Partitioning 2-connected graphs into connected subgraphs.

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Abstract

A classic result of Lovasz and Györi states that if G is a k-connected graph of order n, and $n = n_1 + n_2 + \cdots + n_k$ is any partition of n into k parts, then $V(G)$ can be partitioned into k parts $V_1, V_2, \ldots, V_k$ such that $|V_i| = n_i$ and V_i induces a connected subgraph of G, for all $1 \leq i \leq k$. We call such a partition a k-partition of G and G is said to be k-partitionable. The complete bipartite graph $K_{k,k+2}$ shows that k-connected graphs are not necessarily $(k+1)$-partitionable. In this article we consider k-connected graphs with maximum degree at most $k+1$. We show that 2-connected graphs with maximum degree at most 3 are 4-partitionable. As a consequence, 2-connected claw-free graphs are 4-partitionable, which implies that the edges of a 2-edge-connected graph can be 4-partitioned into connected parts of specified sizes. We show that k-connected graphs with maximum degree $\leq k+1$ are not $(2k+1)$-partitionable in general. Thus the result is best possible for 2-connected graphs and we suggest possible generalizations for k-connected graphs.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. All graphs considered are simple, finite and undirected. Let $n_1, n_2, \ldots, n_k$ be positive integers such that $|V(G)| = n_1 + n_2 + \cdots + n_k$. A k-partition of G into parts of sizes $n_1, n_2, \ldots, n_k$ is a partition of $V(G)$ into k parts $V_1, V_2, \ldots, V_k$ such that $|V_i| = n_i$ and V_i induces a connected subgraph of G for $1 \leq i \leq k$. A graph G of order $n \geq k$ is said to be k-partitionable if for every partition $n = n_1 + n_2 + \cdots + n_k$ of n into k parts, there exists a k-partition of G into parts of sizes $n_1, n_2, \ldots, n_k$.

A classic result of Lovasz [6] and independently, Györi [4] is that k-connected graphs are k-partitionable, for all $k \geq 1$. Polynomial-time algorithms are known for finding a 2-partition of a 2-connected graph, a 3-partition of a 3-connected graph and a 4-partition of a 4-connected planar graph, into parts of specified sizes [7, 8]. However, no such algorithms are known for $k \geq 4$ for general k-connected graphs, although the partitions are known to exist.

The complete bipartite graph $K_{k,k+2}$, which has no perfect matching, shows that in general, k-connected graphs are not $(k+1)$-partitionable. In this article, we show that 2-connected graphs with maximum degree ≤ 3 are 4-partitionable. Our proof is constructive and it also gives a polynomial-time algorithm for finding a 4-partition into parts of specified sizes. It is known that every 2-connected claw-free graph contains a spanning 2-connected subgraph with maximum degree ≤ 3 (lemma 3.5 in [5]). Thus our result also implies that 2-connected claw-free graphs are 4-partitionable. In particular, the edges of any 2-edge-connected graph can be partitioned into 4 parts of specified sizes such that each part induces a connected subgraph. In [5], it is shown that 2-connected claw-free graphs of order $3k$ can be k-partitioned into parts of size 3.

In the more general case, we show that k-connected graphs with maximum degree $\leq k+1$ are not necessarily $(2k+1)$-partitionable. Thus the result for 2-connected graphs is best possible. It is also the best possible for partitioning edges of 2-edge-connected graphs into connected parts. We believe that k-connected graphs with maximum degree $\leq k+1$ are $2k$-partitionable. We also do not know any example of a k-connected, k-regular graph which is not l-partitionable, for any l, and believe that there is no such graph. It may be noted that in [5], it is conjectured that 3-connected claw-free graphs are l-partitionable for all $l \geq 1$.

In the next section, we introduce our notation and some basic properties of 2-connected and 3-connected graphs. In section 3, we prove the main result for 2-connected graphs and conclude by suggesting possible generalizations in section 4.

2 Basic Results

The notation and terminology used is standard and may be found in, for example, [2]. The vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$. The order of G, denoted $|G|$, is the cardinality of $V(G)$. An edge incident with vertices u and v is denoted by uv. The degree of a vertex is the number of edges incident with it and $\Delta(G)$ is the maximum degree of vertices in G. If $S \subset V(G)$, $G - S$ is the subgraph of G obtained by deleting all vertices in S and edges incident with them. The subgraph induced by S is $G - V(G) \setminus S$. We write $G - u$ instead of $G - \{u\}$. The graph obtained by deleting an edge uv in G is denoted $G - uv$. If u, v are two distinct vertices in G, $G + uv$ is the graph obtained by adding the edge uv to G, if not already present. If G_1 and G_2 are two graphs, $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If uv is any edge in a graph G, the graph obtained from $G - uv$ by adding a new vertex w and the edges uw, vw is said to be obtained from G by subdividing edge uv. G is said to be a subdivision of $G^{'}$ if it can be obtained from $G^{'}$ by repeatedly subdividing edges.

