Colouring planar graphs with a precoloured induced cycle

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Abstract

Let C be a cycle and $f : V(C) \rightarrow \{c_1, c_2, \ldots, c_k\}$ a proper k-colouring of C for some $k \geq 4$. We say the colouring f is *safe* if for any planar graph G in which C is an induced cycle, there exists a proper k-colouring f' of G such that $f'(v) = f(v)$ for all $v \in V(C)$. The only safe 4-colouring is any proper colouring of a triangle. We give a simple necessary condition for a k-colouring of a cycle to be safe and conjecture that it is sufficient for all $k \geq 4$. The sufficiency for $k = 4$ follows from the four colour theorem and we prove it for $k = 5$, independent of the four colour theorem. We show that a stronger condition is sufficient for all $k \geq 4$. As a consequence, it follows that any proper k-colouring of a cycle that uses at most $k - 3$ distinct colours is safe. Also, any proper k-colouring of a cycle of length at most $2k - 5$ that uses at most $k - 1$ distinct colours is safe.

1 Introduction

A proper k-colouring of a graph G is an assignment $f : V(G) \rightarrow \{c_1, c_2, \ldots, c_k\}$ of colours to the vertices of G such that for every edge uv in the graph $f(u) \neq f(v)$. We will refer to proper colourings as simply colourings and a graph is k-colourable if there exists a k-colouring. The study of colourings of planar graphs has a long history, starting with the four colour conjecture and its eventual proof [2] [3] [7]. Many variations of this have since been considered, the most important of which is perhaps the notion of list-colouring. Let $L(v)$ be a list of allowed colours assigned to each vertex $v \in V(G)$. An L-colouring of G is a proper colouring such that each vertex v is assigned a colour in $L(v)$. The usual colouring problem is the special case when the lists $L(v)$ are the same for all vertices. While all planar graphs are four colourable, it is not true that they are L-colourable if $|L(v)| \geq 4$ for all vertices v [9]. Thomassen [8] showed that all planar graphs are L-colourable if $|L(v)| \geq 5$

for each vertex v. Subsequently, there has been a lot of work on finding conditions on the lists $L(v)$ that guarantee any planar graph is L-colourable. A recent result of Zhu [10] shows that if $|L(v)| = 4$ for each vertex v, and $|L(u) \cap L(v)| \leq 2$ for any edge uv, then any planar graph is L-colourable. Many such results with different restrictions were proved earlier, and several conjectured, as mentioned in [10].

Another variation considered is colouring or L-colouring with some precoloured vertices, whose colours are specified. This is equivalent to considering an L-colouring in which some vertices have lists of size 1. This can also be viewed as extending a partial assignment of colours to some vertices to a colouring or L-colouring of the whole graph. Albertson [1] showed that every planar graph with precoloured vertices is five colourable if the precoloured vertices are at distance at least 4 from each other and the precolouring uses at most five distinct colours. No such result is possible for four colours, even if only two vertices are precoloured. The corresponding result for L-colouring was shown in [5]. There exists a constant M such that if the precoloured vertices are at distance at least M from each other, and $|L(v)| = 5$ for all other vertices, then any planar graph is L-colourable.

In most of these results, the restrictions on the lists are in terms of their sizes, or the sizes of their intersections, but not the actual elements in the list. Here, we consider another special case where the vertices of a connected induced subgraph are assigned lists of size 1 and all other vertices have the same list of a specified size. This is equivalent to asking when can a specified k-colouring of a graph H be extended to a k-colouring of any planar graph that contains H as in induced subgraph. We call such a k-colouring of the graph H a *safe* k-colouring. The subgraph we consider is a cycle. If H is any 2-connected planar graph, a k-colouring of H is safe if and only if the colouring of every non-separating induced cycle in H is safe. Thus considering a cycle as the subgraph H is a natural choice.

We give a simple necessary condition for a k-colouring of a cycle to be safe. We conjecture that this condition is sufficient for all $k \geq 4$. The sufficiency for $k = 4$ follows from the four colour theorem. We prove it for $k = 5$. The proof is independent of and much simpler than the proof of the four colour theorem. We prove that a stronger condition is sufficient for all $k \geq 4$. As a consequence, any k-colouring of a cycle that uses at most $k - 3$ distinct colours is safe. Also, any k-colouring of a cycle of length at most $2k-5$ that uses at most $k-1$ colours is safe. If the conjecture is true, then any k-colouring of a cycle of length at most $3k - 10$ that uses at most $k - 2$ colours is safe. However, we have not been able to prove this.

Such results have been considered for 3-colourings of planar triangle-free graphs. In [6] for example, 3-colourings of a cycle of length at most 8 contained in a planar triangle-free graph that cannot be extended to a 3-colouring of the whole graph have been characterized. The same has been done for cycles of length 9 in [4]. However, we do not know any such results for general planar graphs when the number of colours and the cycle length is arbitrary. Also, these results are based on characterizing all critical graphs, that is graphs for which the colouring of the cycle cannot be extended to the whole graph, but can be extended for any proper subgraph that contains the cycle. We do not attempt to do this here. To show that a colouring is not safe, it is enough to show one graph for which the colouring of the cycle cannot be extended. To show it is safe, we use properties of the colouring to extend it to the whole graph.

2 Main Result

Let $C = v_1, v_2, \ldots, v_l$ be a cycle of length $l \geq 3$ and f a safe k-colouring of C. Any k-colouring of C obtained from f by permuting the colours is also safe. The colouring f_i defined by $f_i(v_j) = f(v_{i+j})$, where addition is modulo l, is also safe, for all $1 \leq i < l$. Similarly, the colouring obtained by reversing the cycle, that is $f_r(v_i) = f(v_{l+1-i})$ is safe. We say these colourings of C are equivalent and consider safety to be a property of the equivalence class.

Let $[i, j] = \{i, i+1, \ldots, j\}$ for $1 \leq i, j \leq l$, where again addition is considered modulo l. Let $[i, j) = [i, j] \setminus \{j\}$, $(i, j] = [i, j] \setminus \{i\}$, and $(i, j) = [i, j] \setminus \{i, j\}$. Let $F[i, j] = \{c_t \mid 1 \leq t \leq k, \exists m \in [i, j]\ f(v_m) = c_t\}$ be the subset of colours that occur in the subpath of C from v_i to v_j. The sets $F[i, j), F(i, j], F(i, j)$ are defined similarly.

To prove that a k-colouring f of C is safe, we consider an arbitrary planar graph G such that C is an induced cycle in G, and show that the colouring of C can be extended to a k-colouring of G. It is sufficient to do this in the case C is a non-separating induced cycle in G, since otherwise we can consider each component of $G - V(C)$ separately. Thus we may assume C bounds a face of G and without loss of generality, we can embed G so that C is the boundary of the external face. We may further assume $G - V(C)$ is not empty, and add edges to G that are not chords of C if needed, so that every internal face of G is bounded by a triangle. We call such a graph G a *chordless near-triangulation*. Thus a k-colouring f of a cycle C is safe if and only if for any chordless near-triangulation G with external face bounded by the cycle C, there exists a k-colouring of G that extends f.

We first give some simple necessary conditions for a k-colouring to be safe.

Lemma 1 *Let f be a k-colouring of a cycle C of length l. If f satisfies any one of the following conditions, then f is not safe.*

1. $|F[1, l]| = k$.
2. *There exist indices $1 \leq p < q \leq l$ such that $|F[p, q] \cap F[q, p]| \geq k - 1$.*
3. *There exist indices $1 \leq p < q < r \leq l$ such that $|F[p, q] \cap F[q, r] \cap F[r, p]| \geq k - 2$.*

Proof: If $|F[1, l]| = k$ then the k-colouring cannot be extended to a k-colouring of the wheel W_l, hence it is not safe. If there are indices $p < q$ such that $|F[p, q] \cap F[q, p]| \geq k - 1$, construct a chordless near-triangulation by adding a vertex u that is adjacent to all vertices v_m for $m \in [p, q]$, and a vertex v that is adjacent to u and all vertices v_m for $m \in [q, p]$. The k-colouring of the cycle cannot be extended to this near-triangulation, hence such a colouring is not safe. The same argument holds if there are indices $p < q < r$ such that $|F[p, q] \cap F[q, r] \cap F[r, p]| \geq k - 2$. Construct a near-triangulation by adding a triangle u, v, w with u adjacent to vertices v_m for

$m \in [p, q]$, v adjacent to vertices v_m for $m \in [q, r]$ and w adjacent to vertices v_m for $m \in [r, p]$. Again it is easy to check that the colouring of the cycle cannot be extended to the near-triangulation, hence any such k-colouring is not safe. $\square$

We call a k-colouring of a cycle that satisfies any of the conditions in Lemma 1 a *bad* k-colouring. If $k = 4$ and $l \geq 4$, then every 4-colouring of C is bad. Consider any 4-colouring of a cycle C of length at least 4. Assume $|F[1, l]| < 4$ otherwise condition 1 in Lemma 1 is satisfied. If $|F[1, l]| = 2$, we have $|F[1, 2]| = 2$, and $F[1, 2] = F[2, 3] = F[3, 1]$, which implies f satisfies condition 3 in Lemma 1 with $p = 1, q = 2, r = 3$. If $|F[1, l]| = 3$, there exists an index i such that $|F[i, i+2]| = 3$. If $|F[i + 2, i]| = 3$, then f satisfies condition 2 in Lemma 1 with $\{p, q\} = \{i, i + 2\}$, otherwise we have $l \geq 5$, $f(v_{i+3}) = f(v_i)$ and $f(v_{i+4}) = f(v_{i+2})$. In this case, we have $|F[i + 1, i + 3]| = |F[i + 3, i + 1]| = 3$ and f again satisfies condition 2 in Lemma 1 with $\{p, q\} = \{i + 1, i + 3\}$. Therefore, the only 4-colourings that are not bad are those of the triangle. It follows from the four colour theorem that any 4-colouring of a triangle is safe and hence a 4-colouring is safe if and only if it is not bad. We conjecture that this property holds for all $k \geq 4$.

