Homework 1

Submit by Monday August 9

NOTE: Preferably latex the solutions and submit by email to aad@cse.iitb.ac.in. For the first one or two homeworks, you can submit handwritten answers, but should get familiar with basic latex soon.

Q1. Prove that a graph G is chordal iff G is a complete graph or every minimal subset S of vertices such that $G - S$ is disconnected, induces a complete subgraph.

Q2. Let G be a chordal graph. A maximal clique of G is a complete subgraph of G that is not properly contained in any other complete subgraph of G. Let $\mathcal{C}$ be the collection of all maximal cliques in G. Consider a new weighted complete graph H whose vertices are the maximal cliques of G, and the weight of the edge between C_1 and C_2 is the number of vertices of G that are in $C_1 \cap C_2$. Let T be a maximum weight spanning tree in H. Prove that for every vertex v in G, the maximal cliques containing v form a subtree of T. Show that this gives a representation of the chordal graph as the intersection graph of subtrees of a tree. Prove that this is also the most compact representation, that is, amongst all possible representations, the tree has minimum possible size and for every vertex v, the subtree representing v also has minimum possible size.

Q3 Prove that in any chordal graph with n vertices there are at most n maximal cliques. Show that there is a graph with n vertices in which the number of maximal cliques is exponential in n.

Q4. The intersection graph of a collection of subpaths of a path is called an interval graph. The intersection graph of a collection of subpaths of a tree is called a path graph. Thus an interval graph is also a path graph and a path graph is also a chordal graph. Give examples to show that the inclusion is proper, that is, give a chordal graph that is not a path graph and a path graph that is not an interval graph. Try to find the smallest possible examples. Note that intersection is non-empty iff there is a common vertex.

Q5. Let T be a tree and $\{T_1, T_2, \ldots, T_n\}$ a collection of subtrees of T identified by their edge set. Thus T_1 intersects T_2 iff there is an edge of T common to both. Prove that any graph can be obtained as the intersection graph of subtrees of a tree, if intersection is defined this way.