Colouring planar graphs with a precoloured induced cycle

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Abstract  
Let $C$ be a cycle and $f : V(C) \to \{c_1, c_2, \ldots, c_k\}$ a proper $k$-colouring of $C$ for some $k \geq 4$. We say the colouring $f$ is safe if for any planar graph $G$ in which $C$ is an induced cycle, there exists a proper $k$-colouring $f'$ of $G$ such that $f'(v) = f(v)$ for all $v \in V(C)$. The only safe 4-colouring is any proper colouring of a triangle. We give a simple necessary condition for a $k$-colouring of a cycle to be safe and conjecture that it is sufficient for all $k \geq 4$. The sufficiency for $k = 4$ follows from the four colour theorem and we prove it for $k = 5$, independent of the four colour theorem. We show that a stronger condition is sufficient for all $k \geq 4$. As a consequence, it follows that any proper $k$-colouring of a cycle that uses at most $k - 3$ distinct colours is safe. Also, any proper $k$-colouring of a cycle of length at most $2k - 5$ that uses at most $k - 1$ distinct colours is safe.

1 Introduction

A proper $k$-colouring of a graph $G$ is an assignment $f : V(G) \to \{c_1, c_2, \ldots, c_k\}$ of colours to the vertices of $G$ such that for every edge $uv$ in the graph $f(u) \neq f(v)$. We will refer to proper colourings as simply colourings and a graph is $k$-colourable if there exists a $k$-colouring. The study of colourings of planar graphs has a long history, starting with the four colour conjecture and its eventual proof [2] [3] [7]. Many variations of this have since been considered, the most important of which is perhaps the notion of list-colouring. Let $L(v)$ be a list of allowed colours assigned to each vertex $v \in V(G)$. An $L$-colouring of $G$ is a proper colouring such that each vertex $v$ is assigned a colour in $L(v)$. The usual colouring problem is the special case when the lists $L(v)$ are the same for all vertices. While all planar graphs are four colourable, it is not true that they are $L$-colourable if $|L(v)| \geq 4$ for all vertices $v$ [9]. Thomassen [8] showed that all planar graphs are $L$-colourable if $|L(v)| \geq 5$.
for each vertex $v$. Subsequently, there has been a lot of work on finding conditions on the lists $L(v)$ that guarantee any planar graph is $L$-colourable. A recent result of Zhu [10] shows that if $|L(v)| = 4$ for each vertex $v$, and $|L(u) \cap L(v)| \leq 2$ for any edge $uv$, then any planar graph is $L$-colourable. Many such results with different restrictions were proved earlier, and several conjectured, as mentioned in [10].

Another variation considered is colouring or $L$-colouring with some precoloured vertices, whose colours are specified. This is equivalent to considering an $L$-colouring in which some vertices have lists of size 1. This can also be viewed as extending a partial assignment of colours to some vertices to a colouring or $L$-colouring of the whole graph. Albertson [1] showed that every planar graph with precoloured vertices is five colourable if the precoloured vertices are at distance at least 4 from each other and the precolouring uses at most five distinct colours. No such result is possible for four colours, even if only two vertices are precoloured. The corresponding result for $L$-colouring was shown in [5]. There exists a constant $M$ such that if the precoloured vertices are at distance at least $M$ from each other, and $|L(v)| = 5$ for all other vertices, then any planar graph is $L$-colourable.

In most of these results, the restrictions on the lists are in terms of their sizes, or the sizes of their intersections, but not the actual elements in the list. Here, we consider another special case where the vertices of a connected induced subgraph are assigned lists of size 1 and all other vertices have the same list of a specified size. This is equivalent to asking when can a specified $k$-colouring of a graph $H$ be extended to a $k$-colouring of any planar graph that contains $H$ as in induced subgraph. We call such a $k$-colouring of the graph $H$ a safe $k$-colouring. The subgraph we consider is a cycle. If $H$ is any 2-connected planar graph, a $k$-colouring of $H$ is safe if and only if the colouring of every non-separating induced cycle in $H$ is safe. Thus considering a cycle as the subgraph $H$ is a natural choice.

We give a simple necessary condition for a $k$-colouring of a cycle to be safe. We conjecture that this condition is sufficient for all $k \geq 4$. The sufficiency for $k = 4$ follows from the four colour theorem. We prove it for $k = 5$. The proof is independent of and much simpler than the proof of the four colour theorem. We prove that a stronger condition is sufficient for all $k \geq 4$. As a consequence, any $k$-colouring of a cycle that uses at most $k - 3$ distinct colours is safe. Also, any $k$-colouring of a cycle of length at most $2k - 5$ that uses at most $k - 1$ colours is safe. If the conjecture is true, then any $k$-colouring of a cycle of length at most $3k - 10$ that uses at most $k - 2$ colours is safe. However, we have not been able to prove this.

Such results have been considered for 3-colourings of planar triangle-free graphs. In [6] for example, 3-colourings of a cycle of length at most 8 contained in a planar triangle-free graph that cannot be extended to a 3-colouring of the whole graph have been characterized. The same has been done for cycles of length 9 in [4]. However, we do not know any such results for general planar graphs when the number of colours and the cycle length is arbitrary. Also, these results are based on characterizing all critical graphs, that is graphs for which the colouring of the cycle cannot be extended to the whole graph, but can be extended for any proper subgraph that contains the cycle. We do not attempt to do this here. To show that
a colouring is not safe, it is enough to show one graph for which the colouring of
the cycle cannot be extended. To show it is safe, we use properties of the colouring
to extend it to the whole graph.

2 Main Result

Let $C = v_1, v_2, \ldots, v_l$ be a cycle of length $l \geq 3$ and $f$ a safe $k$-colouring of $C$. Any $k$-colouring of $C$ obtained from $f$ by permuting the colours is also safe. The colouring $f_i$ defined by $f_i(v_j) = f(v_{i+j})$, where addition is modulo $l$, is also safe, for all $1 \leq i < l$. Similarly, the colouring obtained by reversing the cycle, that is $f_r(v_i) = f(v_{i+1-i})$ is safe. We say these colourings of $C$ are equivalent and consider safety to be a property of the equivalence class.

Let $[i, j] = \{i, i + 1, \ldots, j\}$ for $1 \leq i, j \leq l$, where again addition is considered modulo $l$. Let $[i, j] = [i, j] \setminus \{j\}$, $(i, j) = [i, j] \setminus \{i\}$, and $(i, j) = [i, j] \setminus \{i, j\}$. Let $F[i, j] = \{v_t \mid 1 \leq t \leq k, \exists m \in [i, j] f(v_m) = c_t\}$ be the subset of colours that occur in the subpath of $C$ from $v_i$ to $v_j$. The sets $F[i, j], F(i, j), F(i, j)$ are defined similarly.

To prove that a $k$-colouring $f$ of $C$ is safe, we consider an arbitrary planar graph $G$ such that $C$ is an induced cycle in $G$, and show that the colouring of $C$ can be extended to a $k$-colouring of $G$. It is sufficient to do this in the case $C$ is a non-separating induced cycle in $G$, since otherwise we can consider each component of $G - V(C)$ separately. Thus we may assume $C$ bounds a face of $G$ and without loss of generality, we can embed $G$ so that $C$ is the boundary of the external face. We may further assume $G - V(C)$ is not empty, and add edges to $G$ that are not chords of $C$ if needed, so that every internal face of $G$ is bounded by a triangle. We call such a graph $G$ a chordless near-triangulation. Thus a $k$-colouring $f$ of a cycle $C$ is safe if and only if for any chordless near-triangulation $G$ with external face bounded by the cycle $C$, there exists a $k$-colouring of $G$ that extends $f$.

We first give some simple necessary conditions for a $k$-colouring to be safe.

**Lemma 1** Let $f$ be a $k$-colouring of a cycle $C$ of length $l$. If $f$ satisfies any one of the following conditions, then $f$ is not safe.

1. $|F[1, l]| = k$.

2. There exist indices $1 \leq p < q \leq l$ such that $|F[p, q] \cap F[q, p]| \geq k - 1$.

3. There exist indices $1 \leq p < q < r \leq l$ such that $|F[p, q] \cap F[q, r] \cap F[r, p]| \geq k - 2$.

**Proof:** If $|F[1, l]| = k$ then the $k$-colouring cannot be extended to a $k$-colouring of the wheel $W_l$, hence it is not safe. If there are indices $p < q$ such that $|F[p, q] \cap F[q, p]| \geq k - 1$, construct a chordless near-triangulation by adding a vertex $u$ that is adjacent to all vertices $v_m$ for $m \in [p, q]$, and a vertex $v$ that is adjacent to $u$ and all vertices $v_m$ for $m \in [q, p]$. The $k$-colouring of the cycle cannot be extended to this near-triangulation, hence such a colouring is not safe. The same argument holds if there are indices $p < q < r$ such that $|F[p, q] \cap F[q, r] \cap F[r, p]| \geq k - 2$. Construct a near-triangulation by adding a triangle $u, v, w$ with $u$ adjacent to vertices $v_m$ for...
m \in [p,q], v \text{ adjacent to vertices } v_m \text{ for } m \in [q,r] \text{ and } w \text{ adjacent to vertices } v_m \text{ for } m \in [r,p]. \text{ Again it is easy to check that the colouring of the cycle cannot be extended to the near-triangulation, hence any such } k\text{-colouring is not safe. □}

We call a } k\text{-colouring of a cycle that satisfies any of the conditions in Lemma 1 a bad } k\text{-colouring. If } k = 4 \text{ and } l \geq 4, \text{ then every } 4\text{-colouring of } C \text{ is bad. Consider any } 4\text{-colouring of a cycle } C \text{ of length at least 4. Assume } |F[1,l]| < 4 \text{ otherwise condition 1 in Lemma 1 is satisfied. If } |F[1,l]| = 2, \text{ we have } |F[1,2]| = 2, \text{ and } F[1,2] = F[2,3] = F[3,1], \text{ which implies } f \text{ satisfies condition 3 in Lemma 1 with } p = 1, q = 2, r = 3. \text{ If } |F[1,l]| = 3, \text{ there exists an index } i \text{ such that } |F[i,i+2]| = 3. \text{ If } |F[i+2,l]| = 3, \text{ then } f \text{ satisfies condition 2 in Lemma 1 with } \{p,q\} = \{i,i+2\}, \text{ otherwise we have } l \geq 5, f(v_{i+3}) = f(v_i) \text{ and } f(v_{i+4}) = f(v_{i+2}). \text{ In this case, we have } |F[i+1,i+3]| = |F[i+3,i+1]| = 3 \text{ and } f \text{ again satisfies condition 2 in Lemma 1 with } \{p,q\} = \{i+1,i+3\}. \text{ Therefore, the only } 4\text{-colourings that are not bad are those of the triangle. It follows from the four colour theorem that any } 4\text{-colouring of a triangle is safe and hence a } 4\text{-colouring is safe if and only if it is not bad. We conjecture that this property holds for all } k \geq 4.

