Red-Blue cliques in edge-coloured graphs

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Let $r, b \geq 1$ be integers. It is conjectured that if R(red) and B(blue) are graphs on the same vertex set of size n such that $(r-1)||R|| + b||B|| > \frac{r+b-2}{2}n^2$ then there is a clique K of order r in R and a clique L of order b in B disjoint from K, such that for all vertices $u \in K$, $v \in L$, uv is an edge in B. This is a generalization of Turan's theorem. We prove this for $r = 4$, $b = 2$.

To simplify the discussion, we will restate the theorem for $r = 4$, $b = 2$ in a different form. Let $G = R \cup B$ be the edge-coloured multigraph of order n. We consider G to be a simple edge-coloured graph whose edges are coloured red, blue or double (red and blue). Let $r(G)$, $b(G)$ and $d(G)$ denote the number of edges in G that are coloured red, blue and double respectively. Let $w(G) = 3r(G) + 2b(G) + 5d(G)$ be the weight of G. For disjoint nonempty subsets of vertices X, Y, let $w(X,Y) = 3r(X,Y) + 2b(X,Y) + 5d(X,Y)$, where $r(X,Y), b(X,Y)$ and $d(X,Y)$ are the number of red, blue and double edges respectively, with an endpoint in X and Y. For any vertex v, let $w(v) = w(\{v\}, V(G) \setminus \{v\})$. We will not distinguish between a graph and its vertex set when the meaning is clear from the context. Similarly, we will denote a set containing a single element by the element itself. If H is an induced subgraph of G and v a vertex in $G - H$, we denote by $H + v$ the subgraph of G induced by $V(H) \cup \{v\}$.

Let X and Y be disjoint subsets of vertices of G. We say that the ordered pair (X, Y) is a (4,2)-clique in G if

1. $|X| = 4$ and $|Y| = 2$.
2. $G[X \cup Y]$ is a 6-clique.
3. Every red edge in $G[X \cup Y]$ has both endpoints in X.
4. Every blue edge in $G[X \cup Y]$ has at least one endpoint in Y.

Theorem 1 *Any edge-coloured graph G of order n with $w(G) > 2n^2$ contains a (4,2)-clique.*

Proof: Since $w(G) \leq 5n(n-1)/2$, the statement is vacuously true for $n \leq 5$. Let $n > 5$ be any fixed integer and assume the theorem is true for all integers m, $1 \leq m < n$. Suppose there exists a counterexample G of order n.

We will show that G contains certain induced coloured subgraphs, which will finally imply that G must contain a (4,2)-clique. The subgraphs we consider are cliques in G in which all edges are either red or double, which we call *red cliques*. If H is a red clique in G, we denote by H_R the spanning subgraph of H containing all red edges in H. The red degree of a vertex in a red clique H is its degree in H_R. Figures 1, 2, 3, 4, 5, 6 show all red cliques that we consider. Only the red edges in the red cliques are shown. Note that if G contains any one of these cliques H_i^j, then H_i^j is a proper induced subgraph of G, since $w(H_i^j) \le 2|H_i^j|^2$, but $w(G) > 2|G|^2$.

We next prove a few simple lemmas that are the main tools in all subsequent proofs.

Lemma 2 *For every nonempty proper subset of vertices X, $w(X, G-X) > 4n|X| - 2|X|^2 - w(X)$. There exists a vertex v in $G - X$ such that $w(v, X) > 4|X|$. Also, $w(v) > 4n - 2$ for every vertex v.*

Proof: Since G is a counterexample of minimum order, for every nonempty proper subset of vertices X, $w(X) \le 2|X|^2$, otherwise $G[X]$ is a smaller counterexample. The inequality follows from the fact that $w(G) > 2n^2$. Taking X to be the set $\{v\}$, we get $w(v) > 4n - 2$. $\square$

Lemma 3 *Let H be a red clique in G and $X \subset V(H)$ such that $G[X]$ is a clique with all edges red. Let $|X| = d$, $1 \le d \le (|H| - 2)/2$ and let $r \ge 1$ be an integer. Suppose that $w(H) \le 2|H|^2 - r(|H| - 2d - 2)$ and for every vertex u in $G - H$, $w(u, H) \le 4|H| + r$. Then there is a vertex v in $G - H$ such that $w(v, H) \ge 4|H| - r + 1$ and $w(v, X) > 3d$.*

Proof: By Lemma 2, $w(H, G - H) > 4|H|(n - |H|) + r(|H| - 2d - 2)$. Let $A = \{v \in G - H \mid w(v, H) \ge 4|H| - r + 1\}$ and let $|A| = k$. Then

$$w(H, G-H) \le (4|H| + r)k + (4|H| - r)(n - k - |H|) = 4|H|(n - |H|) + r(2k - n + |H|)$$

which implies $k > (n - 2d - 2)/2$. If for every vertex $v \in A$, $w(v, X) \le 3d$, then

$$\sum_{x \in X} w(x) \ \le 3kd + 5d(n - k - d) + 3d(d - 1) = d(5n - 2k - 2d - 3).$$

By Lemma 2,

$$\sum_{x \in X} w(x) \ \ge (4n - 1)d$$

which implies $k \le (n - 2d - 2)/2$, a contradiction. $\square$

Lemma 4 *Let H be a red clique in G and $X \subset V(H)$ such that $G[X]$ is a clique with all edges red. Let $|X| = d$, $1 \le d \le (|H| - 2)/2$ and let $1 \le s \le r$ be integers. Suppose that $s \le (|H| - 2d - 2)/2$, $w(H) \le 2|H|^2 - r(|H| - 2d - 2) + s$ and for every vertex u in $G - H$, $w(u, H) \le 4|H| + r$. Then there is a vertex v in $G - H$ such that either $w(v, H) \ge 4|H| - r + 1$ and $w(v, X) > 3d$, or $w(v, H) = 4|H| + r$ and $w(v, X) = 3d$.*

Proof: The proof is similar to that of Lemma 3. Suppose there is no vertex v in $G - H$ such that $w(v, H) \geq 4|H| - r + 1$ and $w(v, X) > 3d$. By Lemma 2, $w(H, G - H) > 4|H|(n - |H|) + r(|H| - 2d - 2) - s$. Let $A = \{v \in G - H \mid w(v, H) \geq 4|H| - r + 1\}$ and let $|A| = k$. Then

$$w(H, G - H) \leq (4|H| + r)k + (4|H| - r)(n - k - |H|) = 4|H|(n - |H|) + r(2k - n + |H|)$$

which implies $2kr > r(n - 2d - 2) - s$. Since $s \leq r$, this implies $k \geq (n - 2d - 2)/2$. Also,

$$\sum_{x \in X} w(x) \ \leq 3kd + 5d(n - k - d) + 3d(d - 1) = d(5n - 2k - 2d - 3).$$

By Lemma 2,

$$\sum_{x \in X} w(x) \ \geq (4n - 1)d$$

which implies $k \leq (n-2d-2)/2$. Hence $k = (n-2d-2)/2$, $w(v, X) = 3d$ for every vertex $v \in A$, and $4|H|(n-|H|)+r(|H|-2d-2)-s < w(H, G-H) \leq 4|H|(n-|H|)+r(|H|-2d-2)$. Therefore there are less than s vertices $v \in A$ for which $w(v, H) < 4|H| + r$. Since $s \leq (|H| - 2d - 2)/2$, there is at least one vertex $v \in A$ such that $w(v, H) = 4|H| + r$ and $w(v, X) = 3d$. $\square$

Let H be a red clique in G. For a vertex $v \in G - H$, let $D(v, H)$, $R(v, H)$, $B(v, H)$ denote the subset of vertices of H that are joined to v in G by double, red and blue edges respectively. We say that a vertex $x \in X \subset V(H)$ is *deficient* in X if there are at least 4 double edges joining x to vertices in X. A pair of vertices $\{x, y\} \subset V(H)$ is said to be a *deficient pair* of vertices if the edge xy is double and there are at least 3 vertices in $V(H) \setminus \{x, y\}$ that are joined to both x and y by double edges. A subset of at least 2 vertices in H is deficient if every pair of vertices in the subset is a deficient pair in H.