Conjecture 1 *A k-colouring of a cycle is safe if and only if it is not bad, for all $k \geq 4$.*

We prove Conjecture 1 for $k = 5$. Although the statement of Conjecture 1 may be considered to be a generalization of the four colour theorem, the proof for $k = 5$ is much simpler and does not depend on the four colour theorem. We in fact show that k-colourings satisfying a stronger property are safe for all $k \geq 4$. All 5-colourings that are not bad satisfy the stronger property and are therefore safe. Moreover, there exist such k-colourings satisfying the stronger property for cycles of all lengths, for all $k \geq 5$. However, this proof depends crucially on the fact that $k \geq 5$, and cannot be easily adapted to prove the four colour theorem itself. Also, for $k \geq 6$, there are k-colourings that are not bad but do not satisfy the stronger property, so this does not prove Conjecture 1 for $k \geq 6$.

Henceforth, we will assume that $k \geq 5$ is a fixed integer. We call a k-colouring of a cycle of length l a *good* k-colouring if it satisfies one of the following properties.

1. $|F[1, l]| \leq k - 3$.
2. $|F[1, l]| = k - 2$ and there exist indices $1 \leq p < q \leq l$ and a set A of $k - 4$ colours such that $|F[p, q) \setminus A| = 1$ and $|F[q, p) \setminus A| = 1$.
3. $|F[1, l]| = k - 1$ and there exist indices $1 \leq p < q < r \leq l$ and a set A of $k - 4$ colours such that $|F[p, q) \setminus A| = |F[q, r) \setminus A| = |F[r, p) \setminus A| = 1$.

Lemma 2 *A 5-colouring f of a cycle of length l is not bad if and only if it is good.*

Proof: We show that if f is good it cannot satisfy any of the conditions in Lemma 1 and is therefore not bad. This argument holds for all $k \geq 4$, but we state it only for $k = 5$. Clearly $|F[1, l]| < 5$, hence f does not satisfy condition 1 in Lemma 1. Similarly if $|F[1, l]| = 2$, f does not satisfy any of the conditions in Lemma 1.

Suppose $|F[1,l]| = 3$ and f satisfies condition 2 in the definition of a good 5-colouring with indices p_1, q_1. Then f cannot satisfy condition 2 in the statement of Lemma 1. Suppose for contradiction f satisfies condition 3 in Lemma 1 with indices p_2, q_2, r_2. Since $[p_1, q_1), [q_1, p_1)$ is a partition of $[1, l]$, two of the indices p_2, q_2, r_2 are contained in one of the sets $[p_1, q_1)$ or $[q_1, p_1)$. Without loss of generality, assume $p_2, q_2 \in [p_1, q_1)$. However, this implies $3 \leq |F[p_2, q_2]| \leq |F[p_1, q_1)| \leq 2$, a contradiction. So f cannot satisfy any of the conditions in Lemma 1 and is not bad.

Suppose $|F[1,l]| = 4$ and f satisfies condition 3 in the definition of a good 5-colouring with indices p_1, q_1, r_1 and let $A = \{c_1\}$. Suppose for contradiction f satisfies condition 2 in Lemma 1 with indices p_2, q_2. Again, we may assume, without loss of generality, $p_1, q_1 \in [p_2, q_2)$. If $p_1 \neq p_2$, then $[q_2, p_2] \subseteq [q_1, r_1) \cup [r_1, p_1)$, which contradicts the fact that $|F[q_2, p_2]| \geq 4$ but $|F[q_1, r_1) \cup F[r_1, p_1)| \leq 3$. If $p_1 = p_2$ and $r_1 \in [p_2, q_2]$ then $[q_2, p_2] \subseteq [r_1, p_1]$, which is again a contradiction. If $r_1 \notin [p_2, q_2]$ then $[p_2, q_2] \subseteq [p_1, q_1) \cup [q_1, r_1)$, which again contradicts the fact that $|F[p_2, q_2]| \geq 4$ and $|F[p_1, q_1) \cup F[q_1, r_1)| \leq 3$. Finally, suppose f satisfies condition 3 in Lemma 1 with indices p_2, q_2, r_2. Let c_2 be a colour other than c_1 that is contained in $F[p_2, q_2] \cap F[q_2, r_2] \cap F[r_2, p_2]$. Without loss of generality, we may assume any vertex v_m such that $f(v_m) = c_2$ satisfies $m \in [p_1, q_1)$. However, this implies one of the sets $[p_2, q_2], [q_2, r_2], [r_2, p_2]$ must be contained in $[p_1, q_1)$ contradicting the fact that each of $F[p_2, q_2], F[q_2, r_2], F[r_2, p_2]$ has at least three elements, but $|F[p_1, q_1)| \leq 2$. This proves that if f is good, it is not bad.

We next show that if f is not good then it is bad. This argument holds only for $k = 5$. We may assume $3 \leq |F[1,l]| \leq 4$, otherwise f satisfies condition 1 in Lemma 1 and is bad.

Suppose $|F[1,l]| = 3$. Let p, q be indices such that $|[p,q]|$ is maximum and $|F[p,q)| = 2$. Then $f(v_q) \neq f(v_m)$ and $f(v_{p-1}) \neq f(v_m)$ for any $m \in [p,q)$. Since $|F[1,l]| = 3$, we must have $f(v_q) = f(v_{p-1})$ and $|F[q,p)| > 2$, otherwise f satisfies condition 2 in the definition of a good colouring with indices p, q. This implies $q \neq p-1$ and $[q,p)$ has at least four elements. Since $|F[p,q)| = 2$, we must have $q \geq p+2$. If $|F(p,q]| \geq 3$ then $\{p-1, p+1, q\}$ are three indices that satisfy condition 3 in Lemma 1 and f is bad. If $|F(p,q]| = 2$, we must have $q = p+2$. The choice of $[p,q]$ then implies that $f(v_{q+1}) = f(v_p)$ and $f(v_{q+2}) = f(v_{p+1})$. This implies f satisfies condition 3 in Lemma 1 with indices $\{p, q, q+2\}$ and is bad.

Suppose $|F[1,l]| = 4$. Let p, q be indices such that $|F[p,q]| \geq 4$ and $|[p,q]|$ is minimum possible. We have $f(v_p) \neq f(v_m)$ for any $m \in (p,q]$, $f(v_q) \neq f(v_m)$ for any $m \in [p,q)$, and also $f(v_p) \neq f(v_q)$. This implies $|F(p,q)| = 2$. If $|F[q,p]| \geq 4$ then p, q are indices satisfying condition 2 in Lemma 1 and f is bad. So $|F[q,p]| \leq 3$, which implies $|F(p,q) \cap F[q,p]| \leq 1$ and let c_1 be the colour in this set, if it is not empty, otherwise let c_1 be any colour in $F(p,q)$. Suppose there exists an index $r \in [q,p)$ such that $f(v_p) \notin F[q,r]$ and $f(v_q) \notin F(r,p]$. Then $F(p,q), F[q,r], F(r,p]$ are sets of size at most 2, and each contains one element other than c_1. This implies f satisfies condition 3 in the definition of a good colouring, a contradiction. Therefore there is no such index r and we have $p \neq q+1$. This implies there exists an index r in (q,p) such that $f(v_r) = f(v_p)$ and an index r' in (r,p) such that $f(v_{r'}) = f(v_q)$. If $q \geq p+4$ then we have $|F[p+2, r]| \geq 4$ and $|F[r, p+2]| \geq 4$, implying f satisfies condition 2 in Lemma 1 and is bad. Finally, suppose $q = p + 3$. If either the colour $f(v_{p+1}) \in F[q, r']$ or $f(v_{p+2}) \in F[r, p]$, the same argument holds with indices $\{p+2, r'\}$ or $\{p+1, r\}$, respectively. Without loss of generality, assume the colour c_1 is $f(v_{p+1})$ so v_{p+2} is the only vertex with the colour $f(v_{p+2})$. Choose r' so that $|[r', p]|$ is as small as possible. Then $f(v_q) \notin F(r', p]$ and $|F[p + 2, q)| = 1$, $|F[q, r']| \leq 2$ and $|F(r', p + 2)| \leq 2$. Also $|F[q, r'] \setminus \{f(v_p)\}| = 1$ and $|F(r', p + 2) \setminus \{f(v_p)\}| = 1$, which implies the indices $p+2, q, r'+1$ with $A = \{f(v_p)\}$, satisfy condition 3 in the definition of a good colouring, a contradiction. □

The 6-colouring of the 8 cycle defined by $f(v_i) = f(v_{i+4}) = c_i$ for $1 \leq i \leq 4$ is an example of a colouring that is neither good nor bad.