**Conjecture 1** A } k\text{-colouring of a cycle is safe if and only if it is not bad, for all } k \geq 4.

We prove Conjecture 1 for } k = 5. \text{ Although the statement of Conjecture 1 may be considered to be a generalization of the four colour theorem, the proof for } k = 5 \text{ is much simpler and does not depend on the four colour theorem. We in fact show that } k\text{-colourings satisfying a stronger property are safe for all } k \geq 4. \text{ All } 5\text{-colourings that are not bad satisfy the stronger property and are therefore safe. Moreover, there exist such } k\text{-colourings satisfying the stronger property for cycles of all lengths, for all } k \geq 5. \text{ However, this proof depends crucially on the fact that } k \geq 5, \text{ and cannot be easily adapted to prove the four colour theorem itself. Also, for } k \geq 6, \text{ there are } k\text{-colourings that are not bad but do not satisfy the stronger property, so this does not prove Conjecture 1 for } k \geq 6.

Henceforth, we will assume that } k \geq 5 \text{ is a fixed integer. We call a } k\text{-colouring of a cycle of length } l \text{ a good } k\text{-colouring if it satisfies one of the following properties.}

1. } |F[1,l]| \leq k - 3.
2. } |F[1,l]| = k - 2 \text{ and there exist indices } 1 \leq p < q \leq l \text{ and a set } A \text{ of } k - 4 \text{ colours such that } |F[p,q] \setminus A| = 1 \text{ and } |F[q,p] \setminus A| = 1.
3. } |F[1,l]| = k - 1 \text{ and there exist indices } 1 \leq p < q < r \leq l \text{ and a set } A \text{ of } k - 4 \text{ colours such that } |F[p,q] \setminus A| = |F[q,r] \setminus A| = |F[r,p] \setminus A| = 1.

**Lemma 2** A } 5\text{-colouring } f \text{ of a cycle of length } l \text{ is not bad if and only if it is good.}

**Proof:** We show that if } f \text{ is good it cannot satisfy any of the conditions in Lemma 1 and is therefore not bad. This argument holds for all } k \geq 4, \text{ but we state it only for } k = 5. \text{ Clearly } |F[1,l]| < 5, \text{ hence } f \text{ does not satisfy condition 1 in Lemma 1. Similarly if } |F[1,l]| = 2, f \text{ does not satisfy any of the conditions in Lemma 1.
Suppose $|F[1, l]| = 3$ and $f$ satisfies condition 2 in the definition of a good 5-colouring with indices $p_1, q_1$. Then $f$ cannot satisfy condition 2 in the statement of Lemma 1. Suppose for contradiction $f$ satisfies condition 3 in Lemma 1 with indices $p_2, q_2, r_2$. Since $[p_1, q_1], [q_1, p_1]$ is a partition of $[1, l]$, two of the indices $p_2, q_2, r_2$ are contained in one of the sets $[p_1, q_1]$ or $[q_1, p_1]$. Without loss of generality, assume $p_2, q_2 \in [p_1, q_1]$. However, this implies $3 \leq |F[p_2, q_2]| \leq |F[p_1, q_1]| \leq 2$, a contradiction. Therefore are sets of size at most 2, and each contains one element other than $p, q$

If

then $p, q$

Suppose $|F[1, l]| = 4$ and $f$ satisfies condition 3 in the definition of a good 5-colouring with indices $p_1, q_1, r_1$ and let $A = \{c_1\}$. Suppose for contradiction $f$ satisfies condition 2 in Lemma 1 with indices $p_2, q_2$. Again, we may assume, without loss of generality, $p_1, q_1 \in [p_2, q_2]$. If $p_1 \neq p_2$, then $[q_2, p_2] \subseteq [q_1, r_1] \cup [r_1, p_1]$, which contradicts the fact that $|F[q_2, p_2]| \geq 4$ but $|F[q_1, r_1] \cup F[r_1, p_1]| \leq 3$. If $p_1 = p_2$ and $r_1 \in [p_2, q_2]$ then $[q_2, p_2] \subseteq [r_1, p_1]$, which is again a contradiction. If $r_1 \notin [p_2, q_2]$ then $[p_2, q_2] \subseteq [p_1, q_1] \cup [q_1, r_1]$, which again contradicts the fact that $|F[p_2, q_2]| \geq 4$ and $|F[p_1, q_1] \cup F[q_1, r_1]| \leq 3$. Finally, suppose $f$ satisfies condition 3 in Lemma 1 with indices $p_2, q_2, r_2$. Let $c_2$ be a colour other than $c_1$ that is contained in $F[p_2, q_2] \cap F[q_2, r_2] \cap F[r_2, p_2]$. Without loss of generality, we may assume any vertex $v_m$ such that $f(v_m) = c_2$ satisfies $m \in [p_1, q_1]$. However, this implies one of the sets $[p_2, q_2], [q_2, r_2], [r_2, p_2]$ must be contained in $[p_1, q_1]$, contradicting the fact that each of $F[p_2, q_2], F[q_2, r_2], F[r_2, p_2]$ has at least three elements, but $|F[p_1, q_1]| \leq 2$. This proves that if $f$ is good, it is not bad.

We next show that if $f$ is not good then it is bad. This argument holds only for $k = 5$. We may assume $3 \leq |F[1, l]| \leq 4$, otherwise $f$ satisfies condition 1 in Lemma 1 and is bad.

Suppose $|F[1, l]| = 3$. Let $p, q$ be indices such that $|[p, q]|$ is maximum and $|F[p, q]| = 2$. Then $f(v_q) \neq f(v_m)$ and $f(v_{p-1}) \neq f(v_m)$ for any $m \in [p, q]$. Since $|F[1, l]| = 3$, we must have $f(v_q) = f(v_{p-1})$ and $|F[q, p]| > 2$, otherwise $f$ satisfies condition 2 in the definition of a good colouring with indices $p, q$. This implies $q \neq p - 1$ and $[q, p]$ has at least four elements. Since $|F[p, q]| = 2$, we must have $q \geq p + 2$. If $|F[p, q]| \geq 3$ then $\{p - 1, p + 1, q\}$ are three indices that satisfy condition 3 in Lemma 1 and $f$ is bad. If $|F[p, q]| = 2$, we must have $q = p + 2$. The choice of $[p, q]$ then implies that $f(v_{q+1}) = f(v_p)$ and $f(v_{q+2}) = f(v_{p+1})$. This implies $f$ satisfies condition 3 in Lemma 1 with indices $\{p, q, q + 2\}$ and is bad.

Suppose $|F[1, l]| = 4$. Let $p, q$ be indices such that $|F[p, q]| \geq 4$ and $|[p, q]|$ is minimum possible. We have $f(v_p) \neq f(v_m)$ for any $m \in [p, q]$, $f(v_q) \neq f(v_m)$ for any $m \in [p, q]$, and also $f(v_p) \neq f(v_q)$. This implies $|F[p, q]| = 2$. If $|F[p, q]| \geq 4$ then $p, q$ are indices satisfying condition 2 in Lemma 1 and $f$ is bad. So $|F[p, q]| \leq 3$, which implies $|F(p, q) \cap F[q, p]| \leq 1$ and let $c_1$ be the colour in this set, if it is not empty, otherwise let $c_1$ be any colour in $F(p, q)$. Suppose there exists an index $r \in [q, p]$ such that $f(v_p) \notin F[q, r]$ and $f(v_q) \notin F(r, p)$. Then $F(p, q), F[q, r], F(r, p)$ are sets of size at most 2, and each contains one element other than $c_1$. This implies $f$ satisfies condition 3 in the definition of a good colouring, a contradiction. Therefore there is no such index $r$ and we have $p \neq q + 1$. This implies there exists an index $r$ in $(q, p)$ such that $f(v_r) = f(v_p)$ and an index $r'$ in $(r, p)$ such that $f(v_{r'}) = f(v_q)$. If $q \geq p + 4$ then we have $|F[p + 2, r]| \geq 4$ and $|F[r, p + 2]| \geq 4$, implying $f$ satisfies
condition 2 in Lemma 1 and is bad. Finally, suppose \( q = p + 3 \). If either the colour \( f(v_{p+1}) \in F[q,r'] \) or \( f(v_{p+2}) \in F[r,p] \), the same argument holds with indices \( \{p+2,r'\} \) or \( \{p+1,r\} \), respectively. Without loss of generality, assume the colour \( c_1 \) is \( f(v_{p+1}) \) so \( v_{p+2} \) is the only vertex with the colour \( f(v_{p+2}) \). Choose \( r' \) so that \( ||r',p|| \) is as small as possible. Then \( f(v_q) \notin F(r',p) \) and \( |F[p+2,q]| = 1, |F[q,r']| \leq 2 \) and \( |F(r',p+2)| \leq 2 \). Also \( |F[q,r']| \setminus \{f(v_{p+2})\} = 1 \) and \( |F(r',p+2) \setminus \{f(v_{p+2})\}| = 1 \), which implies the indices \( p+2, q, r'+1 \) with \( A = \{f(v_{p+2})\} \), satisfy condition 3 in the definition of a good colouring, a contradiction.