Lemma 5 *If H is a red clique in G and v a vertex in $G-H$, then no vertex in $B(v, H) \cup D(v, H)$ is deficient in $B(v, H) \cup D(v, H)$. If $V(H) = R(v, H) \cup D(v, H)$ and $\{x, y\}$ is a deficient pair of vertices in H then $R(v, H) \cap \{x, y\}$ is nonempty.*

Proof: If $x \in B(v, H) \cup D(v, H)$ is a deficient vertex in $B(v, H) \cup D(v, H)$ then there is a subset A of 4 vertices in $B(v, H) \cup D(v, H)$ such that $(A, \{x, v\})$ is a (4,2)-clique in G. Similarly, if $V(H) = R(v, H) \cup D(v, H)$ and $\{x, y\}$ is a deficient pair of vertices in H such that $\{x, y\} \cap R(v, H) = \emptyset$, then there is a subset B of 3 vertices in H such that $(B \cup \{v\}, \{x, y\})$ is a (4,2)-clique in G. $\square$

Lemma 6 *G does not contain a 6-clique K with at least 13 double edges.*

Proof: Suppose there is such a clique K in G. We may assume K has two single edges, that is, edges that are not double. We show that it is always possible to choose a subset $X \subset V(K)$ such that $(X, V(K) \setminus X)$ is a (4,2)-clique in G. If both single edges are red, choose X to be a subset of 4 vertices including all endpoints of the two red edges. If one edge is red and the other blue, choose X to be a subset of 4 vertices including both endpoints of the red edge and excluding at least one endpoint of the blue edge. If both the edges are blue, let X be a subset of 4 vertices excluding at least one endpoint of each of the two blue edges. This contradicts the assumption that G is a counterexample. $\square$

Claim 7 *G contains H_3^0.*

Proof: Since $2n^2 < w(G) = 2b(G) + 3r(G) + 5d(G) \leq 3(r(G) + b(G) + d(G)) + 2d(G)$ and $r(G) + b(G) + d(G) \leq n(n-1)/2$, we have $d(G) > n^2/4$ and hence G contains a 3-clique H with all edges double. □

Claim 8 *G contains either H_4^0 or H_4^1.*

Proof: By Claim 7, G contains H_3^0. Let H be a 3-clique with all edges double. By Lemma 2, there exists a vertex v_4 in $G - H$ such that $w(v_4, H) > 12$. This implies that either $|D(v_4, H)| = 3$ or $|D(v_4, H)| = 2$ and $|R(v_4, H)| = 1$. Then $H + v_4$ is a red 4-clique with at most one red edge and is isomorphic to either H_4^0 or H_4^1. □

Claim 9 *G contains either H_5^0 or H_5^1.*

Proof: By Claim 8, G contains either H_4^0 or H_4^1. We consider the two cases separately.

Lemma 10 *If G contains H_4^0 then G contains H_5^0.*

Proof: Let H be a red 4-clique in G with all edges double. By Lemma 2, there is a vertex v_5 in $G - H$ such that $w(v_5, H) > 16$. If $|R(v_5, H)| = 1$, then $|D(v_5, H)| = 3$ and $H + v_5$ is a red 5-clique with one red edge, isomorphic to H_5^0. If either $|D(v_5, H)| = 4$ or $|B(v_5, H)| = 1$, then $H' = H + v_5$ is a 5-clique with at least 9 double edges and no red edge. By Lemma 2, there is a vertex v_6 in $G - H'$ such that $w(v_6, H') > 20$. If $|D(v_6, H')| \geq 4$, or if H' has all edges double, then $H' + v_6$ is a 6-clique with at least 13 double edges, contradicting Lemma 6. The only other possibility is that $|D(v_6, H')| = 3$, $|R(v_6, H')| = 2$ and H' has one blue edge. If the blue edge in H' does not have both endpoints in $R(v_6, H')$, then $H' + v_6$ contains a (4,2)-clique. Otherwise, deleting an endpoint of the blue edge gives a graph isomorphic to H_5^0. □

Lemma 11 *If G contains H_4^1 then G contains either H_5^0 or H_5^1.*

Proof: Let H be an induced subgraph of G isomorphic to H_4^1. Let v_1v_2 be the red edge in H, and let v_3, v_4 be the isolated vertices in H_R. By Lemma 2, there is a vertex v_5 in $G - H$ such that $w(v_5, H) > 16$. This implies $|D(v_5, H)| \geq 3$. If $\{v_3, v_4\} \subset D(v_5, H)$ then G contains H_4^0 and by Lemma 10, G contains H_5^0. We may assume v_3v_5 is not a double edge. If v_3v_5 is a red edge then $H + v_5$ is the required red 5-clique isomorphic to H_5^1.

Suppose v_3v_5 is a blue edge and let $H' = H + v_5$. By Lemma 2, there is a vertex v_6 in $G - H'$ such that $w(v_6, H') > 20$. If $|B(v_6, H') \cup D(v_6, H')| = 5$ then $(\{v_1, v_2, v_3, v_4\}, \{v_5, v_6\})$ is a (4,2)-clique in G. This implies $V(H') = R(v_6, H') \cup D(v_6, H')$ and $1 \leq |R(v_6, H')| \leq 2$. If neither v_3 nor v_5 is in $R(v_6, H')$ then $(\{v_1, v_2, v_4, v_6\}, \{v_3, v_5\})$ is a (4,2)-clique in G. Suppose $v_3 \in R(v_6, H')$. If $|R(v_6, H')| = 1$ or the other vertex in $R(v_6, H')$ is either v_4 or v_5, then $(H' + v_6) - v_3$ is isomorphic to H_5^1. If the other vertex in $R(v_6, H')$ is either v_1 or v_2, then $(\{v_1, v_2, v_3, v_6\}, \{v_4, v_5\})$ is a (4,2)-clique in G, a contradiction. □

Lemmas 10, 11 complete the proof of Claim 9. □

Claim 12 *G contains one of the graphs H_6^0, H_6^1, H_6^2, H_6^3.*

Proof: By Claim 9, G contains one of the graphs H_5^0, H_5^1. Again consider both cases separately.