We can now state the main result.

Theorem 1 *Any good k-colouring of a cycle is safe for all $k \geq 5$.*

In order to prove Theorem 1 by induction, we need to consider sets of k-colourings of a cycle, rather than a single k-colouring. We also need to consider near-triangulations G with a given external boundary C that may have chords. Let $C = v_1, v_2, \dots, v_l$ be a cycle and $L(v_i) \subseteq \{c_1, c_2, \dots, c_k\}$ a non-empty list of at most k colours assigned to the vertex v_i, for $1 \leq i \leq l$. A near-triangulation G with external boundary C is said to be *consistent* with the list assignment if for any two vertices $u, v \in V(C)$ such that uv is an edge or chord of C, both u and v are not assigned the same list of size 1. We will only consider list assignments such that there are no two adjacent vertices u, v in C such that $|L(u)| = 1$ and $L(u) = L(v)$. We show that for certain kinds of list assignments to the vertices of C, for any near-triangulation G with external boundary C that is consistent with it, there exists an L-colouring of C that can be extended to a k-colouring of G.

Let $S_i = \{c_i\} \cup \{c_j | 5 \leq j \leq k\}$ for $i \in \{1, 2, 3\}$. A vertex v is said to be of type T_i for $i \in \{1, 2, 3\}$ if either $L(v) = \{c_1, c_2, c_3, c_4\} \setminus \{c_i\}$ or $|L(v)| = 1$ and $L(v) \subset S_i$. The list assignments that we consider will be such that every vertex will be of one of the three types.

Lemma 3 *Let $C = v_1, v_2, \dots, v_l$ be a cycle of length l and L an assignment of list of colours such that every vertex is of type T_1. Let $(s, t) \in L(v_1) \times L(v_l)$ be any ordered pair of distinct colours. Then for any near-triangulation G with boundary C that is consistent with the list assignment, there exists an L-colouring f of C that extends to a k-colouring of G with $f(v_1) = s$ and $f(v_l) = t$.*

Proof: Suppose there is a counterexample and let G be one with minimum number of edges.

Case 1. Suppose G has a chord $v_i v_j$ for some $i < j$. Then both (i, j) and (j, i) are non-empty sets. Let C_1 be the cycle $v_1, v_2, \dots, v_i, v_j, v_{j+1}, \dots, v_l$ and G_1 the near-triangulation induced by the vertices on or in the interior of C_1. Then G_1 is consistent with the list assignment L restricted to the vertices of C_1, and by the minimality of G, there exists an L-colouring f_1 of C_1 that can be extended to a k-colouring of G_1 with $f_1(v_1) = s$ and $f_1(v_l) = t$. Let C_2 be the cycle $v_i, v_{i+1}, \dots, v_j$ and let G_2 be the near-triangulation induced by the vertices on or in the interior of

C_2. Again, the list assignment to the vertices of C_2 satisfies the induction hypothesis and G_2 is consistent with it. The minimality of G implies there exists an L-colouring f_2 of C_2 that can be extended to a k-colouring of G_2 with $f_2(v_i) = f_1(v_i)$ and $f_2(v_j) = f_1(v_j)$. Defining $f(v) = f_1(v)$ for all vertices $v \in V(G_1)$ and $f(v) = f_2(v)$ for all vertices $v \in V(G_2)$, gives a k-colouring of G that extends an L-colouring of C, a contradiction.

Case 2. A similar argument holds if G has a separating triangle uvw, that is a triangle whose interior as well as exterior contain vertices of G. Let G_1 be the near-triangulation obtained by deleting the vertices in the interior of the separating triangle uvw. Then G_1 has the same boundary C as G and is consistent with the list assignment L. The minimality of G implies there exists an L-colouring f_1 of C that extends to a k-colouring of G_1, with $f_1(v_1) = s$ and $f_1(v_l) = t$. Rename the colours so that $f_1(u) = c_2$, $f_1(v) = c_3$ and $f_1(w) = c_4$. Let G_2 be the triangulation induced by the vertices on and in the interior of the triangle uvw and let $L(u) = L(v) = L(w) = \{c_2, c_3, c_4\}$. Then the triangle uvw satisfies the induction hypothesis and by induction, there exists an L-colouring f_2 of the triangle that extends to a k-colouring of G_2, with $f_2(u) = c_2$ and $f_2(v) = c_3$. The list assigned to w ensures $f_2(w) = c_4$. Rename the colours in the colouring f_2 so that $f_2(x) = f_1(x)$ for all $x \in \{u, v, w\}$. Again, setting $f(v) = f_1(v)$ for all vertices $v \in V(G_1)$ and $f(v) = f_2(v)$ for all vertices $v \in V(G_2)$, gives a k-colouring of G that extends an L-colouring of C, a contradiction.

Case 3. We may now assume that G is chordless and has no separating triangle. If G is a triangle, let $f(v_1) = s$ and $f(v_3) = t$ with $s \neq t$ by assumption. If $|L(v_1)| = 3$ then $s \in \{c_2, c_3, c_4\}$ and the same holds for v_3. This implies that if $|L(v_2)| = 1$, then $s, t \notin L(v_2)$, and otherwise $|L(v_2)| = 3$ and $L(v_2)$ contains a colour other than s, t. Assigning this colour to v_2 gives an L-colouring of C.

Suppose G is not a triangle, which implies that for every edge v_iv_{i+1} in C, there exists an internal vertex v in G such that vv_iv_{i+1} is an internal face in G. Suppose there exists such an edge v_iv_{i+1} for some $1 \leq i < l$ such that $L(v_i) \cap L(v_{i+1}) = \emptyset$. Let G_1 be obtained from G by deleting the edge v_iv_{i+1}. Then G_1 is a near-triangulation bounded by the cycle $C_1 = v_1, \ldots, v_i, v, v_{i+1}, \ldots, v_l$. Setting $L(v) = \{c_2, c_3, c_4\}$ gives a list assignment to the vertices of C_1 that satisfies the induction hypothesis, and G_1 is consistent with it. The minimality of G implies there exists an L-colouring f of C_1 that extends to a k-colouring of G_1 with $f(v_1) = s$ and $f(v_l) = t$. Since $L(v_i) \cap L(v_{i+1}) = \emptyset$ by assumption, this gives the required L-colouring of C that extends to a k-colouring of G. If there is no such edge v_iv_{i+1} in C, then for all $m \in (1, l)$, $L(v_m) = \{c_2, c_3, c_4\}$, for otherwise $L(v_i) = \{a\}$ for some $a \in S_1$ and $a \notin L(v_{i+1})$, which implies $L(v_i) \cap L(v_{i+1}) = \emptyset$. This also implies $L(v_1) = L(v_l) = \{c_2, c_3, c_4\}$.

Let $v_1 = u_1, u_2, \ldots, u_r = v_3$ be the vertices adjacent to v_2 such that $v_2u_iu_{i+1}$ is an internal face of G, for some $r \geq 3$ and all $1 \leq i < r$. Let G_1 be the graph obtained from G by deleting the vertex v_2. Then G_1 is a near-triangulation bounded by the cycle $C_1 = v_1, u_2, \ldots, u_{r-1}, v_3, \ldots, v_l$. Let $L(u_i) = \{c_1\}$ if i is odd and $L(u_i) = \{c_5\}$ if i is even, for all $1 < i < r$. The list assignment L to the vertices of C_1 satisfies the induction hypothesis and G_1 is consistent with it. The minimality of G implies there exists an L-colouring f of C_1 that extends to a k-colouring of G_1 with $f(v_1) = s$ and $f(v_l) = t$. Since $L(v_2) = \{c_2, c_3, c_4\}$, setting $f(v_2)$ to be a colour in $\{c_2, c_3, c_4\} \setminus (\{f(v_1)\} \cup \{f(v_3)\})$, gives the required L-colouring of C that extends to a k-colouring of G. This completes all cases and the proof of the lemma. $\square$

We next consider the case when the vertices can be of two types. In this case, we need to put more restrictions on the L-colouring of the cycle. If uv is an edge in the cycle C, we place restrictions on the ordered pair of colours $(f(u), f(v))$ assigned to the vertices u and v in an L-colouring f. In the usual list colouring, this can be any pair in $L(u) \times L(v)$ such that the two colours are distinct. Here we allow only specified subsets of such pairs and assign a list of allowed pairs from $L(u) \times L(v)$ to the edge uv. We call these sets of ordered pairs the labels of the edges. Let a denote any colour in S_1, b denote any colour in S_2, and x denote any colour in $\{c_3, c_4\}$ with $\{x, y\} = \{c_3, c_4\}$. The allowed label sets are the following and we call the collection of these $\mathcal{L}_{12}$.