The 6-colouring of the 8 cycle defined by \( f(v_i) = f(v_{i+4}) = c_i \) for \( 1 \leq i \leq 4 \) is an example of a colouring that is neither good nor bad.

We can now state the main result.

**Theorem 1** Any good \( k \)-colouring of a cycle is safe for all \( k \geq 5 \).

In order to prove Theorem 1 by induction, we need to consider sets of \( k \)-colourings of a cycle, rather than a single \( k \)-colouring. We also need to consider near-triangulations \( G \) with a given external boundary \( C \) that may have chords. Let \( C = v_1, v_2, \ldots, v_l \) be a cycle and \( L(v_i) \subseteq \{c_1, c_2, \ldots, c_k\} \) a non-empty list of at most \( k \) colours assigned to the vertex \( v_i \), for \( 1 \leq i \leq l \). A near-triangulation \( G \) with external boundary \( C \) is said to be consistent with the list assignment if for any two vertices \( u, v \in V(C) \) such that \( uv \) is an edge or chord of \( C \), both \( u \) and \( v \) are not assigned the same list of size 1. We will only consider list assignments such that there are no two adjacent vertices \( u, v \) in \( C \) such that \( |L(u)| = 1 \) and \( L(u) = L(v) \). We show that for certain kinds of list assignments to the vertices of \( C \), for any near-triangulation \( G \) with external boundary \( C \) that is consistent with it, there exists an \( L \)-colouring of \( C \) that can be extended to a \( k \)-colouring of \( G \).

Let \( S_i = \{c_i\} \cup \{c_j|5 \leq j \leq k\} \) for \( i \in \{1, 2, 3\} \). A vertex \( v \) is said to be of type \( T_i \) for \( i \in \{1, 2, 3\} \) if either \( L(v) = \{c_1, c_2, c_3, c_4\} \setminus \{c_i\} \) or \( |L(v)| = 1 \) and \( L(v) \subset S_i \). The list assignments that we consider will be such that every vertex will be of one of the three types.

**Lemma 3** Let \( C = v_1, v_2, \ldots, v_l \) be a cycle of length \( l \) and \( L \) an assignment of list of colours such that every vertex is of type \( T_1 \). Let \( (s,t) \in L(v_1) \times L(v_l) \) be any ordered pair of distinct colours. Then for any near-triangulation \( G \) with boundary \( C \) that is consistent with the list assignment, there exists an \( L \)-colouring of \( f \) of \( C \) that extends to a \( k \)-colouring of \( G \) with \( f(v_1) = s \) and \( f(v_l) = t \).

**Proof:** Suppose there is a counterexample and let \( G \) be one with minimum number of edges.

**Case 1.** Suppose \( G \) has a chord \( v_iv_j \) for some \( i < j \). Then both \((i,j) \) and \((j,i) \) are non-empty sets. Let \( C_1 \) be the cycle \( v_1, v_2, \ldots, v_i, v_j, v_j+1, \ldots, v_l \) and \( G_1 \) the near-triangulation induced by the vertices on or in the interior of \( C_1 \). Then \( G_1 \) is consistent with the list assignment \( L \) restricted to the vertices of \( C_1 \), and by the minimality of \( G \), there exists an \( L \)-colouring \( f_1 \) of \( C_1 \) that can be extended to a \( k \)-colouring of \( G_1 \) with \( f_1(v_1) = s \) and \( f_1(v_l) = t \). Let \( C_2 \) be the cycle \( v_i, v_{i+1}, \ldots, v_j \) and let \( G_2 \) be the near-triangulation induced by the vertices on or in the interior of
C2. Again, the list assignment to the vertices of C2 satisfies the induction hypothesis and G2 is consistent with it. The minimality of G implies there exists an L-colouring f2 of C2 that can be extended to a k-colouring of G2 with f2(v1) = f1(v1) and f2(vj) = f1(vj). Defining f(v) = f1(v) for all vertices v ∈ V(G1) and f(v) = f2(v) for all vertices v ∈ V(G2), gives a k-colouring of G that extends an L-colouring of C, a contradiction.

Case 2. A similar argument holds if G has a separating triangle uvw, that is a triangle whose interior as well as exterior contain vertices of G. Let G1 be the near-triangulation obtained by deleting the vertices in the interior of the separating triangle uvw. Then G1 has the same boundary C as G and is consistent with the list assignment L. The minimality of G implies there exists an L-colouring f1 of C that extends to a k-colouring of G1, with f1(v1) = s and f1(vl) = t. Rename the colours so that f1(u) = c2, f1(v) = c3 and f1(w) = c4. Let G2 be the triangulation induced by the vertices on and in the interior of the triangle uvw and let L(u) = L(v) = L(w) = {c2, c3, c4}. Then the triangle uvw satisfies the induction hypothesis and by induction, there exists an L-colouring f2 of the triangle that extends to a k-colouring of G2, with f2(u) = c2 and f2(v) = c3. The list assigned to w ensures f2(w) = c4. Rename the colours in the colouring f2 so that f2(x) = f1(x) for all x ∈ {u, v, w}. Again, setting f(v) = f1(v) for all vertices v ∈ V(G1) and f(v) = f2(v) for all vertices v ∈ V(G2), gives a k-colouring of G that extends an L-colouring of C, a contradiction.

Case 3. We may now assume that G is chordless and has no separating triangle. If G is a triangle, let f(v1) = s and f(v3) = t with s ≠ t by assumption. If |L(v1)| = 3 then s ∈ {c2, c3, c4} and the same holds for v3. This implies that if |L(v2)| = 1, then s, t ∉ L(v2), and otherwise |L(v2)| = 3 and L(v2) contains a colour other than s, t. Assigning this colour to v2 gives an L-colouring of C.

Suppose G is not a triangle, which implies that for every edge vi+1 in C, there exists an internal vertex v in G such that vivi+1 is an internal face in G. Suppose there exists such an edge vi+1 for some 1 ≤ i < l such that L(vi) ∩ L(vi+1) = ∅. Let G1 be obtained from G by deleting the edge vi+1. Then G1 is a near-triangulation bounded by the cycle C1 = v1, . . ., vi, vi+1, . . ., vl. Setting L(v) = {c2, c3, c4} gives a list assignment to the vertices of C1 that satisfies the induction hypothesis, and G1 is consistent with it. The minimality of G implies there exists an L-colouring f of C1 that extends to a k-colouring of G1 with f(v1) = s and f(vl) = t. Since L(vi) ∩ L(vi+1) = ∅ by assumption, this gives the required L-colouring of C that extends to a k-colouring of G. If there is no such edge vi+1 in C, then for all m ∈ {1, l}, L(vm) = {c2, c3, c4}, for otherwise L(vl) = {a} for some a ∈ S1 and a ∉ L(vi+1), which implies L(vi) ∩ L(vi+1) = ∅. This also implies L(v1) = L(vl) = {c2, c3, c4}.

Let v1 = u1, u2, . . ., ur = v3 be the vertices adjacent to v2 such that v2ui+1 is an internal face of G, for some r ≥ 3 and all 1 ≤ i < r. Let G1 be the graph obtained from G by deleting the vertex v2. Then G1 is a near-triangulation bounded by the cycle C1 = v1, u2, . . ., ur−1, v3, . . ., vl. Let L(u1) = {c1} if i is odd and L(ui) = {c3} if i is even, for all 1 < i < r. The list assignment L to the vertices of C1 satisfies the induction hypothesis and G1 is consistent with it. The minimality
of $G$ implies there exists an $L$-colouring $f$ of $C_1$ that extends to a $k$-colouring of $G_1$ with $f(v_1) = s$ and $f(v_l) = t$. Since $L(v_2) = \{c_2, c_3, c_4\}$, setting $f(v_2)$ to be a colour in $\{c_2, c_3, c_4\} \setminus \{(f(v_1)) \cup \{f(v_3)\}\}$, gives the required $L$-colouring of $C$ that extends to a $k$-colouring of $G$. This completes all cases and the proof of the lemma.

We next consider the case when the vertices can be of two types. In this case, we need to put more restrictions on the $L$-colouring of the cycle. If $uv$ is an edge in the cycle $C$, we place restrictions on the ordered pair of colours $(f(u), f(v))$ assigned to the vertices $u$ and $v$ in an $L$-colouring $f$. In the usual list colouring, this can be any pair in $L(u) \times L(v)$ such that the two colours are distinct. Here we allow only specified subsets of such pairs and assign a list of allowed pairs from $L(u) \times L(v)$ to the edge $uv$. We call these sets of ordered pairs the labels of the edges. Let $a$ denote any colour in $S_1$, $b$ denote any colour in $S_2$, and $x$ denote any colour in $\{c_3, c_4\}$ with $\{x, y\} = \{c_3, c_4\}$. The allowed label sets are the following and we call the collection of these $L_{12}$,

(i) $\{(a, b), a \neq b\}$  
(ii) $\{(a, x)\}$  
(iii) $\{(x, b)\}$  
(iv) $\{(c_2, x), (y, x)\}$  
(v) $\{(x, c_1), (x, y)\}$  
(vi) $\{(c_2, x), (x, c_1)\}$

We consider list assignments to a cycle $C$ in which every vertex $v$ is assigned a list of colours and exactly two edges $u_1 v_1$ and $u_2 v_2$ are assigned labels $L(u_1 v_1), L(u_2 v_2) \in L_{12}$, respectively, where $L(u_1 v_1) \subseteq L(u_1) \times L(v_1)$ and $L(u_2 v_2) \subseteq L(u_2) \times L(v_2)$. Let $G$ be a near-triangulation with boundary $C$ that is consistent with the lists assigned to the vertices in $C$. We say the list assignment is feasible for $G$ if for every ordered pair $(s_1, t_1) \in L(u_1 v_1)$ there exists an ordered pair $(s_2, t_2) \in L(u_2 v_2)$ and an $L$-colouring $f$ of $C$ that can be extended to a $k$-colouring of $G$ with $f(u_i) = s_i$ and $f(v_i) = t_i$ for $i \in \{1, 2\}$. We call $(L(u_1 v_1), L(u_2 v_2))$ a feasible pair of labels for $G$.