Lemma 13 *If G contains H_5^0 then G contains H_6^0.*

Proof: Let H be an induced subgraph of G isomorphic to H_5^0. Let v_1v_2 be the red edge in H and let v_3, v_4, v_5 be the isolated vertices in H_R. By Lemma 2, there exists a vertex v_6 in $G - H$ such that $w(v_6, H) > 20$. If $|D(v_6, H)| \geq 4$, then $H + v_6$ is a 6-clique with at least 13 double edges, contradicting Lemma 6. Otherwise, we have $|D(v_6, H)| = 3$ and $|R(v_6, H)| = 2$. Since $\{v_3, v_4, v_5\}$ is a deficient set in H, $R(v, H)$ must contain two vertices from $\{v_3, v_4, v_5\}$ (Lemma 5), which implies $H + v_6$ is isomorphic to H_6^0. □

Lemma 14 *If G contains H_5^1 then G contains one of the graphs H_6^0, H_6^1, H_6^2, H_6^3.*

Proof: Let H be an induced subgraph of G isomorphic to H_5^1. Let v_1v_2 and v_3v_4 be the red edges in H and v_5 the isolated vertex in H_R. By Lemma 2, there exists a vertex v_6 in $G - H$ such that $w(v_6, H) \geq 21$. If $|B(v_6, H) \cup D(v_6, H)| = 5$ then v_5 is a deficient vertex in $B(v_6, H) \cup D(v_6, H)$. This implies $V(H) = R(v_6, H) \cup D(v_6, H)$ and $1 \leq |R(v_6, H)| \leq 2$.

Suppose $|D(v_6, H)| = 4$. If $v_5 \in D(v_6, H)$, then G contains H_5^0, and by Lemma 13, G contains H_6^0. Otherwise $v_5 \in R(v_6, H)$ and $H + v_6$ is isomorphic to H_6^1.

Suppose $|D(v_6, H)| = 3$. If $R(v_6, H) = \{v_1, v_2\}$ or $R(v_6, H) = \{v_3, v_4\}$ then $H + v_6$ is isomorphic to H_6^2. If $v_5 \in R(v_6, H)$, then $H + v_6$ is isomorphic to H_6^3. In all other cases, G contains H_4^0 and hence H_6^0, by Lemmas 10, 13. □

Lemmas 13, 14 complete the proof of Claim 12. □

Claim 15 *G contains one of the graphs H_7^0, H_7^1, H_7^2, H_7^3, H_7^4, H_7^5.*

Proof: By Claim 12, G contains one of the graphs H_6^0, H_6^1, H_6^2, H_3^6. We consider each case.

Lemma 16 *If G contains H_6^0 then G contains H_7^0.*

Proof: Let H be an induced subgraph of G isomorphic to H_6^0. Let the components of H_R be the edge v_1v_2, the path v_3, v_4, v_5 and the isolated vertex v_6.

Suppose there is a vertex v_7 in $G - H$ such that $w(v_7, H) \geq 26$. If $|B(v_7, H) \cup D(v_7, H)| = 6$ then v_6 is a deficient vertex in $B(v_7, H) \cup D(v_7, H)$, contradicting Lemma 5. This implies $V(H) = R(v_7, H) \cup D(v_7, H)$ and $1 \leq |R(v_7, H)| \leq 2$. Since $\{v_3, v_5, v_6\}, \{v_1, v_6\}, \{v_2, v_6\}$ are deficient sets in H, we have $v_6 \in R(v_7, H)$ and any one of v_3, v_5 is in $R(v_7, H)$. This implies $H + v_7$ is isomorphic to H_7^0.

Suppose for every vertex v in $G - H$, $w(v, H) \leq 25$. Since $w(H) = 69$, by Lemma 3, with $X = \{v_6\}$ and $r = 1$, there exists a vertex v_7 in $G - H$ such that $w(v_7, H) \geq 24$ and $w(v_7, v_6) > 3$, that is, v_6v_7 is a double edge. If $|B(v_7, H) \cup D(v_7, H)| \geq 5$ then v_6 is a deficient vertex in $B(v_7, H) \cup D(v_7, H)$. This implies $V(H) = R(v_7, H) \cup D(v_7, H)$ and $2 \leq |R(v_7, H)| \leq 3$. However, since $\{v_3, v_5, v_6\}, \{v_1, v_6\}, \{v_2, v_6\}$ are deficient sets, and $v_6 \notin R(v_7, H)$, we must have $|R(v_7, H)| \geq 4$, a contradiction. □

Lemma 17 *If G contains H_6^1 then G contains one of the graphs H_7^0, H_7^1.*

Proof: Let H be an induced subgraph of G isomorphic to H_6^1, and let v_1v_2, v_3v_4, v_5v_6 be the red edges in H. By Lemma 2, there is a vertex v_7 in $G - H$ such that $w(v_7, H) > 24$. If $|B(v_7, H) \cup D(v_7, H)| \geq 5$, then $B(v_7, H) \cup D(v_7, H)$ contains a deficient vertex, a contradiction. This implies $|D(v_7, H)| = 4$ and $|R(v_7, H)| = 2$. If $R(v_7, H)$ contains both endpoints of some red edge in H then $H + v_7$ is isomorphic to H_7^1, otherwise it is isomorphic to H_7^0. □

Lemma 18 *If G contains H_6^2 then G contains one of the graphs H_7^0, H_7^1, H_7^2, H_7^3, H_7^4.*

Proof: Let H be an induced subgraph of G isomorphic to H_6^2. Let the components of H_R be the edge v_1v_2, the triangle v_3, v_4, v_5, and the isolated vertex v_6.

Suppose there is a vertex v_7 in $G - H$ such that $w(v_7, H) \geq 26$. If $|B(v_7, H) \cup D(v_7, H)| = 6$ then v_6 is a deficient vertex in $B(v_7, H) \cup D(v_7, H)$, a contradiction. This implies $V(H) = R(v_7, H) \cup D(v_7, H)$ and $1 \leq |R(v_7, H)| \leq 2$. If $|D(v_7, H)| = 5$, then $R(v_7, H) = \{v_6\}$ otherwise v_6 is a deficient vertex in $D(v_7, H)$. Hence $H + v_7$ is isomorphic to H_7^1. If $|D(v_7, H)| = 4$ and $|R(v_7, H)| = 2$, then since $\{v_1, v_6\}$ and $\{v_2, v_6\}$ are deficient pairs of vertices, either $R(v_7, H) = \{v_1, v_2\}$ or $v_6 \in R(v_7, H)$. If $R(v_7, H) = \{v_1, v_2\}$ then $H + v_7$ is isomorphic to H_7^2. If $v_6 \in R(v_7, H)$, and if any of the vertices v_3, v_4, v_5 is in $R(v_7, H)$, say v_3, then $(H + v_7) - v_3$ is isomorphic to H_6^1. Lemma 17 now implies that G contains H_7^0 or H_7^1. If $v_6 \in R(v_7, H)$, and the other vertex in $R(v_7, H)$ is either v_1 or v_2, then $H + v_7$ is isomorphic to H_7^3.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 25$. Since $w(H) = 67$, by Lemma 3, with $X = \{v_6\}$ and $r = 1$, there is a vertex v_7 in $G - H$ such that $w(v_7, H) \geq 24$ and $w(v_7, v_6) > 3$. If $|B(v_7, H) \cup D(v_7, H)| \geq 5$ then v_6 is a deficient vertex in $B(v_7, H) \cup D(v_7, H)$. This implies $V(H) = R(v_7, H) \cup D(v_7, H)$ and $2 \leq |R(v_7, H)| \leq 3$. Since $v_6 \in D(v_7, H)$, and $\{v_1, v_6\}, \{v_2, v_6\}$ are deficient sets, $\{v_1, v_2\} \subseteq R(v_7, H)$. If $|R(v_7, H)| = 2$ then $H + v_7$ is isomorphic to H_7^2 else if $|R(v_7, H)| = 3$ then $H + v_7$ is isomorphic to H_7^4. □