(i) $\{(a, b)\}, a \neq b$ (ii) $\{(a, x)\}$ (iii) $\{(x, b)\}$
(iv) $\{(c_2, x), (y, x)\}$ (v) $\{(x, c_1), (x, y)\}$ (vi) $\{(c_2, x), (x, c_1)\}$

We consider list assignments to a cycle C in which every vertex v is assigned a list of colours and exactly two edges u_1v_1 and u_2v_2 are assigned labels $L(u_1v_1), L(u_2v_2) \in \mathcal{L}_{12}$ respectively, where $L(u_1v_1) \subseteq L(u_1) \times L(v_1)$ and $L(u_2v_2) \subseteq L(u_2) \times L(v_2)$. Let G be a near-triangulation with boundary C that is consistent with the lists assigned to the vertices in C. We say the list assignment is *feasible* for G if for every ordered pair $(s_1, t_1) \in L(u_1v_1)$ there exists an ordered pair $(s_2, t_2) \in L(u_2v_2)$ and an L-colouring f of C that can be extended to a k-colouring of G with $f(u_i) = s_i$ and $f(v_i) = t_i$ for $i \in \{1, 2\}$. We call $(L(u_1v_1), L(u_2v_2))$ a feasible pair of labels for G.

Lemma 4 *Let $C = v_1, v_2, \ldots, v_l$ be a cycle of length l and p an integer such that $1 \leq p < l$. Let L be an assignment of list of colours to each vertex in C such that v_m is of type T_1 for all $m \in [1, p]$, and of type T_2 for all $m \in (p, 1)$. Let $L_2 = L(v_pv_{p+1}) \subseteq L(v_p) \times L(v_{p+1})$ be any label in $\mathcal{L}_{12}$ assigned to the edge v_pv_{p+1}. For any near-triangulation G with boundary C that is consistent with L, there exists a label $L_1 \in \mathcal{L}_{12}$ and $L_1 \subseteq L(v_1) \times L(v_l)$ that can be assigned to the edge v_1v_l such that (L_1, L_2) is a feasible pair of labels for G.*

Proof: The proof is again by induction on the number of edges. Let G be a counterexample with the minimum number of edges.

Case 1. Suppose G has a chord v_iv_j with $1 \leq i < j \leq l$. Let C_1 be the cycle $v_1, v_2, \ldots, v_i, v_j, \ldots, v_l$ and G_1 the near-triangulation induced by the vertices on and in the interior of C_1. Let G_2 be the near-triangulation induced by the vertices on and in the interior of the cycle $C_2 = v_i, v_{i+1}, \ldots, v_j$.

Case 1.1 Suppose $j \leq p$. The list assignment L restricted to the vertices in C_1, along with the label L_2 assigned to the edge v_pv_{p+1}, satisfies the induction hypothesis, and the minimality of G implies the existence of a feasible pair of labels (L_1, L_2) for G_1. We claim that this is also feasible for G. Consider any L-colouring f_1 of C_1 that can be extended to a k-colouring of G_1. The list assignment L restricted to the vertices of C_2 satisfies the conditions of Lemma 3 and applying it gives an L-colouring f_2 of C_2 that can be extended to a k-colouring of G_2, with $f_2(v) = f_1(v)$

for $v \in \{v_i, v_j\}$. Thus any L-colouring of C_1 that can be extended to a k-colouring of G_1 can also be extended to a k-colouring of G. Therefore the labels (L_1, L_2) form a feasible pair for G. A similar argument holds if $i > p$. In this case, for applying Lemma 3 to the near-triangulation G_2, first swap the colours c_1 and c_2 in the lists assigned to the vertices of C_2. This converts all vertices v_m with $m \in (i, j)$ to type T_1 and Lemma 3 can be applied. After getting the L-colouring of C_2 that can be extended to a k-colouring of G_2, again swap the colours c_1, c_2 in the k-colouring of G_2 to get the required L-colouring of C_2.

Case 1.2 Suppose $i \leq p < j$. In this case, the list assignment L restricted to the vertices of C_2 satisfies the induction hypothesis, considering v_iv_j to be the edge v_1v_l. The minimality of G implies there exists a label L'_1 that can be assigned to the edge v_iv_j such that (L'_1, L_2) is a feasible pair of labels for G_2. The list assignment L restricted to the vertices of C_1, with the label L'_1 assigned to the edge v_iv_j, also satisfies the induction hypothesis, and there exists a label L_1 that can be assigned to the edge v_1v_l such that (L_1, L'_1) is a feasible pair for G_1. This implies (L_1, L_2) is a feasible pair of labels for G.

Case 2. We may assume G is chordless. If G has a separating triangle, the same argument as in Lemma 3 can be used. Delete the vertices in the interior of a separating triangle to get a near-triangulation G_1 with the same boundary as G. By induction, there exists a feasible pair of labels (L_1, L_2) for G_1. It can then be argued as in Lemma 3 that (L_1, L_2) is also feasible for G. We may therefore assume G has no chord and no separating triangle.

Case 2.1 Suppose $l = 3$ and without loss of generality assume $p = 2$, otherwise reverse the cycle and swap the colours c_1 and c_2 to apply the argument. In this case, it is sufficient to show that for every label $L_2 \in \mathcal{L}_{12}$ assigned to the edge v_2v_3, there exists a label L_1 that can be assigned to v_1v_3 such that for every pair $(s, t) \in L_1$, there exists a colour $r \notin \{s, t\}$ such that $(r, t) \in L_2$. Lemma 3 implies that the L-colouring f of C defined by $f(v_1) = s$, $f(v_2) = r$, $f(v_3) = t$, can be extended to a k-colouring of G, hence (L_1, L_2) is a feasible pair for G.

If $|L(v_2)| = |L(v_3)| = 1$ then $L_2 = L(v_2) \times L(v_3)$. If $|L(v_1)| = 1$, then since G is consistent with L, $L(v_1) \neq L(v_2)$ and $L(v_1) \neq L(v_3)$. Setting $L_1 = L(v_1) \times L(v_3)$ gives a feasible pair of labels (L_1, L_2) for G. If $|L(v_1)| = 3$, then since v_1 is of type T_1, $L(v_1) = \{c_2, c_3, c_4\}$. Setting $L_1 = \{c_3\} \times L(v_3)$ gives the required label for the edge v_1v_3.

Suppose $|L(v_2)| = 3$ and $|L(v_3)| = 1$. We can assume $L_2 = \{(x, b)\}$ for some $x \in \{c_3, c_4\}$ and $b \in S_2$. If $|L(v_1)| = 1$, let L_1 be the set $L(v_1) \times \{b\}$. If $|L(v_1)| = 3$, let L_1 be the set $\{(y, b)\}$, where $y \in \{c_3, c_4\} \setminus \{x\}$. In both cases, we get a feasible pair of labels (L_1, L_2) for G.

Suppose $|L(v_2)| = 1$ and $|L(v_3)| = 3$. We can assume that $L_2 = \{(a, x)\}$ for some $x \in \{c_3, c_4\}$ and $a \in S_1$. If $|L(v_1)| = 1$ let $L_1 = L(v_1) \times \{x\}$, otherwise $L(v_1) = \{c_2, c_3, c_4\}$ and let $L_1 = \{(c_2, x), (y, x)\}$. In both cases (L_1, L_2) is a feasible pair of labels for G.

Suppose both $|L(v_2)|$ and $|L(v_3)|$ are 3, which implies $L(v_2) = \{c_2, c_3, c_4\}$ and $L(v_3) = \{c_1, c_3, c_4\}$. Then L_2 can be any one of $\{(c_2, x), (y, x)\}$, $\{(x, c_1), (x, y)\}$ and $\{(c_2, x), (x, c_1)\}$. If $|L(v_1)| = 1$, then $L(v_1) = \{a\}$ for some $a \in S_1$. If $(c_2, x) \in L_2$, let $L_1 = \{(a,x)\}$, otherwise $(x,y) \in L_2$ and let $L_1 = \{(a,y)\}$. In both cases (L_1, L_2) is a feasible pair for G. If $|L(v_1)| = 3$, then $L(v_1) = \{c_2, c_3, c_4\}$. If $L_2 = \{(c_2,x),(y,x)\}$ let $L_1 = L_2$. If $L_2 = \{(x,c_1),(x,y)\}$ let $L_1 = \{(y,c_1),(c_2,y)\}$ and if $L_2 = \{(c_2,x),(x,c_1)\}$ let $L_1 = \{(y,x),(y,c_1)\}$. In each case, it is easy to verify that (L_1, L_2) is a feasible pair of labels for G.

Case 2.2 Suppose now that $l > 3$. Since G is chordless, for every edge v_iv_{i+1} in C, there exists an internal vertex v such that vv_iv_{i+1} is an internal face in G. Suppose there exists such an edge v_iv_{i+1} for some $1 \leq i < l$ and $i \neq p$ such that $L(v_i) \cap L(v_{i+1}) = \emptyset$. Let G_1 be the near-triangulation obtained by deleting the edge v_iv_{i+1}, with the cycle $C_1 = v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_l$ as the boundary. If $i < p$, let $L(v) = \{c_2, c_3, c_4\}$ and if $i > p$ then let $L(v) = \{c_1, c_3, c_4\}$. The list assignment L to the vertices of C_1, along with the label L_2 to the edge v_pv_{p+1}, satisfies the induction hypothesis and G_1 is consistent with it. By induction, there exists a label L_1 for the edge v_1v_l such that (L_1, L_2) is a feasible pair of labels for G_1. Since every L-colouring of C_1 is also an L-colouring of C, (L_1, L_2) is a feasible pair of labels for G also. We may assume there is no such edge in C.