**Lemma 4** Let $C = v_1, v_2, \ldots, v_l$ be a cycle of length $l$ and $p$ an integer such that $1 \leq p < l$. Let $L$ be an assignment of list of colours to each vertex in $C$ such that $v_m$ is of type $T_1$ for all $m \in [1, p]$, and of type $T_2$ for all $m \in (p, 1)$. Let $L_2 = L(v_p v_{p+1}) \subseteq L(v_p) \times L(v_{p+1})$ be any label in $L_{12}$ assigned to the edge $v_p v_{p+1}$. For any near-triangulation $G$ with boundary $C$ that is consistent with $L$, there exists a label $L_1 \in L_{12}$ and $L_1 \subseteq L(v_1) \times L(v_l)$ that can be assigned to the edge $v_1 v_l$ such that $(L_1, L_2)$ is a feasible pair of labels for $G$.

**Proof:** The proof is again by induction on the number of edges. Let $G$ be a counterexample with the minimum number of edges.

**Case 1.** Suppose $G$ has a chord $v_i v_j$ with $1 \leq i < j \leq l$. Let $C_1$ be the cycle $v_1, v_2, \ldots, v_i, v_j, \ldots, v_l$ and $G_1$ the near-triangulation induced by the vertices on and in the interior of $C_1$. Let $G_2$ be the near-triangulation induced by the vertices on and in the interior of the cycle $C_2 = v_i, v_{i+1}, \ldots, v_j$.

**Case 1.1** Suppose $j \leq p$. The list assignment $L$ restricted to the vertices in $C_1$, along with the label $L_2$ assigned to the edge $v_p v_{p+1}$, satisfies the induction hypothesis, and the minimality of $G$ implies the existence of a feasible pair of labels $(L_1, L_2)$ for $G_1$. We claim that this is also feasible for $G$. Consider any $L$-colouring $f_1$ of $C_1$ that can be extended to a $k$-colouring of $G_1$. The list assignment $L$ restricted to the vertices of $C_2$ satisfies the conditions of Lemma 3 and applying it gives an $L$-colouring $f_2$ of $C_2$ that can be extended to a $k$-colouring of $G_2$, with $f_2(v) = f_1(v)$.
for $v \in \{v_i, v_j\}$. Thus any $L$-colouring of $C_1$ that can be extended to a $k$-colouring of $G_1$ can also be extended to a $k$-colouring of $G$. Therefore the labels $(L_1, L_2)$ form a feasible pair for $G$. A similar argument holds if $i > p$. In this case, for applying Lemma 3 to the near-triangulation $G_2$, first swap the colours $c_1$ and $c_2$ in the lists assigned to the vertices of $C_2$. This converts all vertices $v_m$ with $m \in (i, j)$ to type $T_1$ and Lemma 3 can be applied. After getting the $L$-colouring of $C_2$ that can be extended to a $k$-colouring of $G_2$, again swap the colours $c_1, c_2$ in the $k$-colouring of $G_2$ to get the required $L$-colouring of $C_2$.

Case 2. Suppose $i \leq p < j$. In this case, the list assignment $L$ restricted to the vertices of $C_2$ satisfies the induction hypothesis, considering $v_iv_j$ to be the edge $v_1v_1$. The minimality of $G$ implies there exists a label $L'_1$ that can be assigned to the edge $v_iv_j$ such that $(L'_1, L_2)$ is a feasible pair for $G_2$. The list assignment $L$ restricted to the vertices of $C_1$, with the label $L'_1$ assigned to the edge $v_iv_j$, also satisfies the induction hypothesis, and there exists a label $L_1$ that can be assigned to the edge $v_1v_1$ such that $(L_1, L'_1)$ is a feasible pair for $G_1$. This implies $(L_1, L_2)$ is a feasible pair of labels for $G$.

Case 2. We may assume $G$ is chordless. If $G$ has a separating triangle, the same argument as in Lemma 3 can be used. Delete the vertices in the interior of a separating triangle to get a near-triangulation $G_1$ with the same boundary as $G$. By induction, there exists a feasible pair of labels $(L_1, L_2)$ for $G_1$. It can then be argued as in Lemma 3 that $(L_1, L_2)$ is also feasible for $G$. We may therefore assume $G$ has no chord and no separating triangle.

Case 2.1 Suppose $l = 3$ and without loss of generality assume $p = 2$, otherwise reverse the cycle and swap the colours $c_1$ and $c_2$ to apply the argument. In this case, it is sufficient to show that for every label $L_2 \in L_{12}$ assigned to the edge $v_2v_3$, there exists a label $L_1$ that can be assigned to $v_1v_3$ such that for every pair $(s, t) \in L_1$, there exists a colour $r \not\in \{s, t\}$ such that $(r, t) \in L_2$. Lemma 3 implies that the $L$-colouring $f$ of $C$ defined by $f(v_1) = s, f(v_2) = r, f(v_3) = t$, can be extended to a $k$-colouring of $G$, hence $(L_1, L_2)$ is a feasible pair for $G$.

If $|L(v_2)| = |L(v_3)| = 1$ then $L_2 = L(v_2) \times L(v_3)$. If $|L(v_1)| = 1$, then since $G$ is consistent with $L$, $L(v_1) \neq L(v_2)$ and $L(v_1) \neq L(v_3)$. Setting $L_1 = L(v_1) \times L(v_3)$ gives a feasible pair of labels $(L_1, L_2)$ for $G$. If $|L(v_1)| = 3$, then since $v_1$ is of type $T_1$, $L(v_1) = \{c_2, c_3, c_4\}$. Setting $L_1 = \{c_2\} \times L(v_3)$ gives the required label for the edge $v_1v_3$.

Suppose $|L(v_2)| = 3$ and $|L(v_3)| = 1$. We can assume $L_2 = \{(x, b)\}$ for some $x \in \{c_3, c_4\}$ and $b \in S_2$. If $|L(v_1)| = 1$, let $L_1$ be the set $L(v_1) \times \{b\}$. If $|L(v_1)| = 3$, let $L_1$ be the set $\{(y, b)\}$, where $y \in \{c_3, c_4\} \setminus \{x\}$. In both cases, we get a feasible pair of labels $(L_1, L_2)$ for $G$.

Suppose $|L(v_2)| = 1$ and $|L(v_3)| = 3$. We can assume that $L_2 = \{(a, x)\}$ for some $a \in S_1$. If $|L(v_1)| = 1$ let $L_1 = L(v_1) \times \{x\}$, otherwise $L(v_1) = \{c_2, c_3, c_4\}$ and let $L_1 = \{(c_2, x), (y, x)\}$. In both cases $(L_1, L_2)$ is a feasible pair of labels for $G$.

Suppose both $|L(v_2)|$ and $|L(v_3)|$ are 3, which implies $L(v_2) = \{c_2, c_3, c_4\}$ and $L(v_3) = \{c_1, c_3, c_4\}$. Then $L_2$ can be any one of $\{(c_2, x), (y, x)\}, \{(x, c_1), (x, y)\}$ and $\{(c_2, x), (x, c_1)\}$. If $|L(v_1)| = 1$, then $L(v_1) = \{a\}$ for some $a \in S_1$. If $(c_2, x) \in L_2$,
let $L_1 = \{(a, x)\}$, otherwise $(x, y) \in L_2$ and let $L_1 = \{(a, y)\}$. In both cases $(L_1, L_2)$ is a feasible pair for $G$. If $|L(v_1)| = 3$, then $L(v_1) = \{c_2, c_3, c_4\}$. If $L_2 = \{(c_2, x), (y, x)\}$ let $L_1 = L_2$. If $L_2 = \{(x, c_1), (x, y)\}$ let $L_1 = \{(y, c_1), (c_2, y)\}$ and if $L_2 = \{(c_2, x), (x, c_1)\}$ let $L_1 = \{(y, x), (y, c_1)\}$. In each case, it is easy to verify that $(L_1, L_2)$ is a feasible pair of labels for $G$.

**Case 2.2** Suppose now that $l > 3$. Since $G$ is chordless, for every edge $v_iv_{i+1}$ in $C$, there exists an internal vertex $v$ such that $v_iv_{i+1}$ is an internal face in $G$. Suppose there exists such an edge $v_iv_{i+1}$ for some $1 \leq i < l$ and $i \neq p$ such that $L(v_i) \cap L(v_{i+1}) = \emptyset$. Let $G_1$ be the near-triangulation obtained by deleting the edge $v_iv_{i+1}$, with the cycle $C_1 = v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_l$ as the boundary. If $i < p$, let $L(v) = \{c_2, c_3, c_4\}$ and if $i > p$ then let $L(v) = \{c_1, c_3, c_4\}$. The list assignment $L$ to the vertices of $C_1$, along with the label $L_2$ to the edge $v_pv_{p+1}$, satisfies the induction hypothesis and $G_1$ is consistent with it. By induction, there exists a label $L_1$ for the edge $v_iv_l$ such that $(L_1, L_2)$ is a feasible pair of labels for $G_1$. Since every $L$-colouring of $C_1$ is also an $L$-colouring of $C$, $(L_1, L_2)$ is a feasible pair of labels for $G$ also. We may assume there is no such edge in $C$.