Lemma 19 *If G contains H_6^3 then G contains one of the graphs H_7^0, H_7^1, H_7^3, H_7^5.*

Proof: Let H be an induced subgraph of G isomorphic to H_6^3. Let the components of H_R be the edge v_1v_2 and the path v_3, v_4, v_5, v_6. By Lemma 2, there is a vertex v_7 in $G - H$ such that $w(v_7, H) > 24$. If $|D(v_7, H)| \geq 5$, then either $\{v_1, v_3, v_5\}$ or $\{v_2, v_4, v_6\}$ is a subset of $D(v_7, H)$, which implies G contains H_4^0. Lemma's 10, 13, 16 now imply that G contains H_7^0.

Suppose $|D(v_7, H)| = 4$. If the subgraph of H induced by $D(v_7, H)$ does not contain two disjoint red edges, then G contains H_4^0 and by Lemmas 10, 13, 16, G contains H_7^0.

Suppose $D(v_7, H) = \{v_3, v_4, v_5, v_6\}$. If any one of the edges v_1v_7 or v_2v_7 is blue, say v_1v_7, then $(\{v_3, v_4, v_5, v_6\}, \{v_1, v_7\})$ is a (4,2)-clique in G. Thus $R(v_7, H) = \{v_1, v_2\}$ and $H + v_7$ is isomorphic to H_7^3.

Suppose $D(v_7, H) = \{v_1, v_2, v_3, v_4\}$. If the edge v_6v_7 is blue, then $(\{v_1, v_2, v_3, v_4\}, \{v_6, v_7\})$ is a (4,2)-clique in G. Therefore v_6v_7 is red and $(H + v_7) - v_5$ is isomorphic to H_6^1. Lemma 17 now implies that G contains H_7^0 or H_7^1. A similar argument holds if $D(v_7, H) = \{v_1, v_2, v_5, v_6\}$.

Finally, suppose $D(v_7, H) = \{v_1, v_2, v_4, v_5\}$. If $R(v_7, H) = \{v_3, v_6\}$ then $H + v_7$ is isomorphic to H_7^5. At least one of v_3, v_6, say v_3, is in $R(v_7, H)$ and suppose v_6 is in $B(v_7, H)$. Let $H' = H + v_7$. By Lemma 2, there is a vertex v_8 in $G - H'$ such that $w(v_8, H') > 28$.

Suppose $|D(v_8, H')| \geq 5$. If any one of v_1, v_2, say v_1, is in $D(v_8, H')$, then we can find vertices $v_i, v_j \in \{v_3, v_4, v_5, v_6, v_7\}$ such that $\{v_1, v_i, v_j, v_8\}$ induces H_4^0 in G. Lemmas 10, 13, 16 then imply that G contains H_7^0. If $D(v_8, H') = \{v_3, v_4, v_5, v_6, v_7\}$ and any one of the edges v_1v_8, v_2v_8 is blue, say v_1v_8, then $(\{v_3, v_4, v_5, v_6\}, \{v_1, v_8\})$ is a (4,2)-clique in G. Therefore, $R(v_8, H') = \{v_1, v_2\}$, and $(\{v_1, v_2, v_4, v_8\}, \{v_6, v_7\})$ is a (4,2)-clique in G.

The only other possibility is that $|D(v_8, H')| = 4$ and $|R(v_8, H')| = 3$. If neither v_6 nor v_7 is contained in $R(v_8, H')$, then $(\{v_1, v_2, v_4, v_8\}, \{v_6, v_7\})$ is a (4,2)-clique in G.

Suppose exactly one of the vertices v_6, v_7, say v_6, is contained in $R(v_8, H')$. If $v_4 \notin R(v_8, H')$, then $(\{v_1, v_2, v_6, v_8\},\{v_4, v_7\})$ is a (4,2)-clique in G. Therefore $\{v_4, v_6\} \subset R(v_8, H')$ and at least one of v_1, v_2, say v_1, is in $D(v_8, H')$. Then $(\{v_4, v_5, v_6, v_8\}, \{v_1, v_7\})$ is a (4,2)-clique in G.

Finally, suppose both v_6 and v_7 are contained in $R(v_8, H')$, and one of v_1, v_2, say v_1, is in $D(v_8, H')$. If neither v_3 nor v_5 is contained in $R(v_8, H')$ then $\{v_1, v_3, v_5, v_8\}$ induces H_4^0 in G. Lemmas 10, 13, 16 now imply that G contains H_7^0. If $R(v_8, H') = \{v_3, v_6, v_7\}$ then $(H' + v_8) - v_7$ is isomorphic to H_7^5. Similarly, if $R(v_8, H') = \{v_5, v_6, v_7\}$, then $(H' + v_8) - v_6$ is isomorphic to H_7^5. □

Lemmas 16, 17, 18, 19 complete the proof of Claim 15. □

Claim 20 *G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6, H_8^7, H_8^8, H_8^9, H_8^{10}.*

Proof: By Claim 15, G contains one of the graphs H_7^0, H_7^1, H_7^2, H_7^3, H_7^4, H_7^5. We consider each case.

Lemma 21 *If G contains H_7^0 then G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3.*

Proof: Let H be an induced subgraph of G isomorphic to H_7^0 and let the edge v_1v_2 and the path v_3, v_4, v_5, v_6, v_7 be the components of H_R. By Lemma 2, there is a vertex v_8 in $G - H$ such that $w(v_8, H) > 28$. If $|B(v_8, H) \cup D(v_8, H)| \geq 6$ then at least one of the vertices v_1, v_2 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$, a contradiction. This implies $V(H) = R(v_8, H) \cup D(v_8, H)$ and $2 \leq |R(v_8, H)| \leq 3$. Since $\{v_3, v_5, v_7\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_7\}, \{v_2, v_7\}$ are deficient sets, we must have $\{v_3, v_7\} \subset R(v_8, H)$. If $|R(v_8, H)| = 2$ then $H + v_8$ is isomorphic to H_8^0. If either v_1 or v_2 is in $R(v_8, H)$, then $H + v_8$ is isomorphic to H_8^1. If v_4 or v_6 is in $R(v_8, H)$, then $H + v_8$ is isomorphic to H_8^2, and if $v_5 \in R(v_8, H)$, then $H + v_8$ is isomorphic to H_8^3. □

Lemma 22 *If G contains H_7^1 then G contains one of the graphs H_8^4, H_8^5, H_8^6.*