A graph G is k-connected if $|G| \geq k + 1$ and $G - S$ is connected for all subsets $S \subset V(G)$ with $|S| < k$. A *cutvertex* in a connected graph G is a vertex v such that $G - v$ is disconnected. A *block* of a connected graph G is a maximal subgraph of G without a cutvertex. Every block of a connected graph G with $|G| > 1$ is either K_2 or a 2-connected graph. A block is said to be trivial if it is K_2. The *block graph* of a connected graph G is a bipartite graph whose vertices are the blocks and cutvertices of G, and a cutvertex v is adjacent to a block B iff $v \in V(B)$. The block graph of any connected graph G is a tree whose leaf vertices are blocks of G, called the *end-blocks* of G.

An *ear decomposition* of a graph G is a partition of G into edge-disjoint subgraphs $P_0, P_1, \ldots, P_k$ such that P_0 is a cycle and P_i is a path of length at least one with endvertices in $V(G_{i-1})$ and whose internal vertices are not contained in $V(G_{i-1})$, where $G_{i-1} = P_0 \cup \cdots \cup P_{i-1}$, for all $1 \leq i \leq k$. An ear decomposition is said to be *non-separating* if $G - V(G_i)$ is connected or empty, for all $0 \leq i < k$. If st is any edge in a graph G of order n, an st-numbering of G is a numbering of the vertices of G from 1 to n, such that s is numbered 1, t is numbered n, and the vertex with number k is adjacent to a vertex with a number $< k$ and also to a vertex with a number $> k$, for all $1 < k < n$. We recall some well-known properties of 2-connected and 3-connected graphs.

Lemma 2.1 *A graph G has an ear decomposition iff it is* 2*-connected.*

Lemma 2.2 *A graph G has a st-numbering iff it is* 2*-connected.*

Lemma 2.3 *Let uv be any edge in a* 3*-connected graph G and x be any vertex different from u, v. There is a non-separating ear decomposition of G*

such that $uv \in E(P_0)$ and if $x \in V(P_i)$ then P_j has length 1, for all $j > i$. Further, P_0 may be chosen to be an induced cycle.

The proof of lemma 2.1 may be found [2] while [3] gives an efficient algorithm for computing an st-numbering of a 2-connected graph. Lemma 2.3 is proved in [1], which also gives a polynomial-time algorithm for finding such a non-separating ear decomposition of a 3-connected graph.

Since we are interested in graphs with maximum degree ≤ 3, we note the following strengthening of lemma 2.3 for 3-connected cubic graphs. An ear decomposition is said to be *induced* if P_0 is an induced cycle and every path has length > 1. Thus G_i is an induced subgraph of G, for $0 \leq i \leq k$. Note that if a graph G has a non-separating induced ear decomposition then every subdivision of G also has a non-separating induced ear decomposition.

Lemma 2.4 *Let G be a 3-connected cubic graph and uv any edge in G. Then $G - uv$ has a non-separating induced ear decomposition such that $u \in V(P_0)$ and P_k has length 2 with v as the internal vertex.*

Proof: This follows by considering a non-separating ear decomposition of G such that $u \in V(P_0)$, P_0 is an induced cycle and v is the last vertex to be added. At every step choose the shortest possible path that can be added. If there is an edge $xy \notin E(P_i)$ such that x is an internal vertex of P_i and $y \in V(G_i)$, either $V(G_i) = V(G)$ or there is another internal vertex in P_i adjacent to a vertex $\notin V(G_i)$. In either case we can choose a shorter path as P_i. Since u and v are the only vertices of degree 2 in $G - uv$, the last ear P_k must have v as its only internal vertex. $\square$

Finally, for applying induction, we extend the notion of k-partition as follows. Let G be any graph of order $n \geq k$ and let $n_1, n_2, \ldots, n_k$ be positive integers such that $n \leq n_1 + n_2 + \cdots + n_k$. A k-partition of G into parts of sizes *at most* $n_1, n_2, \ldots, n_k$ is a partition of $V(G)$ into k parts $V_1, V_2, \ldots, V_k$ such that V_i induces a connected subgraph of G and $1 \leq |V_i| \leq n_i$, for $1 \leq i \leq k$. We call $n_i - |V_i|$ the *deficiency* of the part and the part is said to be *incomplete* if the deficiency is > 0.