**Case 2.2.1** Suppose there exists a vertex $v_i$ with $|L(v_i)| = 1$. Suppose $i = 1$, which implies $L(v_1) = \{a\}$ for some $a \in S_1$. If $p > 1$, then $L(v_1) \cap L(v_2) = \emptyset$, a contradiction. Therefore $p = 1$ and the label $L_2$ attached to the edge $v_1v_2$ is either $\{(a, b)\}$ for some $b \in S_2$ or $\{(a, x)\}$ for $x \in \{c_3, c_4\}$. Let $v$ be the internal vertex in $G$ such that $v_1v_2$ is an internal face in $G$. Let $G_1$ be the near-triangulation obtained by deleting the edge $v_1v_2$ with the cycle $C_1 = v_1, v_2, \ldots, v_l$ as the boundary. Let $L(v) = \{c_2, c_3, c_4\}$ and assign the label $L_2$ to the edge $v_1v_2$, where $L_2 = \{(c_3, b)\}$ if $L_2 = \{(a, b)\}$ and $L_2 = \{(c_2, x), (y, x)\}$ if $L_2 = \{(a, x)\}$. The list assignment $L$ to the vertices in $C_1$, along with the label $L_2$ assigned to the edge $v_1v_2$, satisfies the induction hypothesis, and by induction, there exists a feasible pair $(L_1, L'_2)$ for $G_1$. Since any $L$-colouring of $C_1$ is also an $L$-colouring of $C$, $(L_1, L_2)$ is a feasible pair for $G$. A symmetrical argument holds if $|L(v_2)| = 1$, by reversing the cycle and swapping the colours $c_1, c_2$.

If $1 < i \leq p$ then $L(v_{i-1}) \cap L(v_i) = \emptyset$ and if $l > i > p$ then $L(v_i) \cap L(v_{i+1}) = \emptyset$, a contradiction.

**Case 2.2.2** Suppose $|L(v_i)| = 3$ for all $i \in [1, l]$ and suppose $p > 2$. Let $v_1 = u_1, u_2, \ldots, u_r = v_3$ be the vertices adjacent to $v_2$ such that $v_2u_iv_{i+1}$ is an internal face in $G$ for all $1 \leq i < r$. Let $G_1$ be obtained from $G$ by deleting the vertex $v_2$. Then $G_1$ is a near-triangulation bounded by the cycle $C_1 = v_1, u_2, \ldots, u_{r-1}, v_3, \ldots, v_l$. Let $L(u_i) = \{c_1\}$ if $i$ is odd and $L(u_i) = \{c_5\}$ if $i$ is even, for all $2 \leq i < r$. This gives a list assignment to the vertices of $C_1$, which along with the label $L_2$ for the edge $v_pv_{p+1}$, satisfies the induction hypothesis and $G_1$ is consistent with it. By induction, there exists a label $L_1$ for the edge $v_1v_l$ such that $(L_1, L_2)$ is a feasible pair of labels for $G_1$. Since any $L$-colouring of $C_1$ can be extended to an $L$-colouring of $C$ by assigning $v_2$ a colour in $\{c_2, c_3, c_4\}$ that is not assigned to $v_1$ or $v_3$, $(L_1, L_2)$ is also a feasible pair of labels for $G$. A symmetrical argument can be used if $p < l - 2$, by relabeling the vertex $v_i$ as $v_{i+1-i}$ and swapping the colours $c_1$ and $c_2$.

**Case 2.2.3** Suppose $|L(v_i)| = 3$ for all $i \in [1, l]$, $p \leq 2$ and $p \geq l - 2$, which implies $l = 4$, $p = 2$. This implies $L_2$ is one of the labels $\{(c_2, x), (y, x)\}, \{(x, c_1), (x, y)\}$ or
deleting the vertex $v_2$. Again consider the near-triangulation $G_1$ obtained by deleting the vertex $v_2$ and bounded by the cycle $C_1 = v_1,u_2,\ldots,v_3,v_4$. Let $L(u_i) = \{c_1\}$ and $L(u_i) = \{c_1,c_3\} \setminus L(u_{i+1})$ for $2 \leq i < r$. Note that the list for $v_3 = u_i$ is modified by removing the elements $c_3,c_4$ from it. Assign the label $L'_v = \{\{c_1,x\}\}$ to the edge $v_3v_4$. Then the list assignment to the vertices of $C_1$ satisfies the induction hypothesis, and $G_1$ is consistent with it. By induction, there exists a label $L_1$ for the edge $v_1v_4$ such that $(L_1,L'_2)$ is a feasible pair of labels for $G_1$. Since in any $L$-colouring $f$ of $C_1$ we must have $f(v_4) = x$, $L_1 = \{(c_2,x),(y,x)\}$. Therefore any $L$-colouring of $C_1$ can be extended to an $L$-colouring of $C$ by assigning colour $x$ to $v_2$. This implies $(L_1,L_2)$ is a feasible pair of labels for $G$. If $(x,c_1) \notin L_2$, we can assume $(c_2,x) \in L_2$, and a symmetrical argument can be used after reversing the cycle and swapping the colours $c_1,c_2$.

This completes all cases and the proof of Lemma 4.

We now consider list assignments with vertices of 3 different types. We define two other sets of possible labels for edges in the cycle. Let $a \in S_1$, $b \in S_2$ and $c \in S_3$ be any colours. The label set $L_{13}$ contains the following sets of ordered pairs.

(i) $\{(a,c)\}$ (ii) $\{(a,c_2),(a,c_3)\}$ (iii) $\{(c_2,c),(c_4,c)\}$
(iv) $\{(c_3,c_2),(c_3,c_4),(c_2,c_4),(c_4,c_2)\}$
(v) $\{(c_2,c_1),(c_4,c_1),(c_2,c_4),(c_4,c_2)\}$
(vi) $\{(c_3,c_2),(c_3,c_4),(c_2,c_1),(c_4,c_1)\}$

A set in $L_{13}$ is obtained from a set in $L_{12}$ by first adding all pairs obtained by swapping colours $c_3$ and $c_4$ and then swapping the colours $c_2$ and $c_3$. Thus if the set in $L_{12}$ is $\{(c_2,c_3),(c_4,c_1)\}$, adding all pairs obtained by swapping $c_3$ and $c_4$ gives the set $\{(c_2,c_3),(c_3,c_1),(c_2,c_4),(c_4,c_1)\}$ and then swapping the colours $c_2$ and $c_3$ gives the label $\{(c_3,c_2),(c_2,c_1),(c_3,c_4),(c_4,c_1)\} \in L_{13}$. The labels in $L_{32}$ are obtained in a similar way from those in $L_{12}$, by first adding all pairs obtained by swapping colours $c_3$ and $c_4$ and then swapping the colours $c_1$ and $c_3$. For the example given, the label obtained is $\{(c_2,c_1),(c_1,c_3),(c_2,c_4),(c_4,c_3)\}$. The label set $L_{32}$ contains the following sets of ordered pairs.

(i) $\{(c,b)\}$ (ii) $\{(c,c_1),(c,c_4)\}$ (iii) $\{(c_1,b),(c_4,b)\}$
(iv) $\{(c_2,c_1),(c_2,c_4),(c_1,c_4),(c_4,c_1)\}$
(v) $\{(c_1,c_3),(c_4,c_3),(c_1,c_4),(c_4,c_1)\}$
(vi) $\{(c_2,c_1),(c_2,c_4),(c_1,c_3),(c_4,c_3)\}$.

The sets in $L_{13}$ and $L_{32}$ correspond to the sets in $L_{12}$ from which they are obtained.

Note that in Lemma 4, the list assignments and labels are symmetric in the colours $c_3$ and $c_4$. Therefore if $(L_1,L_2)$ is a feasible pair for a near-triangulation $G$, then so is $(L'_1,L'_2)$, where $L'_1$ and $L'_2$ are obtained from $L_1$ and $L_2$ by swapping the colours $c_3$ and $c_4$.

**Lemma 5** Let $C = v_1,v_2,\ldots,v_l$ be a cycle of length $l$ and $p,q$ positive integers such that $1 \leq p < q < l$. Let $L$ be an assignment of list of colours to the vertices of $C$ such that the vertex $v_m$ is of type $T_1$ for all $m \in [1,p]$, of type $T_3$ for $m \in (p,q]$ and of type $T_2$ for $m \in (q,l]$. Suppose $L_2 \in L_{13}$ and $L_3 \in L_{32}$ are labels assigned
to the edges $v_pv_{p+1}$ and $v_qv_{q+1}$, respectively, such that $L_2 \subseteq L(v_p) \times L(v_{p+1})$ and $L_3 \subseteq L(v_q) \times L(v_{q+1})$. Then for any near-triangulation $G$ with boundary $C$ that is consistent with the list assignment $L$, there exists a label $L_1 \in \mathcal{L}_{12}$ and $L_1 \subseteq L(v_1) \times L(v_1)$ that can be assigned to the edge $v_1v_1$, such that for any ordered pair $(s_1,t_1) \in L_1$, there exist ordered pairs $(s_2,t_2) \in L_2$, $(s_3,t_3) \in L_3$ and an $L$-colouring $f$ of $C$ that can be extended to a $k$-colouring of $G$ with $f(v_1) = s_1$, $f(v_i) = t_1$, $f(v_p) = s_2$, $f(v_{p+1}) = t_2$, $f(v_q) = s_3$ and $f(v_{q+1}) = t_3$.

**Proof:** The proof is again by induction, and we consider a counterexample $G$ with the minimum number of edges. We call the triple of labels $(L_1, L_2, L_3)$ satisfying the properties in the lemma a feasible triple, and suppose $G$ does not have a feasible triple for some list assignment $L$ to the vertices of $C$ and labels $L_2, L_3$.

**Case 1.** Suppose $G$ has a chord $v_iv_j$ for some $1 \leq i < j \leq l$. Again, let $G_1$ be the near-triangulation induced by the vertices on or in the interior of the cycle $C_1 = v_1, \ldots, v_i, v_j, \ldots, v_l$, and $G_2$ the near-triangulation induced by the vertices on or in the interior of the cycle $C_2 = v_i, v_{i+1}, \ldots, v_j$. Let $f_1$ be any $L$-colouring of $C_1$. The list assignment $L$ restricted to the vertices of $C_2$ satisfies the conditions of Lemma 3, perhaps after renaming colours so that vertices $v_m$ have type $T_1$ for $m \in [i,j]$. Lemma 3 then implies there exists an $L$-colouring $f_2$ of $C_2$ that can be extended to a $k$-colouring of $G_2$ with $f_2(v_i) = f_1(v_i)$ and $f_2(v_j) = f_1(v_j)$. Since this holds for any $L$-colouring of $C_1$, $(L_1, L_2, L_3)$ is a feasible triple for $G$.