Proof: Let H be an induced subgraph of G isomorphic to H_7^1. Let the components of H_R be the edges v_1v_2, v_3v_4 and the triangle v_5, v_6, v_7. By Lemma 2, there is a vertex v_8 in $G - H$ such that $w(v_8, H) > 28$. If $|B(v_8, H) \cup D(v_8, H)| \geq 6$, then at least one of the vertices v_1, v_2 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$. Therefore $V(H) = R(v_8, H) \cup D(v_8, H)$ and $2 \leq |R(v_8, H)| \leq 3$. Since $\{v_i, v_j\}$, $i \in \{1, 2\}$, $j \in \{3, 4\}$, are deficient pairs of vertices, we may assume, without loss of generality, $\{v_3, v_4\} \subseteq R(v_8, H)$. If $|R(v_8, H)| = 2$, then $H + v_8$ is isomorphic to H_8^4. If any one of v_5, v_6, v_7 is in $R(v_8, H)$ then $H + v_8$ is isomorphic to H_8^5 else it is isomorphic to H_8^6. $\square$

Lemma 23 *If G contains H_7^2 then G contains one of the graphs H_8^4, H_8^5, H_8^6.*

Proof: Let H be an induced subgraph of G isomorphic to H_7^2. Let the components of H_R be the triangles v_1, v_2, v_3 and v_4, v_5, v_6, and the isolated vertex v_7.

Suppose there is a vertex v_8 in $G - H$ such that $w(v_8, H) \geq 31$. If $|B(v_8, H) \cup D(v_8, H)| = 7$ then v_7 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$, a contradiction. This implies $V(H) = R(v_8, H) \cup D(v_8, H)$ and $1 \leq |R(v_8, H)| \leq 2$. Since $\{v_i, v_7\}$, $i \in \{1, 2, 3, 4, 5, 6\}$, are deficient pairs of vertices, we must have $v_7 \in R(v_8, H)$. Without loss of generality, suppose v_6 is the other vertex in $R(v_8, H)$ (if any). Then $(H + v_8) - v_6$ is isomorphic to H_7^1 and by Lemma 22, G contains one of the graphs H_8^4, H_8^5, H_8^6.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 30$. Since $w(H) = 93$, by Lemma 4, with $X = \{v_7\}$, $s = 1$, and $r = 2$, there is a vertex v_8 in $G - H$ such that either $w(v_8, H) \geq 27$ and $w(v_8, v_7) > 3$ or $w(v_8, H) = 30$ and $w(v_8, v_7) = 3$.

Suppose $w(v_8, H) \geq 27$ and $w(v_8, v_7) > 3$. If $|B(v_8, H) \cup D(v_8, H)| \geq 5$, then v_7 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$. This implies $V(H) = R(v_8, H) \cup D(v_8, H)$ and $3 \leq |R(v_8, H)| \leq 4$. However, since $v_7 \in D(v_8, H)$ and $\{v_i, v_7\}$, $i \in \{1, 2, 3, 4, 5, 6\}$, are deficient pairs of vertices, we must have $|R(v_8, H)| \geq 6$, a contradiction.

Suppose $w(v_8, H) = 30$ and $w(v_8, v_7) = 3$, that is $v_7 v_8$ is a red edge. Since $w(v_8, H) = 30$, $|B(v_8, H)| = 1$. Without loss of generality, $v_6 \in B(v_8, H)$ and $(H + v_8) - v_6$ is isomorphic to H_7^1. Lemma 22 implies that G contains one of the graphs H_8^4, H_8^5, H_8^6. $\square$

Lemma 24 *If G contains H_7^3 then G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6, H_8^7, H_8^8, H_8^9.*

Proof: Let H be an induced subgraph of G isomorphic to H_7^3. Let the components of H_R be the triangle v_1, v_2, v_3 and the path v_4, v_5, v_6, v_7. By Lemma 2, there is a vertex v_8 in $G - H$ such that $w(v_8, H) \geq 29$. If $|B(v_8, H) \cup D(v_8, H)| \geq 6$, then at least one of v_4, v_7 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$. This implies $V(H) = R(v_8, H) \cup D(v_8, H)$ and $2 \leq |R(v_8, H)| \leq 3$.

Suppose $v_7 \notin R(v_8, H)$. Since $\{v_4, v_7\}, \{v_5, v_7\}$ are deficient pairs of vertices, $\{v_4, v_5\} \subset R(v_8, H)$. If $|R(v_8, H)| = 2$, then $H + v_8$ is isomorphic to H_8^6. If $|R(v_8, H)| = 3$ and any one of the vertices v_1, v_2, v_3, say v_3, is in $R(v_8, H)$ then $(H + v_8) - \{v_3, v_5\}$ is isomorphic to H_6^1. By Lemmas 17, 21, 22, G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6. If $R(v_8, H) = \{v_4, v_5, v_6\}$ then $(H + v_8) - v_6$ is isomorphic to H_7^2 and by Lemma 23, G contains one of the graphs H_8^4, H_8^5, H_8^6. A symmetrical argument holds if $v_4 \notin R(v_8, H)$.

Suppose $\{v_4, v_7\} \subset R(v_8, H)$. If $|R(v_8, H)| = 2$ then $H + v_8$ is isomorphic to H_8^7. If $|R(v_8, H)| = 3$ and any one of the vertices v_1, v_2, v_3 is in $R(v_8, H)$ then $H + v_8$ is isomorphic to H_8^8, otherwise either v_5 or v_6 is in $R(v_8, H)$ and $H + v_8$ is isomorphic to H_8^9. $\square$

Lemma 25 *If G contains H_7^4 then G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6, H_8^{10}.*

Proof: Let H be an induced subgraph of G isomorphic to H_7^4. Let the edge v_3v_6 be the bridge in H_R. Let the triangles v_1, v_2, v_3 and v_4, v_5, v_6, and the isolated vertex v_7 be the components of $H_R - v_3v_6$.

Suppose there is a vertex v_8 in $G - H$ such that $w(v_8, H) \geq 31$. If $|B(v_8, H) \cup D(v_8, H)| = 7$ then v_7 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$, a contradiction. This implies $V(H) = R(v_8, H) \cup D(v_8, H)$ and $1 \leq |R(v_8, H)| \leq 2$. Since $\{v_i, v_7\}$, $i \in \{1, 2, 4, 5\}$, are deficient pairs of vertices, we must have $v_7 \in R(v_8, H)$. If $|R(v_8, H)| = 1$ then $H + v_8$ is isomorphic to H_8^5. If $|R(v_8, H)| = 2$ and the other vertex in $R(v_8, H)$ is either v_3 or v_6, say v_6, then $(H + v_8) - v_6$ is isomorphic to H_7^1, and by Lemma 22, G contains one of the graphs H_8^4, H_8^5, H_8^6. If $R(v_8, H)$ contains any one of the vertices v_1, v_2, v_4, v_5, say v_5, then $(H + v_8) - \{v_3, v_5\}$ is isomorphic to H_6^1, and by Lemmas 17, 21, 22, G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6.