3 Main Result

Lemma 3.1 *Let u, v be two distinct vertices in a graph G of order n such that $G + uv$ is 2-connected. If n_1, n_2 are positive integers such that $n = n_1 + n_2$ then G can be 2-partitioned into parts V_1, V_2 of sizes n_1, n_2 such that $u \in V_1$ and $v \in V_2$.*

Proof: This follows from a uv-numbering of $G + uv$. Let V_1 be the set of vertices numbered from 1 to n_1 and $V_2 = V(G) \setminus V_1$. $\square$

We prove a generalization of this for incomplete 2-partitions.

Lemma 3.2 *Let u, v be two distinct vertices in a graph G of order n such that $G + uv$ is 2-connected. Let p, q be two distinct vertices in G not necessarily different from u, v. If n_1, n_2 are positive integers such that $n \leq n_1 + n_2$ then G can be 2-partitioned into parts V_1, V_2 of sizes at most n_1, n_2 such that $u \in V_1$, $v \in V_2$ and any incomplete part contains at least one of the vertices p, q.*

Proof: Again, uv-number the vertices of $G + uv$. Let the numbers of vertices p and q be i and j respectively and assume $i < j$.

If $n_1 < i$, let V_1 be the set of vertices with numbers from 1 to n_1 and let $V_2 = V(G) \setminus V_1$. In this case V_2 may be an incomplete part containing both p and q.

If $n - n_2 \leq i \leq n_1$, let V_2 be the set of vertices numbered from $i + 1$ to n and let $V_1 = V(G) \setminus V_2$. In this case both V_1 and V_2 may be incomplete with $p \in V_1$ and $q \in V_2$.

Finally, if $i < n - n_2$, let V_2 be the set of vertices numbered from $n - n_2 + 1$ to n and let $V_1 = V(G) \setminus V_2$. In this case V_1 may be an incomplete part containing p. $\square$

We note that lemmas 3.1 and 3.2 also hold trivially if the graph G is K_2.

Lemma 3.3 *Let G be a 2-connected graph of order n with $\Delta(G) \leq 3$. Let u be a vertex of degree 2 in G. If n_1, n_2, n_3 are positive integers such that $n = n_1 + n_2 + n_3$ then G can be 3-partitioned into parts V_1, V_2, V_3 of sizes n_1, n_2, n_3 such that $u \in V_1$.*

Proof: The proof is by induction on n_1. Let u_1, u_2 be the neighbors of u.

If $n_1 = 1$, let $V_1 = \{u\}$. The graph $(G - u) + u_1u_2$ has order $n_2 + n_3$ and is either 2-connected or K_2. Applying lemma 3.1, $G - u$ can be 2-partitioned into parts V_2, V_3 of sizes n_2, n_3 such that $u_1 \in V_2$ and $u_2 \in V_3$.

Suppose $n_1 > 1$. If $G - u$ is 2-connected, by induction, it can be 3-partitioned into parts $V_1^{'}, V_2, V_3$ of sizes $n_1 - 1, n_2, n_3$ such that $u_1 \in V_1^{'}$. Let $V_1 = V_1^{'} \cup \{u\}$.

If $G - u$ is not 2-connected, it must contain a cutvertex and the block graph of $G - u$ is a path of length at least two. Let B_1, B_2 be the end-blocks of $G - u$ such that $u_1 \in V(B_1)$ and $u_2 \in V(B_2)$. Let v_1, v_2 be the cutvertices of $G - u$ contained in B_1, B_2 respectively. Note that v_1 and v_2 may be the same vertex but they are distinct from u_1 and u_2.

If $|B_1| \leq n_1$, let $G^{'} = (G - V(B_1) \setminus \{v_1\}) + uv_1$. The graph $G^{'}$ is 2-connected with $n_1 + 1 - |B_1| + n_2 + n_3$ vertices and by induction, it can be 3-partitioned into parts $V_1^{'}, V_2, V_3$ of sizes $n_1+1-|B_1|, n_2, n_3$ such that $u \in V_1^{'}$. Let $V_1 = V_1^{'} \cup (V(B_1) \setminus \{v_1\})$.