Case 2.2.1 Suppose there exists a vertex v_i with $|L(v_i)| = 1$. Suppose $i = 1$, which implies $L(v_1) = \{a\}$ for some $a \in S_1$. If $p > 1$, then $L(v_1) \cap L(v_2) = \emptyset$, a contradiction. Therefore $p = 1$ and the label L_2 attached to the edge v_1v_2 is either $\{(a,b)\}$ for some $b \in S_2$ or $\{(a,x)\}$ for $x \in \{c_3, c_4\}$. Let v be the internal vertex in G such that vv_1v_2 is an internal face in G. Let G_1 be the near-triangulation obtained by deleting the edge v_1v_2 with the cycle $C_1 = v_1, v, v_2, \ldots, v_l$ as the boundary. Let $L(v) = \{c_2, c_3, c_4\}$ and assign the label L'_2 to the edge vv_2, where $L'_2 = \{(c_3, b)\}$ if $L_2 = \{(a,b)\}$ and $L'_2 = \{(c_2,x),(y,x)\}$ if $L_2 = \{(a,x)\}$. The list assignment L to the vertices in C_1, along with the label L'_2 assigned to the edge vv_2, satisfies the induction hypothesis, and by induction, there exists a feasible pair (L_1, L'_2) for G_1. Since any L-colouring of C_1 is also an L-colouring of C, (L_1, L_2) is a feasible pair for G. A symmetrical argument holds if $|L(v_2)| = 1$, by reversing the cycle and swapping the colours c_1, c_2.

If $1 < i \leq p$ then $L(v_{i-1}) \cap L(v_i) = \emptyset$ and if $l > i > p$ then $L(v_i) \cap L(v_{i+1}) = \emptyset$, a contradiction.

Case 2.2.2 Suppose $|L(v_i)| = 3$ for all $i \in [1, l]$ and suppose $p > 2$. Let $v_1 = u_1, u_2, \ldots, u_r = v_3$ be the vertices adjacent to v_2 such that $v_2u_iu_{i+1}$ is an internal face in G for all $1 \leq i < r$. Let G_1 be obtained from G by deleting the vertex v_2. Then G_1 is a near-triangulation bounded by the cycle $C_1 = v_1, u_2, \ldots, u_{r-1}, v_3, \ldots, v_l$. Let $L(u_i) = \{c_1\}$ if i is odd and $L(u_i) = \{c_5\}$ if i is even, for all $2 \leq i < r$. This gives a list assignment to the vertices of C_1, which along with the label L_2 for the edge v_pv_{p+1}, satisfies the induction hypothesis and G_1 is consistent with it. By induction, there exists a label L_1 for the edge v_1v_l such that (L_1, L_2) is a feasible pair of labels for G_1. Since any L-colouring of C_1 can be extended to an L-colouring of C by assigning v_2 a colour in $\{c_2, c_3, c_4\}$ that is not assigned to v_1 or v_3, (L_1, L_2) is also a feasible pair of labels for G. A symmetrical argument can be used if $p < l - 2$, by relabeling the vertex v_i as v_{l+1-i} and swapping the colours c_1 and c_2.

Case 2.2.3 Suppose $|L(v_i)| = 3$ for all $i \in [1, l]$, $p \leq 2$ and $p \geq l - 2$, which implies $l = 4$, $p = 2$. This implies L_2 is one of the labels $\{(c_2,x),(y,x)\}$, $\{(x,c_1),(x,y)\}$ or

$\{(c_2, x), (x, c_1)\}$.

Suppose $(x, c_1) \in L_2$. Again consider the near-triangulation G_1 obtained by deleting the vertex v_2 and bounded by the cycle $C_1 = v_1, u_2, \ldots, v_3, v_4$. Let $L(u_r) = \{c_1\}$ and $L(u_i) = \{c_1, c_5\} \setminus L(u_{i+1})$ for $2 \leq i < r$. Note that the list for $v_3 = u_r$ is modified by removing the elements c_3, c_4 from it. Assign the label $L'_2 = \{(c_1, x)\}$ to the edge v_3v_4. Then the list assignment to the vertices of C_1 satisfies the induction hypothesis, and G_1 is consistent with it. By induction, there exists a label L_1 for the edge v_1v_4 such that (L_1, L'_2) is a feasible pair of labels for G_1. Since in any L-colouring f of C_1 we must have $f(v_4) = x$, $L_1 = \{(c_2, x), (y, x)\}$. Therefore any L-colouring of C_1 can be extended to an L-colouring of C by assigning colour x to v_2. This implies (L_1, L_2) is a feasible pair of labels for G. If $(x, c_1) \notin L_2$, we can assume $(c_2, x) \in L_2$, and a symmetrical argument can be used after reversing the cycle and swapping the colours c_1, c_2.

This completes all cases and the proof of Lemma 4. □

We now consider list assignments with vertices of 3 different types. We define two other sets of possible labels for edges in the cycle. Let $a \in S_1$, $b \in S_2$ and $c \in S_3$ be any colours. The label set $\mathcal{L}_{13}$ contains the following sets of ordered pairs.

(i) $\{(a, c)\}$ (ii) $\{(a, c_2), (a, c_4)\}$ (iii) $\{(c_2, c), (c_4, c)\}$
(iv) $\{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\}$
(v) $\{(c_2, c_1), (c_4, c_1), (c_2, c_4), (c_4, c_2)\}$
(vi) $\{(c_3, c_2), (c_3, c_4), (c_2, c_1), (c_4, c_1)\}$

A set in $\mathcal{L}_{13}$ is obtained from a set in $\mathcal{L}_{12}$ by first adding all pairs obtained by swapping colours c_3 and c_4 and then swapping the colours c_2 and c_3. Thus if the set in $\mathcal{L}_{12}$ is $\{(c_2, c_3), (c_3, c_1)\}$, adding all pairs obtained by swapping c_3 and c_4 gives the set $\{(c_2, c_3), (c_3, c_1), (c_2, c_4), (c_4, c_1)\}$ and then swapping the colours c_2 and c_3 gives the label $\{(c_3, c_2), (c_2, c_1), (c_3, c_4), (c_4, c_1)\} \in \mathcal{L}_{13}$. The labels in $\mathcal{L}_{32}$ are obtained in a similar way from those in $\mathcal{L}_{12}$, by first adding all pairs obtained by swapping colours c_3 and c_4 and then swapping the colours c_1 and c_3. For the example given, the label obtained is $\{(c_2, c_1), (c_1, c_3), (c_2, c_4), (c_4, c_3)\}$. The label set $\mathcal{L}_{32}$ contains the following sets of ordered pairs.

(i) $\{(c, b)\}$ (ii) $\{(c, c_1), (c, c_4)\}$ (iii) $\{(c_1, b), (c_4, b)\}$
(iv) $\{(c_2, c_1), (c_2, c_4), (c_1, c_4), (c_4, c_1)\}$
(v) $\{(c_1, c_3), (c_4, c_3), (c_1, c_4), (c_4, c_1)\}$
(vi) $\{(c_2, c_1), (c_2, c_4), (c_1, c_3), (c_4, c_3)\}$.

The sets in $\mathcal{L}_{13}$ and $\mathcal{L}_{32}$ correspond to the sets in $\mathcal{L}_{12}$ from which they are obtained.

Note that in Lemma 4, the list assignments and labels are symmetric in the colours c_3 and c_4. Therefore if (L_1, L_2) is a feasible pair for a near-triangulation G, then so is (L'_1, L'_2), where L'_1 and L'_2 are obtained from L_1 and L_2 by swapping the colours c_3 and c_4.

Lemma 5 *Let $C = v_1, v_2, \ldots, v_l$ be a cycle of length l and p, q positive integers such that $1 \leq p < q < l$. Let L be an assignment of list of colours to the vertices of C such that the vertex v_m is of type T_1 for all $m \in [1, p]$, of type T_3 for $m \in (p, q]$ and of type T_2 for $m \in (q, l]$. Suppose $L_2 \in \mathcal{L}_{13}$ and $L_3 \in \mathcal{L}_{32}$ are labels assigned* *to the edges v_pv_{p+1} and v_qv_{q+1}, respectively, such that $L_2 \subseteq L(v_p) \times L(v_{p+1})$ and $L_3 \subseteq L(v_q) \times L(v_{q+1})$. Then for any near-triangulation G with boundary C that is consistent with the list assignment L, there exists a label $L_1 \in \mathcal{L}_{12}$ and $L_1 \subseteq L(v_1) \times L(v_l)$ that can be assigned to the edge v_1v_l, such that for any ordered pair $(s_1, t_1) \in L_1$, there exist ordered pairs $(s_2, t_2) \in L_2$, $(s_3, t_3) \in L_3$ and an L-colouring f of C that can be extended to a k-colouring of G with $f(v_1) = s_1$, $f(v_l) = t_1$, $f(v_p) = s_2$, $f(v_{p+1}) = t_2$, $f(v_q) = s_3$ and $f(v_{q+1}) = t_3$.*

Proof: The proof is again by induction, and we consider a counterexample G with the minimum number of edges. We call the triple of labels (L_1, L_2, L_3) satisfying the properties in the lemma a feasible triple, and suppose G does not have a feasible triple for some list assignment L to the vertices of C and labels L_2, L_3.