Suppose that for every vertex v in $G-H$, $w(v, H) \leq 30$. Since $w(H) = 91$, by Lemma 3, with $X = \{v_7\}$ and $r = 2$, there exists a vertex v_8 in $G-H$, such that $w(v_8, H) \geq 27$ and $w(v_8, v_7) > 3$. If $|B(v_8, H) \cup D(v_8, H)| \geq 5$, then v_7 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$. This implies $V(H) = R(v_8, H) \cup D(v_8, H)$ and $3 \leq |R(v_8, H)| \leq 4$. However, since $v_7 \in D(v_8, H)$ and $\{v_i, v_7\}$, $i \in \{1, 2, 4, 5\}$, are deficient pairs of vertices, we must have $R(v_8, H) = \{v_1, v_2, v_4, v_5\}$, and $H + v_8$ is isomorphic to H_8^{10}. □

Lemma 26 *If G contains H_7^5 then G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6, H_8^7, H_8^8.*

Proof: Let H be an induced subgraph of G isomorphic to H_7^5. Let the components of H_R be the edge v_1v_2 and the 5-cycle v_3, v_4, v_5, v_6, v_7. By Lemma 2, there is a vertex v_8 in $G - H$ such that $w(v_8, H) > 28$. If $|B(v_8, H) \cup D(v_8, H)| \geq 6$ then either v_1 or v_2 is a deficient vertex in $B(v_8, H) \cup D(v_8, H)$, a contradiction. This implies $V(H) = R(v_8, H) \cup D(v_8, H)$ and $2 \leq |R(v_8, H)| \leq 3$.

Suppose $|D(v_8, H)| = 5$. If either v_1 or v_2 is in $D(v_8, H)$, say v_1, then we can find vertices $v_i, v_j \in \{v_3, v_4, v_5, v_6, v_7\}$ such that $\{v_1, v_i, v_j, v_8\}$ induces H_4^0. Lemmas 10, 13, 16, 21 now imply that G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3. Otherwise $R(v_8, H) = \{v_1, v_2\}$ and $H + v_8$ is isomorphic to H_8^7.

Suppose $|D(v_8, H)| = 4$. If the subgraph of H induced by $D(v_8, H)$ does not contain two disjoint red edges, then G contains H_4^0, and Lemmas 10, 13, 16, 21 imply that G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3. If $\{v_1, v_2\} \subset D(v_8, H)$, we may assume, without loss of generality, that $\{v_3, v_7\} \subset D(v_8, H)$. Then $(H + v_8) - \{v_4, v_6\}$ is isomorphic to H_6^1 and by Lemmas 17, 21, 22, G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6. The only other possibility is that $D(v_8, H)$ contains four of the vertices $\{v_3, v_4, v_5, v_6, v_7\}$, in which case, $H + v_8$ is isomorphic to H_8^8. □

Lemmas 21, 22, 23, 24, 25, 26 complete the proof of Claim 20. □

Claim 27 *G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7, H_9^8.*

Proof: By Claim 20, G contains one of the graphs H_8^0, H_8^1, H_8^2, H_8^3, H_8^4, H_8^5, H_8^6, H_8^7, H_8^8, H_8^9, H_8^{10}. We consider each of the graphs.

Lemma 28 *G does not contain H_8^0.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^0. Let the components of H_R be the edge v_1v_2 and the 6-cycle $v_3, v_4, v_5, v_6, v_7, v_8$.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_1 or v_2 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$. However, $\{v_3, v_5, v_7\}$ and $\{v_4, v_6, v_8\}$ are deficient sets in H, which implies $|R(v_9, H)| \geq 4$, a contradiction.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 33$. Since $w(H) = 126$, by Lemma 3, with $X = \{v_1, v_2\}$ and $r = 1$, there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 32$ and $w(v_9, \{v_1, v_2\}) > 6$. We may assume, without loss of generality, that v_1v_9 is a double edge. If $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then v_1 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. However, $\{v_1, v_i\}$, $i \in \{3, 4, 5, 6, 7, 8\}$, are deficient pairs of vertices and $v_1 \in D(v_9, H)$, hence $|R(v_9, H)| \geq 6$, a contradiction. □

Lemma 29 *G does not contain H_8^1.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^1. Let v_2v_8 be the bridge in H_R and let the components of $H_R - v_2v_8$ be the edge v_1v_2 and the 6-cycle $v_3, v_4, v_5, v_6, v_7, v_8$.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_1 or v_2 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$. Since $\{v_1, v_i\}$, $i \in \{3, 4, 5, 6, 7, 8\}$, are deficient pairs of vertices, $v_1 \in R(v_9, H)$. However, $\{v_3, v_5, v_7\}$ and $\{v_4, v_6\}$ are deficient sets in H, which implies $|R(v_9, H)| \geq 4$, a contradiction.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 33$. Since $w(H) = 124$, by Lemma 3, with $X = \{v_1\}$ and $r = 1$, there exists a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 32$ and $w(v_9, v_1) > 3$. If $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then v_1 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. However, $\{v_1, v_i\}$, $i \in \{3, 4, 5, 6, 7, 8\}$, are deficient pairs of vertices and $v_1 \in D(v_9, H)$, hence $|R(v_9, H)| \geq 6$, a contradiction. □

Lemma 30 *If G contains H_8^2 then G contains H_9^0.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^2. Let the components of H_R be the edge v_1v_2 and the 6-cycle $v_3, v_4, v_5, v_6, v_7, v_8$ with the chord v_6v_8.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_1 or v_2 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$. Since $\{v_1, v_i\}, \{v_2, v_i\}$, $i \in \{3, 4, 5, 7\}$, are deficient pairs of vertices in H, $\{v_1, v_2\} \subset R(v_9, H)$. However, $\{v_3, v_5, v_7\}$ is a deficient set in H, which implies $|R(v_9, H)| \geq 4$, a contradiction.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 33$. Since $w(H) = 124$, by Lemma 3, with $X = \{v_1\}$ and $r = 1$, there exists a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 32$ and $w(v_9, v_1) > 3$. If $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then v_1 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$.

This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. Since $\{v_1, v_i\}$, $i \in \{3, 4, 5, 7\}$, are deficient pairs of vertices and $v_1 \in D(v_9, H)$, we must have $R(v_9, H) = \{v_3, v_4, v_5, v_7\}$, and $H + v_9$ is isomorphic to H_9^0. $\square$

Lemma 31 *If G contains H_8^3 then G contains H_9^1.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^3. Let the components of H_R be the edge v_1v_2 and the 6-cycle $v_3, v_4, v_5, v_6, v_7, v_8$ with the chord v_5v_8.

Suppose there is a vertex v_9 in $G-H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_1 or v_2 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$. Since $\{v_1, v_i\}, \{v_2, v_i\}$, $i \in \{3, 4, 6, 7\}$, are deficient pairs of vertices, $\{v_1, v_2\} \subset R(v_9, H)$. However, $\{v_4, v_6\}$ and $\{v_3, v_7\}$ are deficient pairs of vertices in H, which implies $|R(v_9, H)| \geq 4$, a contradiction.