If $n_1 < |B_1| < n_1 + n_2$ then 2-partition B_1 into parts $V_1^{'}, V_2^{'}$ of sizes $n_1 - 1, |B_1| + 1 - n_1$ such that $u_1 \in V_1^{'}$ and $v_1 \in V_2^{'}$. Let $V_1 = V_1^{'} \cup \{u\}$ and let $G^{'} = (G-u) - V(B_1) \setminus \{v_1\}$. The graph $G^{'} + v_1u_2$ is either 2-connected or K_2 and contains $n_2^{''} + n_3$ vertices, where $n_2^{''} = n_1 + n_2 - |B_1|$. Thus $G^{'}$ can be 2-partitioned into parts $V_2^{''}, V_3$ of sizes $n_2^{''}, n_3$ such that $v_1 \in V_2^{''}$ and $u_2 \in V_3$. Let $V_2 = V_2^{'} \cup V_2^{''}$.

Finally, if $n_1 + n_2 \leq |B_1|$ then $|B_2| \leq n_3 < n_1 + n_3$ and we can apply the above argument by replacing B_1 by B_2 and n_2 by n_3. □

We next prove an analog of lemma 3.3 for incomplete 3-partitions.

Lemma 3.4 *Let G be a 2-connected graph of order n with $\Delta(G) \leq 3$ and let u, v, w be distinct vertices of degree 2 in G. If n_1, n_2, n_3 are positive integers such that $n > n_1 + n_2$, $n > n_1 + n_3$ and $n \leq n_1 + n_2 + n_3$ then G can be 3-partitioned into parts V_1, V_2, V_3 of sizes at most n_1, n_2, n_3 such that $u \in V_1$ and any incomplete part contains at least one of the vertices v, w.*

Proof: The proof is again by induction on n_1. Again, let u_1, u_2 be the neighbors of u in G.

If $n_1 = 1$, let $V_1 = \{u\}$. The graph $(G-u)+u_1u_2$ is either 2-connected or K_2 and contains at most n_2+n_3 vertices. Applying lemma 3.2, $G-u$ can be 2-partitioned into parts V_2, V_3 of sizes at most n_2, n_3 such that $u_1 \in V_2, u_2 \in V_3$ and any incomplete part contains at least one of the vertices v, w.

Suppose $n_1 > 1$. If $G-u$ is 2-connected then u_1, u_2 have degree 3 in G and must be distinct from v, w. By induction, $G-u$ can be 3-partitioned into parts $V_1^{'}, V_2, V_3$ of sizes at most $n_1 - 1, n_2, n_3$ such that $u_1 \in V_1^{'}$ and any incomplete part contains at least one of the vertices v, w. Let $V_1 = V_1^{'} \cup \{u\}$.

If $G-u$ is not 2-connected, it must contain a cutvertex and the block graph of $G-u$ is a path of length at least two. Let B_1, B_2 be the end-blocks of $G-u$ such that $u_1 \in V(B_1)$ and $u_2 \in V(B_2)$. Let v_1, v_2 be the cutvertices of $G-u$ contained in B_1, B_2 respectively.

Case 1. Suppose neither v nor w is contained in $B_1 - v_1$. If $|B_1| \leq n_1$, let $G^{'} = (G - V(B_1) \setminus \{v_1\}) + uv_1$. The graph $G^{'}$ satisfies the hypothesis of the lemma with n_1 replaced by $n_1^{'} = n_1 + 1 - |B_1|$ and by induction, it can be 3-partitioned into parts $V_1^{'}, V_2, V_3$ of sizes at most $n_1^{'}, n_2, n_3$ such that $u \in V_1^{'}$ and any incomplete part contains at least one of the vertices v, w. Let $V_1 = V_1^{'} \cup (V(B_1) \setminus \{v_1\})$.

If $n_1 < |B_1| < n_1 + n_2$ then 2-partition B_1 into parts $V_1^{'}, V_2^{'}$ of sizes $n_1 - 1, |B_1| + 1 - n_1$ such that $u_1 \in V_1^{'}$ and $v_1 \in V_2^{'}$. Let $V_1 = V_1^{'} \cup \{u\}$ and let $G^{'} = (G-u) - V(B_1) \setminus \{v_1\}$. The graph $G^{'} + v_1u_2$ is either 2-connected or K_2 and contains at most $n_2'' + n_3$ vertices, where $n_2'' = n_1 + n_2 - |B_1|$. By lemma 3.2, G' can be 2-partitioned into parts V_2'', V_3 of sizes at most n_2'', n_3 such that $v_1 \in V_2'', u_2 \in V_3$ and any incomplete part contains at least one of the vertices v, w. Let $V_2 = V_2' \cup V_2''$.