Case 1. Suppose G has a chord v_iv_j for some $1 \leq i < j \leq l$. Again, let G_1 be the near-triangulation induced by the vertices on or in the interior of the cycle $C_1 = v_1, \ldots, v_i, v_j, \ldots, v_l$ and G_2 the near-triangulation induced by the vertices on or in the interior of the cycle $C_2 = v_i, v_{i+1} \ldots, v_j$.

Case 1.1 Suppose v_i and v_j are vertices of the same type. Then the list assignment L restricted to the vertices of C_1, along with the labels L_2 and L_3 assigned to the edges v_pv_{p+1} and v_qv_{q+1}, satisfies the induction hypothesis. By induction, there exists a feasible triple (L_1, L_2, L_3) for G_1, and we claim that it is also feasible for G. Let f_1 be any L-colouring of C_1. The list assignment L restricted to the vertices of C_2 satisfies the conditions of Lemma 3, perhaps after renaming colours so that vertices v_m have type T_1 for $m \in [i, j]$. Lemma 3 then implies there exists an L-colouring f_2 of C_2 that can be extended to a k-colouring of G_2 with $f_2(v_i) = f_1(v_i)$ and $f_2(v_j) = f_1(v_j)$. Since this holds for any L-colouring of C_1, (L_1, L_2, L_3) is a feasible triple for G.

Case 1.2 Suppose v_i is of type T_1 and v_j of type T_2, which implies $1 \leq i \leq p$ and $q < j \leq l$. The list assignment L restricted to the vertices of C_2, along with the labels L_2, L_3, satisfies the induction hypothesis, and by induction there exists a label $L_1' \in \mathcal{L}_{12}$ that can be assigned to the edge v_iv_j such that (L_1', L_2, L_3) is a feasible triple for G_2. The list assignment L restricted to the vertices of C_1, along with the label L_1' assigned to the edge v_iv_j, satisfies the conditions of Lemma 4. Applying Lemma 4 to G_1, there exists a label L_1 that can be assigned to the edge v_1v_l such that (L_1, L_1') is a feasible pair of labels for G_1. This implies (L_1, L_2, L_3) is a feasible triple for G.

Case 1.3 Suppose v_i is of type T_1 and v_j of type T_3. Then the list assignment L restricted to the cycle C_1 satisfies the induction hypothesis but the edge v_pv_{p+1} is not in C_1. We choose an appropriate label $L_2' \in \mathcal{L}_{13}$ for the edge v_iv_j and the label L_3 for the edge v_qv_{q+1} in C_1. Applying induction gives a feasible triple (L_1, L_2', L_3) for G_1. We choose L_2' so that for any ordered pair $(s_2', t_2') \in L_2'$, there exists an ordered pair $(s_2, t_2) \in L_2$ and an L-colouring f_2 of C_2 that can be extended to a k-colouring of G_2 with $f_2(v_i) = s_2'$, $f_2(v_j) = t_2'$, $f_2(v_p) = s_2$ and $f_2(v_{p+1}) = t_2$. Then (L_1, L_2, L_3) is a feasible triple for G.

The label L_2' for the edge v_iv_j is found by applying Lemma 4 to the near-triangulation G_2. We first swap the colours c_2 and c_3 in all the lists and the ordered pairs in the label so that vertices v_m of type T_3 for $m \in (p, j]$ are converted to type T_2. This swap does not affect the lists for the vertices v_m of type T_1 for $m \in [i, p]$. After swapping colours c_2 and c_3, the labels in $\mathcal{L}_{13}$ are closed under swapping colours c_3 and c_4. Retaining only one of the pairs that can be obtained by such a swap converts the label L_2 into a label L_2'' that is in $\mathcal{L}_{12}$. In other words, L_2'' is the label in $\mathcal{L}_{12}$ corresponding to the label $L_2 \in \mathcal{L}_{13}$. Lemma 4 implies there exists a label $L_2''' \in \mathcal{L}_{12}$ that can be assigned to the edge v_iv_j such that (L_2''', L_2'') is a feasible pair for G_2. Let L_2' be the label in $\mathcal{L}_{13}$ corresponding to the label L_2'''. It follows from Lemma 4, that (L_2', L_2) is a feasible pair of labels for G_2.

The argument in the case v_i is of type T_3 and v_j is of type T_2 is symmetric, and the above argument can be applied after reversing the cycle and swapping the colours c_1 and c_2 in the lists and the labels.

Case 2. We may now assume G has no chords. If G has a separating triangle, exactly the same argument as in Lemmas 3 and 4 can be used. So we may assume G has no chords or separating triangles.

Case 2.1 Suppose $l = 3$, which implies $p = 1$ and $q = 2$. In this case, it is sufficient to show that for any labels L_2, L_3 assigned to the edges v_1v_2 and v_2v_3, respectively, there exists a label L_1 that can be assigned to v_1v_3 such that for every pair $(s, t) \in L_1$, there is colour $r \notin \{s, t\}$ that can be assigned to v_2 such that $(s, r) \in L_2$ and $(r, t) \in L_3$. Lemma 3 implies that any such L-colouring of C can be extended to a k-colouring of G, hence (L_1, L_2, L_3) is a feasible triple for any triangulation G.

Suppose the label L_2 is $\{(a, c)\}$ for some $a \in S_1$ and $c \in S_3$. Then L_3 is either $\{(c, b)\}$ for some $b \in S_2$ or $\{(c, c_1), (c, c_4)\}$. In the first case, the L-colouring is given by $f(v_1) = a$, $f(v_2) = c$ and $f(v_3) = b$, hence $L_1 = \{(a, b)\}$. In the second case, set $f(v_3) = c_4$ and hence $L_1 = \{(a, c_4)\}$ satisfies the requirements.

Suppose L_2 is the label $\{(a, c_2), (a, c_4)\}$ for some $a \in S_1$. If $L_3 = \{(c_1, b), (c_4, b)\}$, for some $b \in S_2$, let f be the L-colouring of C with $f(v_1) = a$, $f(v_2) = c_4$ and $f(v_3) = b$. In this case, the label L_1 is $\{(a, b)\}$. Otherwise if $(c_1, c_3), (c_4, c_3) \in L_3$, set $f(v_3) = c_3$ to get the required label $L_1 = \{(a, c_3)\}$. The remaining possibility is that $L_3 = \{(c_2, c_1), (c_2, c_4), (c_1, c_4), (c_4, c_1)\}$. In this case, choose $f(v_2) = c_2$ and $f(v_3) = c_4$, to get the required label $L_1 = \{(a, c_4)\}$.

We may now assume $|L(v_1)| = 3$ and by symmetry, $|L(v_3)| = 3$. Suppose $|L(v_2)| = 1$, which implies $L(v_2) = \{c\}$ for some $c \in S_3$. In this case $L_2 = \{(c_2, c), (c_4, c)\}$ and $L_3 = \{(c, c_1), (c, c_4)\}$. Then setting $f(v_1) = c_2$, $f(v_2) = c$ and $f(v_3) = c_4$ gives an L-colouring of C. Similarly, $f(v_1) = c_4$, $f(v_2) = c$ and $f(v_3) = c_1$ is an L-colouring of C. This implies $L_1 = \{(c_2, c_4), (c_4, c_1)\}$ gives a feasible triple for G.

Finally, suppose all three vertices have lists of size 3. Suppose $(c_2, c_1), (c_4, c_1) \in L_2$. If $(c_1, c_3), (c_4, c_3) \in L_3$, let $f(v_2) = c_1$, $f(v_3) = c_3$ and $f(v_1)$ can be either c_2 or c_4. These give L-colourings of C, hence setting $L_1 = \{(c_2, c_3), (c_4, c_3)\}$ gives a feasible triple for G. If $(c_1, c_3) \notin L_3$ then $L_3 = \{(c_2, c_1), (c_2, c_4), (c_1, c_4), (c_4, c_1)\}$. Applying a symmetrical argument by reversing the cycle and swapping the colours c_1, c_2, we must have $L_2 = \{(c_2, c_1), (c_4, c_1), (c_2, c_4), (c_4, c_2)\}$. In this case, setting $f(v_1) = c_2$, $f(v_2) = c_1$ and $f(v_3) = c_4$ gives an L-colouring of C, as does setting $f(v_1) = c_4$,

$f(v_2) = c_2$ and $f(v_3) = c_1$. This implies $L_1 = \{(c_2, c_4), (c_4, c_1)\}$ gives a feasible triple for G. The final case to consider is if $(c_2, c_1), (c_4, c_1) \notin L_2$ and by a symmetrical argument $(c_2, c_1), (c_2, c_4) \notin L_3$. This implies $L_2 = \{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\}$ and $L_3 = \{(c_1, c_3), (c_4, c_3), (c_1, c_4), (c_4, c_1)\}$. Therefore $f(v_1) = c_3$, $f(v_2) = c_4$ and $f(v_3) = c_1$ is an L-colouring of C and so is $f(v_1) = c_2$, $f(v_2) = c_4$ and $f(v_3) = c_3$. This implies $L_1 = \{(c_2, c_3), (c_3, c_1)\}$ gives a feasible triple for G. This completes all cases when $l = 3$.