Suppose that for every vertex v in $G-H$, $w(v, H) \leq 33$. Since $w(H) = 124$, by Lemma 3, with $X = \{v_1\}$ and $r = 1$, there exists a vertex v_9 in $G-H$ such that $w(v_9, H) \geq 32$ and $w(v_9, v_1) > 3$. If $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then v_1 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. However, since $\{v_1, v_i\}$, $i \in \{3, 4, 6, 7\}$, are deficient pairs of vertices and $v_1 \in D(v_9, H)$, we must have $R(v_9, H) = \{v_3, v_4, v_6, v_7\}$, and $H + v_9$ is isomorphic to H_9^1. $\square$

Lemma 32 *If G contains H_8^4 then G contains H_9^2 or H_9^3.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^4. Let the components of H_R be the triangles v_1, v_2, v_3 and v_4, v_5, v_6 and the edge v_7v_8.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_7 or v_8 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$. Since $\{v_i, v_7\}$ and $\{v_i, v_8\}$ are deficient pairs in H for $i \in \{1, 2, 3, 4, 5, 6\}$, we have $\{v_7, v_8\} \subset R(v_9, H)$. Then $H + v_9$ is isomorphic to H_9^2 if $|R(v_9, H)| = 2$ and it is isomorphic to H_9^3 otherwise.

Suppose for every vertex v in $G - H$, $w(v, H) \leq 33$. Since $w(H) = 126$, by Lemma 3, with $X = \{v_7, v_8\}$ and $r = 1$, there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 32$ and $w(v_9, \{v_7, v_8\}) > 6$. We may assume, without loss of generality, that $v_7 \in D(v_9, H)$. If $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then v_7 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. However, $\{v_i, v_7\}$, $i \in \{1, 2, 3, 4, 5, 6\}$, are deficient pairs of vertices and $v_7 \in D(v_9, H)$, hence $|R(v_9, H)| \geq 6$, a contradiction. $\square$

Lemma 33 *If G contains H_8^5 then G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^5. Let the components of H_R be the edge v_7v_8 and H'. Let v_3v_6 be the bridge in H' and let the triangles v_1, v_2, v_3 and v_4, v_5, v_6 be the components of $H' - v_3v_6$.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_7 or v_8 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) =$

$R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$. Since $\{v_i, v_7\}$ and $\{v_i, v_8\}$ are deficient pairs in H for $i \in \{1, 2, 4, 5\}$, we have $\{v_7, v_8\} \subset R(v_9, H)$. If $|R(v_9, H)| = 2$ then $H + v_9$ is isomorphic to H_9^3. If $|R(v_9, H)| = 3$ and any one of the vertices v_1, v_2, v_4, v_5 is in $R(v_9, H)$ then $H + v_9$ is isomorphic to H_9^4. If either v_3 or v_6, say v_3, is in $R(v_9, H)$, then $(H + v_9) - v_3$ is isomorphic to H_8^4, and by Lemma 32, G contains one of the graphs H_9^2 or H_9^3.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 33$. Since $w(H) = 124$, by Lemma 3, with $X = \{v_7\}$ and $r = 1$, there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 32$ and $w(v_9, v_7) > 3$. If $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then v_7 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. Since $\{v_i, v_7\}$, $i \in \{1, 2, 4, 5\}$, are deficient pairs of vertices and $v_7 \in D(v_9, H)$, we must have $R(v_9, H) = \{v_1, v_2, v_4, v_5\}$, and $H + v_9$ is isomorphic to H_9^5. □

Lemma 34 *If G contains H_8^6 then G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^6, H_9^7.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^6. Let the components of H_R be the triangle v_1, v_2, v_3 and H'. Let v_4, v_5, v_6 be the triangle in H' and v_7v_8, v_6v_8 the remaining edges in H'.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_7 or v_8 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$. Since $\{v_i, v_7\}$, $i \in \{1, 2, 3, 4, 5, 6\}$, are deficient pairs in H, we have $v_7 \in R(v_9, H)$. If $v_8 \notin R(v_9, H)$ then since $\{v_4, v_8\}, \{v_5, v_8\}$ are deficient pairs, $R(v_9, H) = \{v_4, v_5, v_7\}$ and $H + v_9$ is isomorphic to H_9^6. Suppose $v_8 \in R(v_9, H)$. If $|R(v_9, H)| = 2$ then $H + v_9$ is isomorphic to H_9^3. If $|R(v_9, H)| = 3$ and any one of the vertices v_1, v_2, v_3 is in $R(v_9, H)$ then $H + v_9$ is isomorphic to H_9^4. If v_4 or v_5 is in $R(v_9, H)$ then $H + v_9$ is isomorphic to H_9^7. If $v_6 \in R(v_9, H)$ then $(H + v_9) - v_6$ is isomorphic to H_8^4, and by Lemma 32, G contains one of the graphs H_9^2 or H_9^3.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 33$. Since $w(H) = 124$, by Lemma 3, with $X = \{v_7\}$ and $r = 1$, there is a vertex v_9 in $G-H$ such that $w(v_9, H) \geq 32$ and $w(v_9, v_7) > 3$. If $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then v_7 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. However, $\{v_i, v_7\}$, $i \in \{1, 2, 3, 4, 5, 6\}$, are deficient pairs of vertices and $v_7 \in D(v_9, H)$, hence $|R(v_9, H)| \geq 6$, a contradiction. □

Lemma 35 *If G contains H_8^7 or H_8^8 then G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7.*

Proof: Let H be an induced subgraph of G isomorphic to either H_8^7 or H_8^8. Let v_1, v_2, v_3 and v_4, v_5, v_6, v_7, v_8 be the triangle and the 5-cycle respectively, in H_R. Let v_3v_4 be the remaining edge in H_R, if H is isomorphic to H_8^8. By Lemma 2, there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 33$. This implies that $|D(v_9, H)| \geq 5$.

Suppose $|B(v_9, H) \cup D(v_9, H)| \geq 6$. It is easy to check that any subset of 6 vertices in H induces a red 6-clique containing a vertex with red degree at most 1. Such a vertex is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $|D(v_9, H)| = 5$ and $|R(v_9, H)| = 3$.

Since $\{v_5, v_7\}, \{v_6, v_8\}$ are deficient pairs of vertices, at least one of the vertices v_1, v_2, say v_1, is in $D(v_9, H)$. If there are vertices $v_i, v_j \in D(v_9, H) \cap \{v_4, v_5, v_6, v_7, v_8\}$ such that v_iv_j is a double edge, then $\{v_1, v_i, v_j, v_9\}$ induces H_4^0 in G. Lemmas 10, 13, 16, 21, 28, 29, 30, 31 now imply that G contains one of the graphs H_9^0, H_9^1. Otherwise $\{v_1, v_2, v_3\} \subset D(v_9, H)$ and the other two vertices in $D(v_9, H)$ are the endpoints of a red edge in the 5-cycle v_4, v_5, v_6, v_7, v_8.