Finally, suppose $n_1 + n_2 \leq |B_1|$. If $|B_1| \leq n_2 + n_3$ then 2-partition B_1 into parts V_2, V_3' of sizes $n_2, |B_1| - n_2$ such that $u_1 \in V_2$ and $v_1 \in V_3'$. Let $G' = G - V(B_1) \setminus \{v_1\}$. The graph $G' + uv_1$ is 2-connected and contains at most $n_1 + n_3''$ vertices, where $n_3'' = n_2 + n_3 + 1 - |B_1|$. Thus G' can be 2-partitioned into parts V_1, V_3'' of sizes at most n_1, n_3'' such that $u \in V_1$, $v_1 \in V_3''$ and any incomplete part contains at least one of the vertices v, w. Let $V_3 = V_3' \cup V_3''$.

If $|B_1| > n_2 + n_3$, by lemma 3.3, B_1 can be 3-partitioned into parts V_1', V_2, V_3 of sizes $|B_1| - n_2 - n_3, n_2, n_3$ such that $u_1 \in V_1'$. Let $V_1 = V_1' \cup (V(G) \setminus V(B_1))$.

A symmetrical argument can be used if neither v nor w is contained in $B_2 - v_2$.

Case 2. We may therefore assume, without loss of generality, that $v \in V(B_1) \setminus \{v_1\}$ and $w \in V(B_2) \setminus \{v_2\}$. We may further assume that $|B_1| < n_1 + n_2$, otherwise $|B_2| \leq n_3 < n_1 + n_3$ and a similar argument can be applied, replacing B_1 by B_2 and n_2 by n_3.

If $|B_1| \leq n - n_2 - n_3$, let $G' = (G - V(B_1) \setminus \{v_1\}) + uv_1$. The graph G' is 2-connected with $\Delta(G') \leq 3$ and contains $n_1' + n_2 + n_3$ vertices, where $n_1' = n + 1 - n_2 - n_3 - |B_1|$. By lemma 3.3, G' can be 3-partitioned into parts V_1', V_2, V_3 of sizes n_1', n_2, n_3 such that $u \in V_1'$. Let $V_1 = V_1' \cup (V(B_1) \setminus \{v_1\})$. In this case V_1 may be incomplete with $v \in V_1$.

If $n - n_2 - n_3 < |B_1| \leq n_1$, let $V_1 = (V(B_1) \setminus \{v_1\}) \cup \{u\}$ and let $G' = (G - u) - V(B_1) \setminus \{v_1\}$. Since $n > n_1 + n_2$, $n_2 < |G'| < n_2 + n_3$. Also, $G' + v_1 w$ is either 2-connected or K_2 since $w \in V(B_2) \setminus \{v_2\}$. The graph G' can be 2-partitioned into parts V_2, V_3 of sizes $n_2, |G'| - n_2$ such that $v_1 \in V_2$ and $w \in V_3$. In this case both V_1 and V_3 may be incomplete with $v \in V_1$ and $w \in V_3$.

Finally, if $n_1 < |B_1|$ then 2-partition B_1 into parts V_1', V_2' of sizes $n_1 - 1, |B_1| + 1 - n_1$ such that $u_1 \in V_1'$ and $v_1 \in V_2'$. Let $V_1 = V_1' \cup \{u\}$. The graph $G' = (G - u) - V(B_1) \setminus \{v_1\}$ has order greater than $n_2'' = n_1 + n_2 - |B_1|$ and at most $n_2'' + n_3$ and $G' + v_1 w$ is either 2-connected or K_2. Thus G' can be 2-partitioned into parts V_2'', V_3 of sizes $n_2'', |G'| - n_2''$ such that $v_1 \in V_2''$ and $w \in V_3$. Let $V_2 = V_2' \cup V_2''$. In this case V_3 may be incomplete with $w \in V_3$.

This completes the proof of lemma 3.4. $\square$

Theorem 3.1 *Let G be any 2-connected graph of order $n \geq 4$ with $\Delta(G) \leq 3$. If $n = n_1 + n_2 + n_3 + n_4$ is any partition of n such that $n_1 \geq n_2 \geq n_3 \geq n_4 \geq$*

1 *then* G *can be* 4*-partitioned into parts* V_1, V_2, V_3, V_4 *of sizes* n_1, n_2, n_3, n_4.

