An *asteroidal triple* in a graph is a subset of 3 mutually non-adjacent vertices such that for every pair of vertices, there is a path in the graph that does not pass through the third vertex or any of its neighbors.

Theorem 1 *A graph G is an interval graph iff it is chordal and has no asteroidal triple.*

Proof: Suppose G is an interval graph. By definition, it is the intersection graph of subpaths of a path, and hence is chordal. Suppose it has an asteroidal triple x, y, z. Then the paths representing x, y, z must be disjoint and without loss of generality appear in the order P_x, P_y, P_z in the path P. Then every path from x to z in G must pass through y or a neighbor of y. This contradicts the assumption that x, y, z is an asteroidal triple.

Conversely, we prove that if G is a chordal, asteroidal triple free graph, it must be an interval graph. The proof is by induction on the number of vertices. If $|G| = 1$, this is trivial. Let $v_1, v_2, \ldots, v_n$ be a perfect elimination ordering of G. Then $G - v_1$ is a chordal, asteroidal triple free graph, since an asteroidal triple in $G - v_1$ is also an asteroidal triple in G. By induction, $G - v_1$ is an interval graph. Consider a representation of $G - v_1$ as the intersection graph of subpaths of a path P, such that $|P|$ is minimum. We may assume that in such a representation, every vertex of P is the start point of some subpath and also an endpoint of some subpath. Since the neighbors of v_1 induce a complete graph, the subpaths representing them intersect in a common vertex say x. If x is an endpoint of P, then we can add a new vertex x' to P adjacent to x, and extend all subpaths representing neighbors of v_1 in G, to include the vertex x'. Suppose x is not an endpoint.

Let $u_0, u_1, \ldots, u_m$ be the vertices in P and suppose $x = u_t$ for some $1 \leq t < m$. We add the edge $x'u_t$ to P and to all subpaths representing neighbors of v_1. v_1 itself is represented by the subpath x'. This gives a representation of G as the intersection graph of subtrees of a tree T. We show that if there is no asteroidal triple, the tree can be modified so that it is a path.

We say an edge e in T is covered by vertex v if every subpath representing some vertex of G that contains the edge e also contains the vertex v. In this case, we can get another tree representing G by deleting the edge e and joining one of its endpoints to v, in such a way that the resulting graph is a tree. Note that if 3 independent vertices do not form an asteroidal triple, then there must exist an edge covered by one of the vertices, such that deleting the edge from the tree separates the subpaths representing the other two vertices.

If there is an index i, $1 \leq i \leq t$ such that $u_{i-1}u_i$ is covered by x', let a_0 be the largest such index otherwise let $a_0 = 0$. Similarly, if there is an index $j, t \leq j < m$, such that $u_j u_{j+1}$ is covered by either u_{a_0} or x', let b_0 be the smallest such index, otherwise let $b_0 = m$.

If the edge $x'u_t$ is covered by u_{a_0}, we can delete the edges $x'u_t$ and $u_{a_0-1}u_{a_0}$ (if $a_0 > 0$) in T, and add the edges $x'u_{a_0}$ and $x'u_{a_0-1}$ to get an interval representation of G. The same argument holds if u_{b_0} covers $x'u_t$ and $u_{b_0}u_{b_0+1}$ is covered by x'. If $u_{b_0}u_{b_0+1}$ is covered by u_{a_0}, we replace

$u_{b_0}u_{b_0+1}$ by $u_{a_0}u_{b_0+1}$, $u_{a_0-1}u_{a_0}$ by $u_{a_0-1}x'$ (if $a_0 > 0$) and $x'u_t$ by $x'u_{b_0}$. Again this gives an interval representation of G.

Suppose neither u_{a_0} nor u_{b_0} covers the edge $x'u_t$. This implies that $a_0 < t$ and $b_0 > t$. We say a sequence of indices $a_0, b_0, a_1, b_1, \ldots$ is a bad sequence if it satisfies the following properties.

1. $0 \leq a_0 < a_1 < \cdots < t$ and $m \geq b_0 > b_1 > \cdots > t$.
2. Either $a_0 = 0$ or $u_{a_0-1}u_{a_0}$ is covered by x'.
3. Either $b_0 = m$ or $u_{b_0}u_{b_0+1}$ is covered by x' or u_{a_0}.
4. No edge u_iu_{i+1} for $a_0 \leq i < b_0$ is covered by x'.
5. For all $i \geq 1$, the edge $u_{a_i-1}u_{a_i}$ is covered by $u_{b_{i-1}}$ and the edge $u_{b_i}u_{b_i+1}$ is covered by u_{a_i}.
6. There is no edge u_ju_{j+1} for $a_i \leq j < t$ that is covered by $u_{b_{i-1}}$ and there is no edge $u_{j-1}u_j$ for $t < j \leq b_i$ that is covered by u_{a_i}.
7. There is no index a_i or b_i such that u_{a_i} or u_{b_i} covers the edge $x'u_t$.

It can be seen that a_0, b_0 is a bad sequence. Suppose $a_0, b_0, \ldots a_l, b_l$ is the longest bad sequence for some $l \geq 0$. There must be a subpath representing some vertex in $G - v_1$ ending at u_{a_l} and this cannot contain the edge $u_{a_l-1}u_{a_l}$, since that edge, if it exists, is covered by $u_{b_{l-1}}$ (or x' if $l = 0$). Thus this subpath must contain only the vertex u_{a_l}. Similarly, there is a subpath containing only the vertex u_{b_l}. Let these subpaths represent the vertices x_l, y_l respectively. Note that no edge in the subpath of P from u_{a_l} to u_{b_l} is covered by x' and no edge in the path from u_t to u_{b_l} is covered by u_{a_l}. Further u_{a_l} and u_{b_l} do not cover the edge $x'u_t$.

Since x_l, y_l, v_1 do not form an asteroidal triple, there must be an index $j, a_l < j \leq t$ such that the edge $u_{j-1}u_j$ is covered by b_l. Let a_{l+1} be the largest such index. If $u_{a_{l+1}}$ does not cover $x'u_t$, we get a longer bad sequence $a_0, b_0, \ldots, a_l, b_l, a_{l+1}$ contradicting the choice of the longest bad sequence.

Suppose $u_{a_{l+1}}$ covers $x'u_t$. Now we construct an interval representation of G as follows. Replace all edges $u_{a_i-1}u_{a_i}$ for $1 \leq i \leq l+1$ by $u_{a_i-1}u_{b_{i-1}}$ and $u_{b_i}u_{b_i+1}$ for $1 \leq i \leq l$ by $u_{b_i+1}u_{a_i}$. Replace $x'u_t$ by $x'u_{a_{l+1}}$. If $u_{b_0}u_{b_0+1}$ is covered by x' replace $u_{b_0}u_{b_0+1}$ by $x'u_{b_0+1}$, else if it is covered by u_{a_0}, replace it by $u_{a_0}u_{b_0+1}$ and $u_{a_0-1}u_{a_0}$ by $x'u_{a_0-1}$ (if $a_0 > 0$). This gives an interval representation of G.

A similar argument holds if the longest bad sequence is of the form $a_0, b_0, a_1, \ldots, a_l, b_l, a_{l+1}$ for some $l \geq 0$. Therefore if G is asteroidal-triple free, we get an interval representation of G, and thus G is an interval graph.