**Case 1.2** Suppose $v_i$ is of type $T_1$ and $v_j$ of type $T_2$, which implies $1 \leq i \leq p$ and $q < j \leq l$. The list assignment $L$ restricted to the vertices of $C_2$, along with the lists $L_2, L_3$, satisfies the induction hypothesis, and by induction there exists a label $L'_1 \in \mathcal{L}_{12}$ that can be assigned to the edge $v_iv_j$ such that $(L'_1, L_2, L_3)$ is a feasible triple for $G_2$. The list assignment $L$ restricted to the vertices of $C_1$, along with the label $L'_1$ assigned to the edge $v_iv_j$, satisfies the conditions of Lemma 4. Applying Lemma 4 to $G_1$, there exists a label $L_1$ that can be assigned to the edge $v_1v_1$ such that $(L_1, L'_1)$ is a feasible pair of labels for $G_1$. This implies $(L_1, L_2, L_3)$ is a feasible triple for $G$.

**Case 1.3** Suppose $v_i$ is of type $T_1$ and $v_j$ of type $T_3$. Then the list assignment $L$ restricted to the cycle $C_1$ satisfies the induction hypothesis but the edge $v_pv_{p+1}$ is not in $C_1$. We choose an appropriate label $L'_2 \in \mathcal{L}_{13}$ for the edge $v_iv_j$ and the label $L_3$ for the edge $v_qv_{q+1}$ in $C_1$. Applying induction gives a feasible triple $(L_1, L_2, L_3)$ for $G_1$. We choose $L'_2$ so that for any ordered pair $(s'_1, t'_1) \in L'_2$, there exists an ordered pair $(s_2, t_2) \in L_2$ and an $L$-colouring $f_2$ of $C_2$ that can be extended to a $k$-colouring of $G_2$ with $f_2(v_i) = s'_1$, $f_2(v_j) = t'_1$, $f_2(v_p) = s_2$ and $f_2(v_{p+1}) = t_2$. Then $(L_1, L_2, L_3)$ is a feasible triple for $G$.

The label $L'_2$ for the edge $v_iv_j$ is found by applying Lemma 4 to the near-triangulation $G_2$. We first swap the colours $c_2$ and $c_3$ in all the lists and the ordered
pairs in the label so that vertices $v_m$ of type $T_3$ for $m \in (p, j]$ are converted to type $T_2$. This swap does not affect the lists for the vertices $v_m$ of type $T_1$ for $m \in [i, p]$. After swapping colours $c_2$ and $c_3$, the labels in $L_{13}$ are closed under swapping colours $c_3$ and $c_4$. Retaining only one of the pairs that can be obtained by such a swap converts the label $L_2$ into a label $L'_2$ that is in $L_{12}$. In other words, $L'_2$ is the label in $L_{12}$ corresponding to the label $L_2 \in L_{13}$. Lemma 4 implies there exists a label $L''_2 \in L_{12}$ that can be assigned to the edge $v_i v_j$ such that $(L''_2, L'_2)$ is a feasible pair for $G_2$. Let $L'_2$ be the label in $L_{13}$ corresponding to the label $L''_2$. It follows from Lemma 4, that $(L'_2, L_2)$ is a feasible pair of labels for $G_2$.

The argument in the case $v_i$ is of type $T_3$ and $v_j$ is of type $T_2$ is symmetric, and the above argument can be applied after reversing the cycle and swapping the colours $c_1$ and $c_2$ in the lists and the labels.

**Case 2.** We may now assume $G$ has no chords. If $G$ has a separating triangle, exactly the same argument as in Lemmas 3 and 4 can be used. So we may assume $G$ has no chords or separating triangles.

**Case 2.1** Suppose $l = 3$, which implies $p = 1$ and $q = 2$. In this case, it is sufficient to show that for any labels $L_2, L_3$ assigned to the edges $v_1 v_2$ and $v_2 v_3$, respectively, there exists a label $L_1$ that can be assigned to $v_1 v_3$ such that for every pair $(s, t) \in L_1$, there is a colour $r \notin \{s, t\}$ that can be assigned to $v_2$ such that $(s, r) \in L_2$ and $(r, t) \in L_3$. Lemma 3 implies that any such $L$-colouring of $C$ can be extended to a $k$-colouring of $G$, hence $(L_1, L_2, L_3)$ is a feasible triple for any triangulation $G$.

Suppose the label $L_2$ is $\{(c, a)\}$ for some $a \in S_1$ and $c \in S_3$. Then $L_3$ is either $\{(c, b)\} \in S_2$ or $\{(c, c_1), (c, c_4)\}$. In the first case, the $L$-colouring is given by $f(v_1) = a$, $f(v_2) = c$ and $f(v_3) = b$, hence $L_1 = \{(a, b)\}$. In the second case, set $f(v_3) = c_4$ and hence $L_1 = \{(a, c_4)\}$ satisfies the requirements.

Suppose $L_2$ is the label $\{(a, c_2), (a, c_4)\}$ for some $a \in S_1$. If $L_3 = \{(c_1, b), (c_1, b)\}$, for some $b \in S_2$, let $f$ be the $L$-colouring of $C$ with $f(v_1) = a$, $f(v_2) = c_4$ and $f(v_3) = b$. In this case, the label $L_1$ is $\{(a, b)\}$. Otherwise if $(c_1, c_3), (c_1, c_4) \in L_3$, set $f(v_3) = c_3$ to get the required label $L_1 = \{(a, c_3)\}$. The remaining possibility is that $L_3 = \{(c_2, c_1), (c_2, c_4), (c_1, c_4), (c_4, c_1)\}$. In this case, choose $f(v_2) = c_2$ and $f(v_3) = c_4$, to get the required label $L_1 = \{(a, c_4)\}$.

We may now assume $|L(v_1)| = 3$ and by symmetry, $|L(v_3)| = 3$. Suppose $|L(v_2)| = 1$, which implies $L(v_2) = \{c\}$ for some $c \in S_3$. In this case $L_2 = \{(c_2, c), (c_4, c)\}$ and $L_3 = \{(c, c_1), (c, c_4)\}$. Then setting $f(v_1) = c_2$, $f(v_2) = c$ and $f(v_3) = c_4$ gives an $L$-colouring of $C$. Similarly, $f(v_1) = c_4$, $f(v_2) = c$ and $f(v_3) = c_1$ is an $L$-colouring of $C$. This implies $L_1 = \{(c_2, c_4), (c_4, c_1)\}$ gives a feasible triple for $G$.

Finally, suppose all three vertices have lists of size 3. Suppose $(c_2, c_1), (c_4, c_1) \in L_2$. If $(c_1, c_3), (c_4, c_3) \in L_3$, let $f(v_2) = c_1$, $f(v_3) = c_3$ and $f(v_1)$ can be either $c_2$ or $c_4$. These give $L$-colourings of $C$, hence setting $L_1 = \{(c_2, c_3), (c_4, c_3)\}$ gives a feasible triple for $G$. If $(c_1, c_3) \notin L_3$ then $L_3 = \{(c_2, c_1), (c_2, c_4), (c_1, c_4), (c_4, c_1)\}$. Applying a symmetrical argument by reversing the cycle and swapping the colours $c_1, c_2$, we must have $L_2 = \{(c_2, c_1), (c_4, c_1), (c_2, c_4), (c_4, c_2)\}$. In this case, setting $f(v_1) = c_2$, $f(v_2) = c_1$ and $f(v_3) = c_4$ gives an $L$-colouring of $C$, as does setting $f(v_1) = c_4$, as does setting $f(v_1) = c_4$,
\(f(v_2) = c_2\) and \(f(v_3) = c_1\). This implies \(L_1 = \{(c_2, c_4), (c_4, c_1)\}\) gives a feasible triple for \(G\). The final case to consider is if \((c_2, c_1), (c_4, c_1) \notin L_2\) and by a symmetrical argument \((c_2, c_1), (c_2, c_4) \notin L_3\). This implies \(L_2 = \{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\}\) and \(L_3 = \{(c_1, c_3), (c_4, c_3), (c_1, c_4), (c_4, c_1)\}\). Therefore \(f(v_1) = c_3, f(v_2) = c_4\) and \(f(v_3) = c_1\) is an \(L\)-colouring of \(C\) and so is \(f(v_1) = c_2, f(v_2) = c_4\) and \(f(v_3) = c_3\). This implies \(L_1 = \{(c_2, c_3), (c_3, c_1)\}\) gives a feasible triple for \(G\). This completes all cases when \(l = 3\).

**Case 2.2** We may now assume \(l > 3\) and \(G\) is a chordless near-triangulation with no separating triangles. Therefore for every edge \(v_iv_{i+1}\) in \(C\) there exists an internal vertex \(v\) in \(G\) such that \(vv_{i+1}\) is an internal face in \(G\). As in the proof of Lemma 4, we may assume that \(L(v_i) \cap L(v_{i+1}) \neq \emptyset\) for any \(i \notin \{p, q\}\), otherwise we can get the required feasible triple of labels by deleting such an edge and applying induction.

