Solutions to Homework 4

1. The proof is by induction on n. We will show that there is a representation in dimension $n-2$ such that the threshold vector is all-zero and the coordinates of all vectors are in $\{0, 1, -1\}$. If $n = 3$, then the graph is either K_3, P_3 or their complement. The vectors $\{1, 1, 1\}$, $\{-1, 1, -1\}$, $\{0, 0, -1\}$ and $\{-1, -1, -1\}$ represent the four possible graphs. Assume it is true for graphs with $n-1$ vertices. Let v be any vertex and consider a $n-3$ dimensional representation of $G - v$ satisfying the above properties. Now let the $(n-2)$th coordinate of the vector representing vertex $u \neq v$ be 1 if u is adjacent to v and 0 if it is not adjacent. Let the vector representing v have 1 in all coordinates except the $(n-2)$th, where its value is -1. This gives the required representation of G.

Another way of viewing the threshold dimension is that it is the minimum number of threshold graphs whose intersection is the given graph. So, if $G_1, G_2, \ldots, G_{n-3}$ are threshold graphs with $n-1$ vertices whose intersection is $G - v$, then G is the intersection of the graphs $G'_1, G'_2, \ldots, G'_{n-3}, G'_{n-2}$, where G'_i is obtained from G_i by adding vertex v and joining it to all vertices of G_i, for $1 \leq i \leq n-3$, and G'_{n-2} is a complete graph on all vertices other than v, with vertex v adjacent to those vertices to which it is adjacent in G.

A better bound is $n - \max(\alpha(n), \omega(n)) + 1$. Start with one-dimensional vectors $-1(+1)$ assigned to all vertices in a maximum independent set (maximum clique), and then add a new dimension for every additional vertex as above.

An example for which the threshold dimension is large is a graph consisting of two disjoint cliques of size $n/2$. If this graph is the intersection of threshold graphs, then in any of these threshold graphs the independent set has size at most two. Thus the independent set consists of at most one vertex from each clique and this must be joined to all other vertices in its clique. Because there is no induced P_4, this implies the threshold graph contains an $n-1$ clique with one vertex joined to all other vertices in its clique. Therefore one dimension can take care of absent edges incident with exactly one vertex in one of the cliques. This implies the threshold dimension is $n/2$.

2. If the threshold dimension is 2, G is the intersection of two threshold graphs G_1 and G_2. Let (C_1, I_1) and (C_2, I_2) be the partition of the vertex set into a clique and independent set in G_1 and G_2 respectively.

Let $C = C_1 \cap C_2$, $I = I_1 \cap I_2$, $A = I_1 \cap C_2$ and $B = I_2 \cap C_1$. Then C induces a clique in G. I is an independent set in G such that all neighbors of vertices in I are in C. Similarly A and B are independent sets in G. Thus a maximum clique in G is either C, or some vertex not in C together with its neighbors in C, or an edge joining a vertex in A to a vertex in B together with their common neighbors in C. The largest of these can be clearly found in polynomial-time. (It can be done better in linear time.)

The chromatic number can also be found the same way. First delete all vertices in $G - C$ that are isolated in $G - C$. This will delete all vertices in I and some others in $A \cup B$. These are simplicial vertices in G and can be colored easily once the rest of G is colored optimally. Either one of the existing colors can be assigned to them, or we get a clique of size equal to the number of colors used.

In the threshold graph G_1, the vertices of I_1 can be ordered $a_1, a_2, \ldots, a_p$ such that $N_1(a_1) \subseteq N_1(a_2) \subseteq \cdots \subseteq N_1(a_p)$. A similar ordering $b_1, b_2, \ldots, b_q$ holds for vertices in I_2 such that $N_2(b_i) \subseteq N_2(b_{i+1})$ for $1 \leq i < q$. Let i be the largest index such that a_i is not isolated in $G - C$ and let j be the largest such that b_j is not isolated. Then $a_i \in A$ and $b_j \in B$. We claim that $a_i b_j$ must be an edge in G. Otherwise, let $a_i b_m$ and $a_l b_j$ be edges in G for $m < j$ and $l < i$. This implies $a_l \in A$ and $b_m \in B$. Since a_l is adjacent to b_j in G_1, by the ordering of the vertices in I_1, a_i must also be adjacent to b_j in G_1. Similarly, since b_m is adjacent to a_i in G_2, b_j is adjacent to a_i in G_2. This implies a_i is adjacent to b_j in G.