Proof: We will prove the theorem by induction on the number of vertices of degree 3 in G. If there is no vertex of degree 3 in G then G is a cycle and it is trivial to find a 4-partition of G. If there is an edge uv in G such that $G - uv$ is 2-connected then by induction, $G - uv$ can be 4-partitioned since it has fewer vertices of degree 3 than G. A 4-partition of $G - uv$ is also a 4-partition of G.

We may therefore assume that G has at least 2 vertices of degree 3 and that G is minimally 2-connected. Note that the block graph of $G - uv$ is a path. We can enumerate the blocks and cutvertices of $G - uv$ as $B_1, B_2, \ldots, B_k$ and $v_1, v_2, \ldots, v_{k-1}$ respectively, such that $V(B_i) \cap V(B_{i+1}) = \{v_i\}, 1 \leq i < k$ and $u \in V(B_1)$, $v \in V(B_k)$. Let $v_0 = u$, $v_k = v$ and denote by B_0 the graph containing the vertices u, v and the edge uv. We note that since v_i has degree at most 3 in G, either B_i or B_{i+1} must be a trivial block, $0 \leq i \leq k$, where subscript addition is modulo k+1.

Case 1. Suppose there is an edge uv in G such that $G - uv$ contains a non-trivial block B_i of order at most n_4. Let G' be the graph obtained from $G - V(B_i) \setminus \{v_{i-1}, v_i\}$ by adding a path P of order $|B_i|$ with endvertices v_{i-1}, v_i and whose internal vertices are disjoint from $V(G)$. Since G' has fewer vertices of degree 3 than G, by induction, it can be 4-partitioned into parts of sizes n_1, n_2, n_3, n_4.

Any part in this 4-partition which contains a vertex of P must also contain either v_{i-1} or v_i, otherwise it can have size at most $n_4 - 2$, a contradiction. If $V(P)$ is contained in only one part in the 4-partition of G' then replacing $V(P)$ by $V(B_i)$ in that part gives a 4-partition of G. If there are two parts containing vertices of P, one must contain v_{i-1} and the other v_i. Suppose the part containing v_{i-1} contains n_1' vertices of P and the other part n_2'. Since B_i is 2-connected, we can find a 2-partition of B_i into parts V_1', V_2' of sizes n_1', n_2' such that $v_{i-1} \in V_1'$ and $v_i \in V_2'$. Replace the vertices of P in the part containing v_{i-1} in the 4-partition of G' by V_1' and the vertices of P in the other part by V_2'. This gives the required 4-partition of G.

We can therefore assume that for every edge uv, all non-trivial blocks of $G - uv$ have order $> n_4$.

Case 2. Suppose there exists an edge uv in G such that $G - uv$ contains a non-trivial block B_i of order at most $n_1 + n_2 + n_3$.

Suppose $n_4 + n_3 < |B_i| \leq n_4 + n_3 + n_2$. Applying lemma 3.3, B_i can be 3-partitioned into parts V_2', V_3, V_4 of sizes $|B_i| - n_3 - n_4, n_3, n_4$ such that $v_{i-1} \in V_2'$. Since B_i is non-trivial, B_{i+1} must be trivial. Let $G' = G - V(B_i) \setminus \{v_{i-1}\}$. The graph $G' + v_{i-1}v_{i+1}$ is either 2-connected or K_2 and hence G' can be 2-partitioned into parts V_2'', V_1 of sizes $n + 1 - |B_i| - n_1, n_1$ such that $v_{i-1} \in V_2''$

and $v_{i+1} \in V_1$. Let $V_2 = V_2^{'} \cup V_2^{''}$.

If $|B_i| > n_4 + n_3 + n_2$ then $n_2 + n_3 < |B_i| \leq n_2 + n_3 + n_1$. We can now apply the previous argument, replacing n_4 by n_2, n_2 by n_1 and n_1 by n_4.

Suppose $n_4 < |B_i| \leq n_4 + n_3$. Let $G_{p,q} = B_p \cup B_{p+1} \cup \cdots \cup B_q, 0 \leq p, q \leq k$ be a subgraph of G, where subscript addition is again modulo k+1. Note that $G_{p,q} + v_{p-1}v_q$ is either 2-connected or K_2 for all $0 \leq p, q \leq k$. Let j be such that $|G_{i,j}| > n_4 + n_3$ but $|G_{i,j-1}| \leq n_4 + n_3$. Note that $j \notin \{i, i-1\}$.