Case 2.2 We may now assume $l > 3$ and G is a chordless near-triangulation with no separating triangles. Therefore for every edge $v_i v_{i+1}$ in C there exists an internal vertex v in G such that $v v_i v_{i+1}$ is an internal face in G. As in the proof of Lemma 4, we may assume that $L(v_i) \cap L(v_{i+1}) \neq \emptyset$ for any $i \notin \{p, q\}$, otherwise we can get the required feasible triple of labels by deleting such an edge and applying induction.

Case 2.2.1 Suppose there exists a vertex v_i with $|L(v_i)| = 1$. Suppose $i = 1$, which implies $L(v_1) = \{a\}$ for some $a \in S_1$. Then we must have $p = 1$ otherwise v_2 is also a vertex of type T_1 and $L(v_1) \cap L(v_2) = \emptyset$. This implies the label L_2 assigned to the edge $v_1 v_2$ is either $\{(a, c)\}$ for some $c \in S_3$ or $\{(a, c_2), (a, c_4)\}$. Let v be the vertex such that $v v_1 v_2$ is an internal face in G. Let G_1 be the near-triangulation obtained by deleting the edge $v_1 v_2$ having $C_1 = v_1, v, v_2, \ldots, v_l$ as the boundary. Let $L(v) = \{c_2, c_3, c_4\}$ and assign the label L'_2 to the edge $v v_2$, where $L'_2 = \{(c_2, c), (c_4, c)\}$ if $L_2 = \{(a, c)\}$ and $L'_2 = \{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\}$ otherwise. The list assignment to the vertices of C_1 satisfies the induction hypothesis, and by induction, there exists a label L_1 that can be assigned to $v_1 v_l$ such that (L_1, L'_2, L_3) is a feasible triple for G_1. Since any L-colouring of C_1 is also an L-colouring of C, (L_1, L_2, L_3) is a feasible triple for G. A symmetrical argument holds if $L(v_l) = \{b\}$ for some $b \in S_2$.

Suppose $i \in (1, l)$. If v_i is of type T_1 then $L(v_i) \cap L(v_{i-1}) = \emptyset$, and if it is of type T_2, $L(v_i) \cap L(v_{i+1}) = \emptyset$, a contradiction. If it is of type T_3 and v_{i-1} is also of type T_3, the same argument holds. The only possibility is that $i = p + 1$, $L(v_{p+1}) = \{c_3\}$ and $L(v_p) = \{c_2, c_3, c_4\}$. The label L_2 assigned to the edge $v_p v_{p+1}$ must be $\{(c_2, c_3), (c_4, c_3)\}$. Let v be the vertex such that $v v_p v_{p+1}$ is an internal face in G. Let G_1 be the near-triangulation obtained by deleting the edge $v_p v_{p+1}$ with the cycle $C_1 = v_1, \ldots, v_p, v, v_{p+1}, \ldots, v_l$ as the boundary. Let $L(v) = \{c_1, c_2, c_4\}$ and assign the label $L'_2 = \{(c_2, c_1), (c_4, c_1), (c_2, c_4), (c_4, c_2)\}$ to the edge $v_p v$. The list assignment to the vertices of C_1, along with the label L'_2, satisfies the induction hypothesis, and by induction there exists a feasible triple (L_1, L'_2, L_3) for G_1. In any L-colouring of C_1, the label L'_2 ensures that v_p is coloured c_2 or c_4, hence it is also an L-colouring of C. Therefore (L_1, L_2, L_3) is a feasible triple for G.

Case 2.2.2 We may now assume $|L(v_i)| = 3$ for all $i \in [1, l]$.

Case 2.2.2.1 Suppose $p > 1$ and $q > p + 1$. Suppose the label L_2 contains the pairs (c_2, c_1) and (c_4, c_1). Let $v_{p-1} = u_1, \ldots, u_r = v_{p+1}$ be the neighbours of v_p. Consider the near-triangulation G_1 obtained by deleting v_p having the cycle $v_1, \ldots, v_{p-1}, u_2, \ldots, u_{r-1}, v_{p+1}, \ldots, v_l$ as the boundary. Remove the elements c_2, c_4 from $L(v_{p+1})$ so that $L(v_{p+1}) = \{c_1\}$, which converts v_{p+1} to a vertex of type T_1. Let $L(u_i) = \{c_1, c_5\} \setminus L(u_{i+1})$ for $2 \leq i < r$. Assign the label $L'_2 = \{(c_1, c_2), (c_1, c_4)\}$ to the edge $v_{p+1} v_{p+2}$ and keep the label L_3 for the edge $v_q v_{q+1}$. The resulting list assignment to the vertices of C_1 satisfies the induction hypothesis, and by induction, there exists a feasible triple (L_1, L_2', L_3) for G_1. Since any L-colouring of C_1 can be extended to an L-colouring of C by assigning v_p a colour in $\{c_2, c_4\}$ that is not assigned to v_{p-1}, and since $(c_2, c_1), (c_4, c_1) \in L_2$, (L_1, L_2, L_3) is a feasible triple for G.

If $(c_2, c_1) \notin L_2$ then both (c_3, c_2) and (c_3, c_4) are in L_2. Now let $v_p = u_1, \ldots, u_r = v_{p+2}$ be the neighbours of v_{p+1}. Let G_1 be the near-triangulation obtained by deleting the vertex v_{p+1} with the cycle $C_1 = v_1, \ldots, v_p, u_2, \ldots, u_{r-1}, v_{p+1}, \ldots, v_l$ as the boundary. Remove the elements c_2, c_4 from $L(v_p)$ so that $L(v_p) = \{c_3\}$, which converts v_p to a vertex of type T_3. Let $L(u_i) = \{c_3, c_5\} \setminus L(u_{i-1})$ for $2 \leq i < r$. Assign the label $L_2' = \{(c_2, c_3), (c_4, c_3)\}$ to the edge $v_{p-1}v_p$ and retain the label L_3 for the edge $v_q v_{q+1}$ in C_1. The list assignment to the vertices of C_1 satisfies the induction hypothesis and by induction there exists a feasible triple (L_1, L_2', L_3) for G_1. Any L-colouring of C_1 can be extended to an L-colouring of C by assigning v_{p+1} a colour in $\{c_2, c_4\}$ that is not assigned to v_{p+2}. This gives an L-colouring of C in which v_p is coloured c_3 and v_{p+1} is coloured c_2 or c_4, hence (L_1, L_2, L_3) is a feasible triple for G.

Case 2.2.2.2 Suppose $p > 1$ and $q = p + 1$. The argument is similar to that in Case 2.2.2.1. Suppose $(c_2, c_1) \in L_2$, and again consider the near-triangulation obtained by deleting the vertex v_p. If $(c_1, c_3) \in L_3$, assign $L(v_{p+1}) = \{c_1\}$ and $L(u_i) = \{c_1, c_5\} \setminus L(u_{i+1})$ for $2 \leq i < r$. There is no vertex of type T_3 in C_1 now. Instead of the label L_3 assigned to the edge $v_{p+1}v_{p+2}$, assign the corresponding label $L_3' = \{(c_1, c_3)\}$. Applying Lemma 4, there exists a feasible pair (L_1, L_3') for G_1. Assigning v_p a colour in $\{c_2, c_4\}$ that is not assigned to v_{p-1} extends any L-colouring of C_1 to an L-colouring of C. Therefore (L_1, L_2, L_3) is a feasible triple for G. If $(c_1, c_3) \notin L_3$, then $L_3 = \{(c_2, c_1), (c_2, c_4), (c_1, c_4), (c_4, c_1)\}$ and the same argument holds by choosing $L_3' = \{(c_1, c_4)\}$. The remaining possibility is $(c_2, c_1) \notin L_2$, and by symmetry, $(c_2, c_1) \notin L_3$ which implies $L_2 = \{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\}$ and $L_3 = \{(c_1, c_3), (c_4, c_3), (c_1, c_4), (c_4, c_1)\}$. In this case, swap the colours c_3 and c_4 in all the lists and labels and assign list $\{c_3\}$ to the vertex v_{p+1}. The lists for the other vertices remain the same and the label L_2 changes to $\{(c_2, c_3), (c_4, c_3)\}$ and L_3 to $\{(c_3, c_1), (c_3, c_4)\}$. This reduces to the Case 2.2.1 considered previously.

Cases 2.2.2.1 and 2.2.2.2 cover all possibilities with $p > 1$. A symmetrical argument can be used if $q < l - 1$ by reversing the cycle and swapping the colours c_1 and c_2.

Case 2.2.2.3 The only case that remains now is when $p = 1$ and $q = l - 1$, which means v_1 is the only vertex of type T_1 and v_l is the only vertex of type T_2.

Suppose $l \geq 5$ and let $v_2 = u_1, u_2, \ldots, u_r = v_4$ be the neighbours of v_3 such that $v_3 u_i u_{i+1}$ is an internal face in G, for $1 \leq i < r$. Let G_1 be the near-triangulation obtained by deleting the vertex v_3 with the cycle $C_1 = v_1, v_2, u_2, \ldots, u_{r-1}, v_4, \ldots, v_l$ as the boundary. Let $L(u_i) = \{c_3\}$ if i is odd and $L(u_i) = \{c_5\}$ if i is even for $2 \leq i < r$. The list assignment to the vertices of C_1, together with the labels L_2 and L_3 for the edges $v_1 v_2$ and $v_{l-1} v_l$ respectively, satisfies the induction hypothesis, and by induction there exists a feasible triple (L_1, L_2, L_3) for G_1. Any L-colouring of C_1 can be extended to an L-colouring of C by assigning v_3 a colour in $\{c_1, c_2, c_4\}$

that is not assigned to v_2 or v_4. This implies (L_1, L_2, L_3) is also a feasible triple for G.