So to color the graph obtained by deleting isolated vertices in $G - C$, it is enough to color the graph induced by $C \cup \{a_i, b_j\}$. Any other vertex a_l that is not isolated in $G - C$ can be given the same color as a_i and all non-isolated vertices b_m can be given the same color as b_j. Note that the neighbors of a_l in C must be a subset of the neighbors of a_i in C. Thus the number of colors used is either $|C|$, or $|C| + 1$, or $|C| + 2$ and is equal to the size of the largest clique.

3. A cograph is defined as a graph that does not contain an induced P_4. We show that G is a cograph iff for every subset S of at least two vertices, either the subgraph of G induced by S is disconnected or its complement is. Clearly if G satisfies this property, it cannot contain an induced P_4 as these 4 vertices induce a subgraph which is connected and also its complement is connected.

Conversely, suppose G is a cograph. For any set $S \subset V(G)$ of vertices, we can apply induction by removing a vertex v not in S, since $G - v$ is also a cograph. Hence it remains to show that either G or its complement is disconnected. This is essentially the proof that the vertex set of a connected cograph can be partitioned into two nonempty parts A, B such that every vertex in A is adjacent to every vertex in B. This was done in class as part of the characterization of cographs.

4. Suppose G is a cograph of order n. Let n_1, n_2, n_3 be numbers such that $n_1 + 2n_2 + 3n_3 = n$. We say that G has a (n_1, n_2, n_3) partition iff the vertices of G can be partitioned into n_i parts of size i such that each part induces a clique. So the question is to determine whether G has a $(0, 0, n/3)$ partition. We solve the more general problem.

Consider a binary tree representation of the cograph. Suppose the root node is white. Then G is the disjoint union of two cographs G_1 and G_2. Thus G has a (n_1, n_2, n_3) partition iff there exist numbers (m_1, m_2, m_3) such that $0 \leq m_i \leq n_i$, $\sum_{1\leq i\leq 3} im_i = |G_1|$ and G_1 has a (m_1, m_2, m_3) partition and G_2 has a $(n_1 - m_1, n_2 - m_2, n_3 - m_3)$ partition. There are at most $O(n^2)$ possible choices for (m_1, m_2, m_3) and we try all of them.

Suppose the root node of the binary tree is black. Then G is the join of two disjoint graphs G_1 and G_2. Now, G has a (n_1, n_2, n_3) partition iff there exist numbers $(m_1, m_2, m_3, m_4, m_5, m_6)$ such that G_1 has a $(m_1 + m_2 + m_3, m_4 + m_5, m_6)$ partition and G_2 has a $(n_1 - m_1 + m_2 + m_5, n_2 - m_2 - m_4 + m_3, n_3 - m_3 - m_5 - m_6)$ partition. Here m_1 is the number of isolated nodes from G_1, m_2 is the number of isolated nodes from G_1 that are combined with isolated nodes from G_2 to form edges in G, m_3 is the number of isolated nodes in G_1 combined with edges in G_2 to form triangles, etc. While a simple algorithm can try all possible values of the m_i, it is possible to restrict the choices significantly. For example, m_6 can be assumed to be either 0 or $n_3 - m_3 - m_5$. Also m_4 is either 0 or $n_2 - m_2 + m_3$. Even if all possibilities are tried, this is still a polynomial-time algorithm.

5. The algorithm is similar to the previous problem. Suppose the cograph is represented by a binary tree. Let $S[r, i]$ denote the maximum number of vertices in an induced subgraph of the cograph represented by the subtree rooted at node r having maximum independent set at most i. If r is a black node, then $S[r, i] = S[r \to left, i] + S[r \to right, i]$, if r is a white node, then $S[r, i] = \max_{0\leq j\leq i} S[r \to left, j] + S[r \to right, i - j]$, while if r is a leaf node $S[r, i] = 1$ if $i > 0$ and $S[r, 0] = 0$.