**Case 2.2.1** Suppose there exists a vertex \(v_i\) with \(|L(v_i)| = 1\). Suppose \(i = 1\), which implies \(L(v_1) = \{a\}\) for some \(a \in S_1\). Then we must have \(p = 1\) otherwise \(v_2\) is also a vertex of type \(T_1\) and \(L(v_1) \cap L(v_2) = \emptyset\). This implies the label \(L_2\) assigned to the edge \(v_1v_2\) is either \(\{(a, c)\}\) for some \(c \in S_3\) or \(\{(a, c_2), (a, c_4)\}\). Let \(v\) be the vertex such that \(vv_1v_2\) is an internal face in \(G\). Let \(G_1\) be the near-triangulation obtained by deleting the edge \(v_1v_2\) having \(C_1 = v_1, v, v_2, \ldots, v_i\) as the boundary. Let \(L(v) = \{c_2, c_3, c_4\}\) and assign the label \(L_2\) to the edge \(vv_2\), where \(L_2 = \{(c_2, c), (c_4, c)\}\) if \(L_2 = \{(a, c)\}\) and \(L_2 = \{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\}\) otherwise. The list assignment to the vertices of \(C_1\) satisfies the induction hypothesis, and by induction, there exists a label \(L_1\) that can be assigned to \(v_1v_i\) such that \((L_1, L_2, L_3)\) is a feasible triple for \(G_1\). Since any \(L\)-colouring of \(C_1\) is also an \(L\)-colouring of \(C\), \((L_1, L_2, L_3)\) is a feasible triple for \(G\). A symmetrical argument holds if \(L(v_i) = \{b\}\) for some \(b \in S_2\).

Suppose \(i \in (1, l]\). If \(v_i\) is of type \(T_1\) then \(L(v_i) \cap L(v_{i-1}) = \emptyset\), and if it is of type \(T_2, L(v_i) \cap L(v_{i+1}) = \emptyset\), a contradiction. If it is of type \(T_3\) and \(v_{i-1}\) is also of type \(T_3\), the same argument holds. The only possibility is that \(i = p + 1\), \(L(v_{p+1}) = \{c_3\}\) and \(L(v_p) = \{c_2, c_3, c_4\}\). The label \(L_2\) assigned to the edge \(v_pv_{p+1}\) must be \(\{(c_2, c_3), (c_4, c_3)\}\). Let \(v\) be the vertex such that \(vv_pv_{p+1}\) is an internal face in \(G\). Let \(G_1\) be the near-triangulation obtained by deleting the edge \(v_pv_{p+1}\) with the cycle \(C_1 = v_1, \ldots, v, v_p, v_{p+1}, \ldots, v_i\) as the boundary. Let \(L(v) = \{c_1, c_2, c_4\}\) and assign the label \(L_2' = \{(c_2, c_1), (c_4, c_1), (c_2, c_4), (c_4, c_2)\}\) to the edge \(v_pv\). The list assignment to the vertices of \(C_1\), along with the label \(L_2'\), satisfies the induction hypothesis, and by induction there exists a feasible triple \((L_1, L_2, L_3)\) for \(G_1\). In any \(L\)-colouring of \(C_1\), the label \(L_2'\) ensures that \(v_p\) is coloured \(c_2\) or \(c_4\), hence it is also an \(L\)-colouring of \(C\). Therefore \((L_1, L_2, L_3)\) is a feasible triple for \(G\).

**Case 2.2.2** We may now assume \(|L(v_i)| = 3\) for all \(i \in [1, l]\).

**Case 2.2.2.1** Suppose \(p > 1\) and \(q > p + 1\). Suppose the label \(L_2\) contains the pairs \((c_2, c_1)\) and \((c_4, c_1)\). Let \(v_{p-1} = u_1, \ldots, u_r = v_{p+1}\) be the neighbours of \(v_p\). Consider the near-triangulation \(G_1\) obtained by deleting \(v_p\) having the cycle \(v_1, \ldots, v_{p-1}, u_2, \ldots, u_{r-1}, v_{p+1}, \ldots, v_i\) as the boundary. Remove the elements \(c_2, c_4\) from \(L(v_{p+1})\) so that \(L(v_{p+1}) = \{c_1\}\), which converts \(v_{p+1}\) to a vertex of type \(T_1\). Let \(L(u_i) = \{c_1, c_3\}\) \(\setminus L(u_{i+1})\) for \(2 \leq i < r\). Assign the label \(L_2' = \{(c_1, c_2), (c_1, c_4)\}\) to the edge \(v_{p+1}v_{p+2}\) and keep the label \(L_3\) for the edge \(v_qv_{q+1}\). The resulting list
assignment to the vertices of \( C_1 \) satisfies the induction hypothesis, and by induction, there exists a feasible triple \((L_1, L_2^2, L_3)\) for \( G_1 \). Since any \( L \)-colouring of \( C_1 \) can be extended to an \( L \)-colouring of \( C \) by assigning \( v_p \) a colour in \( \{c_2, c_1\} \) that is not assigned to \( v_{p-1} \), and since \((c_2, c_1), (c_4, c_1)\) \( \in L_2 \), \((L_1, L_2, L_3)\) is a feasible triple for \( G \).

If \((c_2, c_1) \notin L_2 \) then both \((c_3, c_2)\) and \((c_3, c_4)\) are in \( L_2 \). Now let \( v_p = u_1, \ldots, u_r = v_{p+2} \) be the neighbours of \( v_{p+1} \). Let \( G_1 \) be the near-triangulation obtained by deleting the vertex \( v_{p+1} \) with the cycle \( C_1 = v_1, \ldots, v_p, u_2, \ldots, u_{r-1}, v_{p+1}, \ldots, v_l \) as the boundary. Remove the elements \( c_2, c_4 \) from \( L v_p \) so that \( L(v_p) = \{c_3\} \), which converts \( v_p \) to a vertex of type \( T_3 \). Let \( L(u_i) = \{c_3, c_5\} \setminus L(u_{i-1}) \) for \( 2 \leq i < r \). Assign the label \( L_2 = \{(c_2, c_3), (c_4, c_3)\} \) to the edge \( v_{p-1} v_p \) and retain the label \( L_3 \) for the edge \( v_{p+1} v_{p+2} \) in \( C_1 \). The list assignment to the vertices of \( C_1 \) satisfies the induction hypothesis and by induction, there exists a feasible triple \((L_1, L_2^2, L_3)\) for \( G_1 \). Any \( L \)-colouring of \( C_1 \) can be extended to an \( L \)-colouring of \( C \) by assigning \( v_{p+1} \) a colour in \( \{c_2, c_4\} \) that is not assigned to \( v_{p+2} \). This gives an \( L \)-colouring of \( C \) in which \( v_p \) is coloured \( c_3 \) and \( v_{p+1} \) is coloured \( c_2 \) or \( c_4 \), hence \((L_1, L_2, L_3)\) is a feasible triple for \( G \).

Case 2.2.2.2 Suppose \( p > 1 \) and \( q = p + 1 \). The argument is similar to that in Case 2.2.2.1. Suppose \((c_2, c_1) \in L_2 \), and again consider the near-triangulation obtained by deleting the vertex \( v_p \). If \((c_1, c_3) \in L_3 \), assign \( L(v_{p+1}) = \{c_1\} \) and \( L(u_i) = \{c_1, c_3\} \setminus L(u_{i-1}) \) for \( 2 \leq i < r \). There is no vertex of type \( T_3 \) in \( C_1 \) now. Instead of the label \( L_3 \) assigned to the edge \( v_{p+1} v_{p+2} \), assign the corresponding label \( L'_3 = \{(c_1, c_3)\} \). Applying Lemma 4, there exists a feasible pair \((L_1, L'_2)\) for \( G_1 \). Assigning \( v_p \) a colour in \( \{c_2, c_4\} \) that is not assigned to \( v_{p-1} \) extends any \( L \)-colouring of \( C_1 \) to an \( L \)-colouring of \( C \). Therefore \((L_1, L_2, L_3)\) is a feasible triple for \( G \). If \((c_1, c_3) \notin L_3 \), then \( L_3 = \{(c_2, c_3), (c_2, c_4), (c_1, c_4), (c_4, c_1)\} \) and the same argument holds by choosing \( L'_3 = \{(c_1, c_4)\} \). The remaining possibility is \((c_2, c_1) \notin L_2 \), and by symmetry, \((c_2, c_1) \notin L_3 \) which implies \( L_2 = \{(c_3, c_2), (c_3, c_4), (c_2, c_4), (c_4, c_2)\} \) and \( L_3 = \{(c_1, c_3), (c_4, c_3), (c_1, c_4), (c_4, c_1)\} \). In this case, swap the colours \( c_3 \) and \( c_4 \) in
all the lists and labels and assign list \( \{c_3\} \) to the vertex \( v_{p+1} \). The lists for the other vertices remain the same and the label \( L_2 \) changes to \( \{(c_2, c_3), (c_4, c_3)\} \) and \( L_3 \) to \( \{(c_3, c_1), (c_3, c_4)\} \). This reduces to the Case 2.2.1 considered previously.

Cases 2.2.2.1 and 2.2.2.2 cover all possibilities with \( p > 1 \). A symmetrical argument can be used if \( q < l - 1 \) by reversing the cycle and swapping the colours \( c_1 \) and \( c_2 \).

Case 2.2.2.3 The only case that remains now is when \( p = 1 \) and \( q = l - 1 \), which means \( v_1 \) is the only vertex of type \( T_1 \) and \( v_l \) is the only vertex of type \( T_2 \).

Suppose \( l \geq 5 \) and let \( v_2 = u_1, u_2, \ldots, v_r = v_4 \) be the neighbours of \( v_3 \) such that \( v_3 u_i u_{i+1} \) is an internal face in \( G \), for \( 1 \leq i < r \). Let \( G_1 \) be the near-triangulation obtained by deleting the vertex \( v_3 \) with the cycle \( C_1 = v_1, v_2, u_2, \ldots, u_{r-1}, v_4, \ldots, v_l \) as the boundary. Let \( L(u_i) = \{c_3\} \) if \( i \) is odd and \( L(u_i) = \{c_5\} \) if \( i \) is even for \( 2 \leq i < r \). The list assignment to the vertices of \( C_1 \), together with the labels \( L_2 \) and \( L_3 \) for the edges \( v_1 v_2 \) and \( v_{l-1} v_l \) respectively, satisfies the induction hypothesis, and by induction, there exists a feasible triple \((L_1, L_2, L_3)\) for \( G_1 \). Any \( L \)-colouring of \( C_1 \) can be extended to an \( L \)-colouring of \( C \) by assigning \( v_3 \) a colour in \( \{c_1, c_2, c_4\} \).
that is not assigned to \(v_2\) or \(v_4\). This implies \((L_1,L_2,L_3)\) is also a feasible triple for \(G\).