If $|G_{i,j}| \leq n_4 + n_3 + n_2$ then 2-partition B_j into parts $V_3^{'}, V_2^{'}$ of sizes $n_4 + n_3 + 1 - |G_{i,j-1}|, |G_{i,j}| - n_4 - n_3$ such that $v_{j-1} \in V_3^{'}$ and $v_j \in V_2^{'}$. The graph $G_{i,j-1}$ can be 2-partitioned into parts $V_4, V_3^{''}$ of sizes $n_4, |G_{i,j-1}| - n_4$ such that $v_{i-1} \in V_4$ and $v_{j-1} \in V_3^{''}$. Let $V_3 = V_3^{'} \cup V_3^{''}$. Since the block B_{i-1} is trivial, the graph $G_{j+1,i-2}$ can be 2-partitioned into parts $V_1, V_2^{''}$ of sizes $n_1, |G_{j+1,i-2}| - n_1$ such that $v_{i-2} \in V_1$ and $v_j \in V_2^{''}$. Let $V_2 = V_2^{'} \cup V_2^{''}$.

If $|G_{i,j}| > n_4 + n_3 + n_2$ then $|B_j| > n_2 + 1$ and B_j must be non-trivial. If $|B_j| > n_4 + n_3$ then we can apply the earlier argument considering B_j instead of B_i, since $|B_j| < n_1 + n_2 + n_3$. If $|B_j| \leq n_4 + n_3 \leq n_2 + n_1$ then 2-partition B_j into parts $V_2, V_1^{'}$ of sizes $n_2, |B_j| - n_2$ such that $v_{j-1} \in V_2$ and $v_j \in V_1^{'}$. The block B_{j-1} must be trivial and hence $|G_{i,j-2}| = |G_{i,j}| - |B_j| > (n_4 + n_3 + n_2) - (n_4 + n_3) = n_2 \geq n_3$. The graph $G_{i,j-2}$ can be 2-partitioned into parts $V_3, V_4^{'}$ of sizes $n_3, |G_{i,j-2}| - n_3$ such that $v_{j-2} \in V_3$ and $v_{i-1} \in V_4^{'}$. Finally, 2-partition $G_{j+1,i-1}$ into parts $V_1^{''}, V_4^{''}$ of sizes $n_1 + n_2 + 1 - |B_j|, n_3 + n_4 + 1 - |G_{i,j-2}|$ such that $v_j \in V_1^{''}$ and $v_{i-1} \in V_4^{''}$. Let $V_1 = V_1^{'} \cup V_1^{''}$ and $V_4 = V_4^{'} \cup V_4^{''}$.

Case 3. We may now assume that for every edge uv, any non-trivial block in $G - uv$ has order $> n_1 + n_2 + n_3$, which implies $G - uv$ contains exactly one non-trivial block. Such a graph is a subdivision of a 3-edge-connected graph and since $\Delta(G) \leq 3$, it is a subdivision of a 3-edge-connected cubic graph $G^{'}$. If G contains exactly 2 vertices of degree 3, it has a Hamiltonian path and can be 4-partitioned easily. Thus we may assume that $G^{'}$ contains at least 4 vertices and is therefore also 3-connected. Every edge uv in $G^{'}$ corresponds to a unique path in G from u to v such that all internal vertices of the path have degree 2 in G. We consider $G^{'}$ to be a weighted graph, where the weight of an edge uv is the number of internal vertices in the path in G corresponding to uv. We note that the weight of every edge in $G^{'}$ is at least 1 and strictly less than n_4.

Let uv be an edge with maximum weight in $G^{'}$. Let $P_0^{'}, P_1^{'}, \ldots, P_k^{'}$ be a non-separating induced ear decomposition of $G^{'} - uv$, as given by lemma 2.4. Let $P_{k+1}^{'}$ be the path containing just the edge uv. Let $P_0, \ldots, P_k, P_{k+1}$ be the corresponding non-separating ear decomposition of G, where P_i is obtained by subdividing each edge of $P_i^{'}$ as many times as its weight. Let $G_i = P_0 \cup \cdots \cup P_i$ and $\overline{G_i} = G - V(G_i)$ be subgraphs of G. Note that G_i is 2-connected and $\overline{G_i}$ is connected for $0 \leq i \leq k$.

Let i be the smallest integer such that $|G_i| > n_4+n_3+n_2$. Since the weight of edge uv is less than n_4, we have $i \leq k$. If $i = 0$, we can 4-partition P_0, which is a cycle, into parts $V_1^{'}, V_2, V_3, V_4$ of sizes $|P_0| - n_2 - n_3 - n_4, n_2, n_3, n_4$ such that $u \in V_1^{'}$. Let $V_1 = V_1^{'} \cup (V(G) \setminus V(P_0))$. We may therefore assume $i > 0$, $u \in V(G_{i-1})$ and $v \in V(\overline{G_{i-1}})$.