Suppose $l = 4$, $(c_2, c_1), (c_4, c_1) \in L_2$ and $(c_1, c_3), (c_4, c_3) \in L_3$. Let $v_4 = u_1, u_2, \ldots, u_r = v_2$ be the neighbours of v_1 such that $v_1 u_i u_{i+1}$ is an internal face in G for $1 \leq i < r$. Consider the near-triangulation G_1 obtained by deleting the vertex v_1 with the cycle $C_1 = u_2, \ldots, v_2, v_3, v_4$ as the boundary. Assign the list $L(u_r) = L(v_2) = \{c_1\}$ and $L(u_i) = \{c_1, c_5\} \setminus L(u_{i+1})$ for $2 \leq i < r$. Assign the list $\{c_2, c_3, c_4\}$ to v_3 and v_4 so that all vertices in C_1 are of type T_1. By Lemma 3, there exists an L-colouring f of C_1 that extends to a k-colouring of G_1 with $f(v_4) = c_3$, $f(v_3) = c_4$ and $f(v_2) = c_1$. This can be extended to an L-colouring of C by assigning the colour c_2 or c_4 to the vertex v_1. Then with $L_1 = \{(c_2, c_3), (c_4, c_3)\}$, (L_1, L_2, L_3) is a feasible triple for G. A symmetrical argument can be used if $\{(c_3, c_2), (c_3, c_4)\} \in L_2$ and $\{(c_2, c_1), (c_2, c_4)\} \in L_3$.

Suppose $(c_2, c_1), (c_4, c_1) \in L_2$ but $(c_1, c_3), (c_4, c_3) \notin L_3$. This implies $L_3 = \{(c_2, c_1), (c_2, c_4), (c_1, c_4), (c_4, c_1)\}$ and by the previous argument, $(c_3, c_2), (c_3, c_4) \notin L_2$. This implies $L_2 = \{(c_2, c_1), (c_4, c_1), (c_4, c_2), (c_2, c_4)\}$. In this case, the previous argument gives an L-colouring of C_1 that extends to a k-colouring of G_1 with $f(v_4) = c_4$, $f(v_3) = c_2$ and $f(v_1) = c_1$. Assigning colour c_2 to v_1 gives an L-colouring f of C. Again using symmetry after reversing the cycle and swapping colours c_1, c_2, we get an L-colouring f of C that extends to a k-colouring of G with $f(v_1) = c_4$, $f(v_2) = c_1$, $f(v_3) = c_2$ and $f(v_4) = c_1$. Therefore setting $L_1 = \{(c_2, c_4), (c_4, c_1)\}$ gives a feasible triple for G.

The remaining case to be considered is if $(c_2, c_1), (c_4, c_1) \notin L_2$, and by symmetry, $(c_2, c_1), (c_2, c_4) \notin L_3$. Then $L_2 = \{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\}$ and $L_3 = \{(c_1, c_3), (c_4, c_3), (c_1, c_4), (c_4, c_1)\}$. Consider the triangulation G_1, and assign the lists $L(v_4) = \{c_1\}$, $L(u_i) = \{c_1, c_5\} \setminus L(u_{i-1})$ for $2 \leq i < r$, and $L(v_2) = L(v_3) = \{c_2, c_3, c_4\}$. Then all vertices are of type T_1 and there exists an L-colouring f of C_1 that extends to a k-colouring of G_1 with $f(v_4) = c_1$, $f(v_2) = c_2$ and $f(v_3) = c_4$. Assigning $f(v_1) = c_3$ gives an L-colouring f of C that extends to a k-colouring of G. A symmetrical argument, after reversing the cycle and swapping the colours c_1, c_2 gives an L-colouring f of C that extends to a k-colouring of G with $f(v_1) = c_2$, $f(v_2) = c_4$, $f(v_3) = c_1$ and $f(v_4) = c_3$. This implies setting $L_1 = \{(c_3, c_1), (c_2, c_3)\}$ gives a feasible triple of labels (L_1, L_2, L_3). This completes all cases when $l = 4$ and hence the proof of Lemma 5. □

Proof of Theorem 1: The proof follows from Lemmas 3, 4 and 5. Let f be a good k-colouring of a cycle C of length l. Suppose f satisfies condition 1 in the definition of a good colouring. By permuting the colours, we can assume that $F[1, l] \subseteq S_1$. Then assigning the list $L(v_i) = \{f(v_i)\}$ for all $1 \leq i \leq l$ gives a list assignment to the vertices of C that satisfies the conditions of Lemma 3. Since any chordless near-triangulation G with boundary C is consistent with the list assignment to C, Lemma 3 implies the colouring of C can be extended to a k-colouring of G.

Suppose f satisfies condition 2 in the definition of a good colouring, with indices p and q. By relabeling the vertices and permuting the colours, we may assume that $p = 1$, $F[1, q) \subseteq S_1$, $F[q, 1) \subseteq S_2$, and the set $A = \{c_j | 5 \leq j \leq k\}$. Then the list assignment $L(v_i) = \{f(v_i)\}$ for $1 \leq i \leq l$, with the label $L_2 = L(v_{q-1}) \times L(v_q)$

assigned to the edge $v_{q-1}v_q$, satisfies the conditions of Lemma 4 and the result follows.

Suppose f satisfies condition 3 in the definition of a good colouring, with indices p, q, r. Again by relabeling vertices and permuting colours, we may assume that $p = 1$, $F[1, q) \subseteq S_1$, $F[q, r) \subseteq S_3$ and $F[r, 1) \subseteq S_2$, where again $A = \{c_j | 5 \leq j \leq k\}$. The list assignment $L(v_i) = \{f(v_i)\}$ satisfies the conditions of Lemma 5 with the labels $L_2 = L(v_{q-1}) \times L(v_q)$ and $L_3 = L(v_{r-1}) \times L(v_r)$ assigned to the edges $v_{q-1}v_q$ and $v_{r-1}v_r$, respectively. The result follows from Lemma 5.

This completes the proof of Theorem 1. $\square$

Corollary 1 *Any k-colouring f of a cycle of length $l \leq 2k-5$ such that $|F[1, l]| \leq k-1$ is safe. There exists a k-colouring of a cycle of length $2k-4$ that is not safe and uses $k-1$ distinct colours.*

Proof: If $|F[1, l]| \leq k-3$ then f is good and the result follows from Theorem 1. If $|F[1, l]| = k-2$, we may assume $F[1, l] = \{c_1, \ldots, c_{k-2}\}$. Since $l \leq 2k-5$, at least one colour occurs only once in the cycle. Without loss of generality, by relabeling vertices and permuting colours, we may assume v_l is the only vertex with colour c_{k-2}. Then f satisfies condition 2 in the definition of a good k-colouring, with $p = 1$, $q = l$ and $A = \{c_1, \ldots, c_{k-4}\}$. If $|F[1, l]| = k-1$, there are at least three distinct colours that occur exactly once in the colouring of the cycle. Without loss of generality, we may assume these colours are $c_{k-3}, c_{k-2}, c_{k-1}$ and the colour c_k does not occur. Let $1 < p' < q' \leq l$ be indices such that $f(v_1) = c_{k-3}$, $f(v_{p'}) = c_{k-2}$ and $f(v_{q'}) = c_{k-1}$. Then f satisfies condition 3 in the definition of a good colouring with $p = 1, q = p', r = q'$ and $A = \{c_1, \ldots, c_{k-4}\}$. Theorem 1 implies f is safe.

An example of a k-colouring f of a cycle of length $2k-4$ that is not safe is given by $f(v_i) = f(v_{k-2+i}) = c_i$ for $1 \leq i \leq k-3$, $f(v_{k-2}) = c_{k-2}$ and $f(v_{2k-4}) = c_{k-1}$. Note that this colouring satisfies condition 2 in Lemma 1, and is bad. $\square$

If Conjecture 1 is true, then every k-colouring of a cycle of length at most $3k-10$ that uses at most $k-2$ distinct colours is safe. However, Theorem 1 does not prove this since the k-colouring of a cycle of length $2k-4$ defined by $f(v_i) = f(v_{i+k-2}) = c_i$ for $1 \leq i \leq k-2$ is not good.

3 Remarks

In principle, it may be possible to extend this approach to prove the four colour theorem itself. However, it is not sufficient to use a set of 4-colourings of a cycle defined by list-colourings. A more general way of defining sets of colourings is required. A set $\mathcal{F}$ of 4-colourings of a cycle C of length l is safe, if for every chordless near-triangulation G with C as the boundary, there exists a 4-colouring $f \in \mathcal{F}$ of C that can be extended to a 4-colouring of G. While it would be nice to characterize exactly the safe sets of 4-colorings, proving the four colour theorem only requires showing that any proper colouring of a triangle is safe. It may be possible to do this by finding a simpler sufficient condition that ensures safety of a set of colourings.

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