Suppose \(l = 4\), \((c_2,c_1),(c_4,c_1) \in L_2\) and \((c_1,c_3),(c_4,c_3) \in L_3\). Let \(v_4 = u_1,u_2,\ldots,u_r = v_2\) be the neighbours of \(v_1\) such that \(v_1u_iu_{i+1}\) is an internal face in \(G\) for \(1 \leq i < r\). Consider the near-triangulation \(G_1\) obtained by deleting the vertex \(v_1\) with the cycle \(C_1 = u_2,\ldots,v_2,v_3,v_4\) as the boundary. Assign the list 
\[
L(u_i) = \{c_1\} \text{ and } L(u_i) = \{c_1,c_3\} \setminus L(u_{i+1}) \text{ for } 2 \leq i < r
\]
Assign the list \(\{c_2,c_3,c_4\}\) to \(v_3\) and \(v_4\) so that all vertices in \(C_1\) are of type \(T_1\). By Lemma 3, there exists an \(L\)-colouring \(f\) of \(C_1\) that extends to a \(k\)-colouring of \(G_1\) with \(f(v_4) = c_3\), \(f(v_3) = c_4\) and \(f(v_2) = c_1\). This can be extended to an \(L\)-colouring of \(C\) by assigning the colour \(c_2\) or \(c_3\) to the vertex \(v_1\). Then with \(L_1 = \{(c_2,c_3),(c_4,c_3)\}, (L_1,L_2,L_3)\) is a feasible triple for \(G\). A symmetrical argument can be used if \(\{(c_3,c_2),(c_3,c_4)\} \in L_2\) and \(\{(c_2,c_1),(c_2,c_4)\} \in L_3\).

Suppose \((c_2,c_1),(c_4,c_1) \in L_2\) but \((c_1,c_3),(c_4,c_3) \not\in L_3\). This implies \(L_3 = \{(c_2,c_1),(c_2,c_4),(c_1,c_4),(c_4,c_1)\}\) and by the previous argument, \((c_3,c_2),(c_3,c_4) \not\in L_2\). This implies \(L_2 = \{(c_2,c_1),(c_4,c_1),(c_4,c_2),(c_2,c_4)\}\). In this case, the previous argument gives an \(L\)-colouring of \(C_1\) that extends to a \(k\)-colouring of \(G_1\) with \(f(v_4) = c_1\), \(f(v_3) = c_2\) and \(f(v_2) = c_1\). Assigning colour \(c_2\) to \(v_1\) gives an \(L\)-colouring \(f\) of \(C\). Again using symmetry after reversing the cycle and swapping colours \(c_1,c_2\), we get an \(L\)-colouring \(f\) of \(C\) that extends to a \(k\)-colouring of \(G\) with \(f(v_1) = c_4\), \(f(v_2) = c_4\), \(f(v_3) = c_2\) and \(f(v_4) = c_1\). Therefore setting \(L_1 = \{(c_2,c_4),(c_4,c_1)\}\) gives a feasible triple for \(G\).

The remaining case to be considered is if \((c_2,c_1),(c_4,c_1) \not\in L_2\), and by symmetry, \((c_2,c_1),(c_2,c_4) \not\in L_3\). Then \(L_2 = \{(c_3,c_2),(c_3,c_4),(c_2,c_4),(c_4,c_2)\}\) and \(L_3 = \{(c_1,c_3),(c_4,c_3),(c_1,c_4),(c_4,c_1)\}\). Consider the triangulation \(G_1\), and assign the lists \(L(v_4) = \{c_1\}\), \(L(u_i) = \{c_1,c_3\} \setminus L(u_{i-1})\) for \(2 \leq i < r\), and \(L(v_2) = L(v_3) = \{c_2,c_3,c_4\}\). Then all vertices are of type \(T_1\) and there exists an \(L\)-colouring \(f\) of \(C_1\) that extends to a \(k\)-colouring of \(G_1\) with \(f(v_4) = c_1\), \(f(v_2) = c_2\) and \(f(v_3) = c_4\). Assigning \(f(v_1) = c_3\) gives an \(L\)-colouring \(f\) of \(C\) that extends to a \(k\)-colouring of \(G\). A symmetrical argument, after reversing the cycle and swapping the colours \(c_1,c_2\) gives an \(L\)-colouring \(f\) of \(C\) that extends to a \(k\)-colouring of \(G\) with \(f(v_1) = c_2\), \(f(v_2) = c_4\), \(f(v_3) = c_1\) and \(f(v_4) = c_3\). This implies setting \(L_1 = \{(c_3,c_1),(c_2,c_3)\}\) gives a feasible triple of labels \((L_1,L_2,L_3)\). This completes all cases when \(l = 4\) and hence the proof of Lemma 5.

**Proof of Theorem 1:** The proof follows from Lemmas 3, 4 and 5. Let \(f\) be a good \(k\)-colouring of a cycle \(C\) of length \(l\). Suppose \(f\) satisfies condition 1 in the definition of a good colouring. By permuting the colours, we can assume that \(F[1,l] \subseteq S_1\). Then assigning the list \(L(v_i) = \{f(v_i)\}\) for all \(1 \leq i \leq l\) gives a list assignment to the vertices of \(C\) that satisfies the conditions of Lemma 3. Since any chordless near-triangulation \(G\) with boundary \(C\) is consistent with the list assignment to \(C\), Lemma 3 implies the colouring of \(C\) can be extended to a \(k\)-colouring of \(G\).

Suppose \(f\) satisfies condition 2 in the definition of a good colouring, with indices \(p\) and \(q\). By relabeling the vertices and permuting the colours, we may assume that \(p = 1\), \(F[1,q] \subseteq S_1\), \(F[q,1] \subseteq S_2\), and the set \(A = \{c_j | 5 \leq j \leq k\}\). Then the list assignment \(L(v_i) = \{f(v_i)\}\) for \(1 \leq i \leq l\), with the label \(L_2 = L(v_{q-1}) \times L(v_q)\)
assigned to the edge $v_{q-1}v_q$, satisfies the conditions of Lemma 4 and the result follows.

Suppose $f$ satisfies condition 3 in the definition of a good colouring, with indices $p,q,r$. Again by relabeling vertices and permuting colours, we may assume that $p = 1$, $F[1,q] \subseteq S_1$, $F[q,r] \subseteq S_2$ and $F[r,1] \subseteq S_2$, where again $A = \{c_j | 5 \leq j \leq k\}$. The list assignment $L(v_i) = \{f(v_i)\}$ satisfies the conditions of Lemma 5 with the labels $L_2 = L(v_{q-1}) \times L(v_q)$ and $L_3 = L(v_{r-1}) \times L(v_r)$ assigned to the edges $v_{q-1}v_q$ and $v_{r-1}v_r$, respectively. The result follows from Lemma 5.

This completes the proof of Theorem 1.

\[\square\]

**Corollary 1** Any $k$-colouring $f$ of a cycle of length $l \leq 2k - 5$ such that $|F[1,l]| \leq k - 1$ is safe. There exists a $k$-colouring of a cycle of length $2k - 4$ that is not safe and uses $k - 1$ distinct colours.

**Proof:** If $|F[1,l]| \leq k - 3$ then $f$ is good and the result follows from Theorem 1. If $|F[1,l]| = k - 2$, we may assume $F[1,l] = \{c_1, \ldots, c_{k-2}\}$. Since $l \leq 2k - 5$, at least one colour occurs only once in the cycle. Without loss of generality, by relabeling vertices and permuting colours, we may assume $v_1$ is the only vertex with colour $c_{k-2}$. Then $f$ satisfies condition 2 in the definition of a good $k$-colouring, with $p = 1$, $q = l$ and $A = \{c_1, \ldots, c_{k-4}\}$. If $|F[1,l]| = k - 1$, there are at least three distinct colours that occur exactly once in the colouring of the cycle. Without loss of generality, we may assume these colours are $c_{k-3}, c_{k-2}, c_{k-1}$ and the colour $c_k$ does not occur. Let $1 < p' < q' \leq l$ be indices such that $f(v_1) = c_{k-3}, f(v_{p'}) = c_{k-2}$ and $f(v_{q'}) = c_{k-1}$. Then $f$ satisfies condition 3 in the definition of a good colouring with $p = 1, q = p', r = q'$ and $A = \{c_1, \ldots, c_{k-4}\}$. Theorem 1 implies $f$ is safe.

An example of a $k$-colouring $f$ of a cycle of length $2k - 4$ that is not safe is given by $f(v_1) = f(v_{k-2+i}) = c_i$ for $1 \leq i \leq k - 3$, $f(v_{k-2}) = c_{k-2}$ and $f(v_{2k-4}) = c_{k-1}$. Note that this colouring satisfies condition 2 in Lemma 1, and is bad. \[\square\]

If Conjecture 1 is true, then every $k$-colouring of a cycle of length at most $3k - 10$ that uses at most $k - 2$ distinct colours is safe. However, Theorem 1 does not prove this since the $k$-colouring of a cycle of length $2k - 4$ defined by $f(v_i) = f(v_{i+k-2}) = c_i$ for $1 \leq i \leq k - 2$ is not good.

### 3 Remarks

In principle, it may be possible to extend this approach to prove the four colour theorem itself. However, it is not sufficient to use a set of 4-colourings of a cycle defined by list-colourings. A more general way of defining sets of colourings is required. A set $F$ of 4-colourings of a cycle $C$ of length $l$ is safe, if for every chordless near-triangulation $G$ with $C$ as the boundary, there exists a 4-colouring $f \in F$ of $C$ that can be extended to a 4-colouring of $G$. While it would be nice to characterize exactly the safe sets of 4-colorings, proving the four colour theorem only requires showing that any proper colouring of a triangle is safe. It may be possible to do this by finding a simpler sufficient condition that ensures safety of a set of colourings.
References


