Solutions to Homework 3

1. This is an alternative proof that a chordal graph whose complement has a transitive orientation is an interval graph. The proof is by constructing an interval representation of the graph, given a perfect elimination ordering of G and a transitive orientation of the complement.

In fact, we can look only at the complement and construct the sequence of maximal cliques. Take all vertices of in-degree 0 in the transitive orientation of the complement. They form a maximal clique in G and put this clique first. There must be at least one vertex in this clique that has outgoing edges in G^c to all vertices not in the clique. If not, let v be a vertex with in-degree 0 and maximum out-degree in G^c. Let u be a vertex with in-degree > 0 such that there is no directed edge vu. Let w be a vertex such that wu is an edge. Then we claim that wx must be an edge for all vertices x such that vx is an edge. If not then the vertices v, x, w, u induce a chordless 4-cycle in G, a contradiction. Thus w has out-degree more than the out-degree of v. If it has in-degree 0, we get a contradiction, otherwise it has a predecessor w' whose out-degree must be strictly greater than that of w. Continuing backwards, we get a vertex of in-degree 0 and out-degree strictly greater than that of v, a contradiction.

All such vertices in the first clique are simplicial vertices in G. Delete all of them and repeat with remaining vertices to find the next clique. The time required to find the representation is proportional to the size of the complement.

This also shows a slightly stronger result. A co-comparability graph with no induced C_4 is an interval graph.

2. Consider the maximal cliques in the graph in order say $C_1, C_2, \ldots, C_m$. Consider a vertex whose finish point is C_1. To dominate this, some vertex that occurs in C_1 must be in the dominating set. Choose the one whose finish point is as far right as possible. Any vertex dominated by a vertex in C_1 is also dominated by this vertex. More generally, consider the smallest i such that C_i contains a vertex v that has not been dominated and whose finish point is i. So for all vertices that occur in $C_1, C_2, \ldots, C_{i-1}$ either they are already dominated or they also occur in C_i. Pick a vertex u that occurs in C_i whose finish point is as large as possible. The optimal solution must contain a vertex that dominates v, and if it is not u, we can replace it by u to get another optimal solution.

To find the minimum weight dominating set, let $S[i]$ denote the minimum weight of a set of vertices that dominates all vertices that finish at or before

C_i. The set itself may include vertices that finish after C_i.

Then

$$S[i] = \min_{v \in C_i} wt(v) + S[l(v) - 1]$$

where $l(v)$ denotes the index of the leftmost clique containing v. Also, $S[0] = 0$ and the final answer is $S[m]$.

3. Let $C_1, C_2, \ldots, C_l$ be the sequence of maximal cliques in the interval ordering. Order the vertices $v_1, v_2, \ldots, v_n$ such that $r(v_{i+1}) \geq r(v_i)$ for $1 \leq i < n$. Here $r(v_i)$ is the index of the rightmost clique containing vertex v_i. Now consider vertices in the order $v_1, \ldots, v_n$ and include v_i in the required graph iff it does not form a clique of size $k+1$ with already included vertices. Consider an optimal solution that agrees with the greedy solution till v_i. If v_{i+1} is excluded by greedy, it forms a $k+1$-clique with already included vertices, which are also included in the optimal. Hence v_{i+1} is also not in optimal. If v_{i+1} is included in greedy but not in the optimal, add it to the optimal. Then there must be $k+1$ vertices (including v_{i+1}) selected in optimal from some maximal clique C_j. Let j be the smallest such index. At least one of these must be v_m with $m > i+1$, otherwise this clique is also in greedy, and v_{i+1} should have been rejected by the greedy algorithm. Since $r(v_m) \geq r(v_{i+1})$, replacing v_m by v_{i+1} gives a new optimal solution including v_{i+1}. Continuing this way, we can find an optimal solution that agrees with greedy for all vertices.

4. Consider a maximal set of paths in a tree that pairwise intersect in at least one edge, that is, form a maximal clique in an EPT-graph. Since they intersect in an edge, they also intersect pairwise in a vertex. Hence there is a vertex v that is contained in all paths in the clique. Suppose v is the endpoint of one of the paths P and let e be the edge in P incident with v. Then we claim all paths must contain e. Any path has an edge and hence at least two vertices in common with P, and if two vertices are common to two paths, then the entire path between them in the tree must be common. Thus the clique contains all paths that include e.

Suppose every path contains two edges incident with v. Since any two paths have at least two vertices in common, one of which is v, at least one edge incident with v must be common to any two paths. This is possible if and only if either one edge incident with v belongs to all paths or there are 3 edges e_1, e_2, e_3 incident with v such that every path contains exactly two of them.

This gives a characterization of maximal cliques in an EPT graph. To find maximum, for each edge find the number of paths containing it, and for each set of 3 edges with a common endpoint in the tree, find the number of paths that contain two of these. The maximum of these numbers is the size of the maximum clique.

5. The reduction is from the Numerical 3-partitioning problem. Given $3n$ numbers $a_1, a_2, \ldots, a_{3n}$, such that $B/4 < a_i < B/2$, $\sum_{1 \leq i \leq 3n} a_i = Bn$, can the numbers be partitioned into n parts, each part having sum exactly B and thus containing exactly 3 of the given numbers. This problem is known to be strongly NP-Complete.

Now construct an interval graph with $3n$ components, in which the ith component is a split graph with a clique of size $n-1$ and an independent set of size a_i, with every vertex in the independent set joined to each vertex in the $(n-1)$-clique. We claim that this interval graph can be colored with n colors such that no color is used more than $B+3n-3$ times iff the given set of numbers can be 3-partitioned.

Suppose there exists such a coloring. Since the maximum clique size in each component has size n, each color must be used at least once in every component. Further all vertices in the independent set of a_i vertices must have the same color. If the independent set in component i has color j put the number a_i in part j. Then since each color is used exactly $B+3n-3$ times, each part must contain exactly 3 numbers whose sum is B. Note that we can assume B is large compared to n. The converse follows the same way.

Here, the number of times a color can be used is specified as part of the input. However, the problem remains NP-Complete for interval graphs, even for fixed values at least 4. For 3 the status is unknown, while for 2 it can be solved easily using maximum matching in the complement of the graph.