Let $P_i = v_0, v_1, \ldots, v_l$ and let j be the smallest positive integer such that v_j has degree 3 in G. There must exist such an integer $< l$ since in the ear decomposition of $G^{'}$, $P_i^{'}$ has length at least 2. Let $m = n_1 - |\overline{G_i}|$. Note that $1 \leq m < l$ and $j \leq n_4$ since the weight of the first edge of $P_i^{'}$ is $< n_4$.

Case 1. If $j \leq m$, let $V_1 = V(\overline{G_i}) \cup \{v_1, \ldots, v_m\}$. If $l - m \leq n_2$ then 3-partition G_{i-1} into parts $V_2^{'}, V_3, V_4$ of sizes $n_2 + 1 - l + m, n_3, n_4$ such that $v_l \in V_2^{'}$. Let $V_2 = V_2^{'} \cup \{v_{m+1}, \ldots, v_{l-1}\}$.

If $n_2 < l - m$, let $V_2 = \{v_{m+1}, \ldots, v_{m+n_2}\}$. The graph $G - (V_1 \cup V_2)$ can be 2-partitioned into parts V_3, V_4 of sizes n_3, n_4 such that $v_0 \in V_3$ and $v_{m+n_2+1} \in V_4$.

Case 2. If $j > m$, let $V_1^{'} = V(\overline{G_i}) \cup \{v_{j-m+1}, \ldots, v_j\}$. If $l - j \leq n_2$, let $n_2^{'} = n_2 + 1 - l + j$. The graph G_{i-1} contains $n^{'} = n_2^{'} + n_3 + n_4 - (j - m) < n_2^{'} + n_3 + n_4$ vertices. Since $j - m < n_4$, $n^{'} > n_2^{'} + n_3$ and $n^{'} > n_2^{'} + n_4$. Applying lemma 3.4, we can 3-partition G_{i-1} into parts $V_2^{'}, V_3^{'}, V_4^{'}$ of sizes at most $n_2^{'}, n_3, n_4$ such that $v_l \in V_2^{'}$ and any incomplete part contains at least one of the vertices u, v_0.

Let $u_0 = u, u_1, \ldots, u_p = v$ be the vertices on the path P_{k+1} corresponding to the edge uv in $G^{'}$. Since uv was an edge with maximum weight in $G^{'}$, we have $j \leq p$.

If the part containing u in the 3-partition of G_{i-1} has deficiency d, $0 \leq d \leq j - m < j$, we add the vertices $u_1, \ldots, u_d$ to that part and replace them by $v_{j-m-d+1}, \ldots, v_{j-m}$ in $V_1^{'}$. If $d < j - m$, there must be another incomplete part with deficiency $j - m - d$ containing the vertex v_0. Add the vertices $v_1, \ldots, v_{j-m-d}$ to that part. This gives the required 4-partition of G.

If $n_2 < l - j$, let $V_2 = \{v_{j+1}, \ldots, v_{j+n_2}\}$. The graph $G - (V_1 \cup V_2)$ can be 2-partitioned into parts V_3, V_4 of sizes n_3, n_4 such that $v_{j-m} \in V_3$ and $v_{j+n_2+1} \in V_4$.

This completes the proof of the theorem. □

4 Remarks

The graph obtained by subdividing every edge of K_4 has 10 vertices and no perfect matching. Thus 2-connected graphs with $\Delta \leq 3$ are not necessarily 5-partitionable. Similarly, if we subdivide every edge of K_4 four times, the resulting graph is 2-edge-connected with 30 edges. It is easy to check that the edges of this graph cannot be partitioned into 5 connected parts of size 6 each.

There exists a k-connected bipartite graph with bipartition A, B such that $|A| = 2k$, $|B| = 2k + 2$ and every vertex in A has degree $k + 1$, while every vertex in B has degree k. Again, this graph does not have a perfect matching and is thus not $(2k + 1)$-partitionable. We do not know any examples of k-connected graphs with $\Delta \leq k + 1$ and order $\geq 2k$, which are not $2k$-partitionable. Similarly, we do not know any examples of k-connected, k-regular graphs of order $\geq l$, which are not l-partitionable, for any $l \geq 1$. In both cases we believe there are no such graphs.

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