incident with C' has weight at most $|C|$. If there is no such edge, we can add such an edge and replace an edge in T incident with C' to get a larger weight spanning tree. So, suppose that $C'C_1$ is an edge in T with weight $|C|$. If this is the only edge incident with C' in T , then the remaining tree is a maximum weight spanning tree in the clique intersection graph of $G - x$, and by induction, maximal cliques containing any given vertex induce a subtree of this tree. The vertex x appears only in the clique C' and all other vertices in C' also appear in C_1 . This implies cliques of G containing a given vertex form a subtree of T .

Suppose the degree of C' in T is more than one. We can modify T to get another maximum weight spanning tree T' in which C' has smaller degree, in such a way, that if cliques containing a given vertex form a subtree of T' , they also form a subtree of T . Let $C'C_1$ be an edge of weight $|C|$ in T and suppose there is another edge $C'C_2$ in T . Note that weight of $C'C_2$ is $|C_2 \cap C| \leq |C_2 \cap C_1|$ since $C \subseteq C_1$. Replace the edge $C'C_2$ by the edge C_1C_2 to get a new tree T' . Since T is a maximum weight spanning tree, weight of $C'C_2$ must be equal to the weight of C_1C_2 and $C \cap C_2 = C_1 \cap C_2$. We show that for any vertex v , if the cliques containing v form a

subtree of T' , then they also form a subtree of T . This can fail only if the vertex is contained in C_1 and C_2 but not in C' . However, due to the maximum weight property, all vertices in $C_1 \cap C_2$ are contained in C . Hence there is no such vertex. Repeating this process we can get a tree in which C' is a leaf node. Then the previous argument holds.

To prove that this is the most compact representation, it is sufficient to observe that in any tree representation of a chordal graph, for every maximal clique in the graph, there must be a vertex in the tree such that all subtrees corresponding to vertices in the clique, contain the vertex. Further, for two different maximal cliques, these vertices must be distinct. Hence the number of vertices in the tree is at least the number of maximal cliques. The same argument holds for maximal cliques containing a given vertex v . For every such clique, there must be a vertex in the intersection of the subtrees, and the same vertex cannot be in the intersection for two different maximal cliques.

3. Let v_1, v_2, \dots, v_n be a perfect elimination ordering. Consider the cliques C_i formed by v_i and its neighbors $v_j, j > i$. Every maximal clique must be of the form C_i for some i . Suppose k is the smallest index such that v_k is in the maximal clique. Then all other vertices in the clique are of the form $v_j, j > k$. Further, every such neighbor of v_k must be in the clique, by maximality. Therefore the maximal clique is C_k . Hence there are at most n maximal cliques. Equality is achieved when the graph has only isolated vertices. An example of a graph with exponentially many maximal cliques is the complete graph K_{2n} with a perfect matching removed. That is, the complement of the graph containing n disjoint edges and $2n$ vertices. The number of maximal cliques is 2^n .

4. The smallest path graph that is not an interval graph is the tree obtained by subdividing exactly once each edge of $K_{1,3}$. This is not an interval graph as it has an asteroidal triple. If we add a new vertex and join it to all vertices of this tree, we get a chordal graph that is not a path graph. For a path graph, the subgraph induced by the neighbors of any vertex must be an interval graph.

5. Let G be any graph. We can represent it as the intersection graph of subtrees of a tree, where two vertices are adjacent iff the corresponding subtrees intersect in an edge. If there are m edges in G , let the tree be $K_{1,m}$ with each edge of the tree corresponding to an edge of the original graph. For every vertex v in G , let T_v be the subtree of T containing exactly those edges that correspond to edges incident with v in G . Then it is easy to see that the intersection graph of the subtrees T_v , with intersection defined as edge intersection, is precisely G .