If $\{v_4, v_5\} \subset D(v_9, H)$ then $(H + v_9) - \{v_3, v_6, v_8\}$ is isomorphic to H_6^1. Lemmas 17, 21, 22, 28, 29, 30, 31, 32, 33, 34 now imply that G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. The same argument holds if $\{v_4, v_8\} \subset D(v_9, H)$. If $\{v_5, v_6\} \subset D(v_9, H)$ then $(H + v_9) - v_4$ is isomorphic to H_8^6 and by Lemma 34, G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^6, H_9^7. The same argument holds if $\{v_7, v_8\} \subset D(v_9, H)$. Finally, if $\{v_6, v_7\} \subset D(v_9, H)$, then $(H + v_9) - \{v_3, v_5, v_8\}$ is isomorphic to H_6^1. Lemmas 17, 21, 22, 28, 29, 30, 31, 32, 33, 34 now imply that G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. □

Lemma 36 *If G contains H_8^9 then G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7, H_9^8.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^9. Let the components of H_R be the triangle v_1, v_2, v_3 and the 5-cycle v_4, v_5, v_6, v_7, v_8 with the chord v_4v_6.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 34$. If $|B(v_9, H) \cup D(v_9, H)| \geq 7$ then either v_1 or v_2 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $2 \leq |R(v_9, H)| \leq 3$.

Suppose one of the vertices v_7, v_8, say v_7, is not in $R(v_9, H)$. Since $\{v_4, v_7\}, \{v_5, v_7\}$ are deficient pairs of vertices, we must have $\{v_4, v_5\} \subset R(v_9, H)$. Since $\{v_6, v_8\}$ is a deficient pair, either v_6 or v_8 is in $R(v_9, H)$. If $v_6 \in R(v_9, H)$ then $(H + v_9) - \{v_4, v_6\}$ is isomorphic to H_7^1. Lemmas 22, 32, 33, 34 imply that G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. If $v_8 \in R(v_9, H)$ then $(H + v_9) - v_4$ is isomorphic to H_8^7 and Lemma 35 implies that G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7.

Suppose $\{v_7, v_8\} \subset R(v_9, H)$. If $|R(v_9, H)| = 2$ then $H + v_9$ is isomorphic to H_9^7. If either v_4 or v_6 is in $R(v_9, H)$, say v_4, then $(H + v_9) - \{v_4, v_7\}$ is isomorphic to H_7^1. Lemmas 22, 32, 33, 34 imply that G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. If $v_5 \in R(v_9, H)$ then $H + v_9$ is isomorphic to H_9^8. If any one of v_1, v_2, v_3, say v_3, is in $R(v_9, H)$ then $(H + v_9) - \{v_3, v_6, v_8\}$ is isomorphic to H_6^1. Lemmas 17, 21, 22, 28, 29, 30, 31, 32, 33, 34 imply that G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 33$. Since $w(H) = 122$, by Lemma 3, with $X = \{v_7\}$ and $r = 1$, there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 32$ and $w(v_9, v_7) > 3$. Note that the only subset of 6 vertices in H that does not contain a deficient vertex is $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Since $v_7 \in D(v_9, H)$, if $|B(v_9, H) \cup D(v_9, H)| \geq 6$ then there is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $3 \leq |R(v_9, H)| \leq 4$. Since $\{v_4, v_7\}, \{v_5, v_7\}$ are deficient pairs of vertices, $\{v_4, v_5\} \subset R(v_9, H)$. Since $\{v_6, v_8\}$ is a deficient pair, at least one of v_6, v_8 is in $R(v_9, H)$.

Suppose $v_6 \in R(v_9, H)$. If $|R(v_9, H)| = 3$ or any one of the vertices v_1, v_2, v_3, say v_3, is in $R(v_9, H)$, then $(H + v_9) - \{v_3, v_4, v_6\}$ is isomorphic to H_6^1. Lemmas 17, 21, 22, 28, 29, 30, 31, 32, 33, 34 imply that G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. If $\{v_6, v_8\} \subset R(v_9, H)$ then $(H + v_9) - \{v_6, v_8\}$ is isomorphic to H_7^2. Lemmas 23, 32, 33, 34 imply that G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. Suppose $v_6 \notin R(v_9, H)$ and $v_8 \in R(v_9, H)$. If $|R(v_9, H)| = 3$ then $(H + v_9) - v_4$ is isomorphic to H_8^7 otherwise it is isomorphic to H_8^8. In either case, Lemma 35 implies that G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6. □

Lemma 37 *If G contains H_8^{10} then G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7.*

Proof: Let H be an induced subgraph of G isomorphic to H_8^{10}. Let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be the 7-cycle in H_R and $v_1v_3, v_2v_7, v_4v_6, v_5v_7$ the remaining edges in H_R. The vertex v_8 is isolated in H_R.

Suppose there is a vertex v_9 in $G - H$ such that $w(v_9, H) \geq 36$. If $|B(v_9, H) \cup D(v_9, H)| = 8$ then v_8 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $1 \leq |R(v_9, H)| \leq 2$. Since $\{v_i, v_8\}$, $i \in \{2, 3, 4, 5, 6, 7\}$, are deficient pairs, $v_8 \in R(v_9, H)$. If $|R(v_9, H)| = 1$ then $H + v_9$ is isomorphic to H_9^5. If $v_7 \in R(v_9, H)$ then $(H + v_9) - v_7$ is isomorphic to H_8^5. Lemma 33 implies that G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5. If any one of the vertices v_3, v_4, say v_3, is in $R(v_9, H)$, then $(H + v_9) - \{v_3, v_7\}$ is isomorphic to H_7^1. Lemmas 22, 32, 33, 34 imply that G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. Finally, if any of the vertices v_1, v_2, v_5, v_6 is in $R(v_9, H)$, say v_1, then $(H + v_9) - \{v_1, v_4, v_7\}$ is isomorphic to H_6^1. Lemmas 17, 21, 22, 28, 29, 30, 31, 32, 33, 34 imply that G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^5, H_9^6, H_9^7.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 35$. Since $w(H) = 118$, by Lemma 4, with $X = \{v_8\}$, $r = 3$ and $s = 2$, there is a vertex v_9 in $G - H$ such that either $w(v_9, H) \geq 30$ and $w(v_9, v_8) > 3$ or $w(v_9, H) = 35$ and $w(v_9, v_8) = 3$.

Suppose $w(v_9, H) \geq 30$ and $w(v_9, v_8) > 3$. If $|B(v_9, H) \cup D(v_9, H)| \geq 5$ then v_8 is a deficient vertex in $B(v_9, H) \cup D(v_9, H)$. This implies $V(H) = R(v_9, H) \cup D(v_9, H)$ and $4 \leq |R(v_9, H)| \leq 5$. However, since $\{v_i, v_8\}$, $i \in \{1, 2, 3, 4, 5, 6\}$, are deficient pairs of vertices and $v_8 \in D(v_9, H)$, we must have $|R(v_9, H)| \geq 6$, a contradiction.

Suppose $w(v_9, H) = 35$ and $w(v_9, v_8) = 3$. Then $R(v_9, H) = \{v_8\}$, $|D(v_9, H)| = 6$ and $|B(v_9, H)| = 1$. If $v_7 \in B(v_9, H)$, then $(H + v_9) - v_7$ is isomorphic to H_8^5 and by Lemma 33, G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5. If any one of the vertices v_1, v_2, v_5, v_6, say v_6, is in $B(v_9, H)$, then $(H + v_9) - \{v_2, v_6\}$ is isomorphic to H_7^5, and by Lemmas 26, 28, 29, 30, 31, 32, 33, 34, 35, G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. If either v_3 or v_4 is in $B(v_9, H)$, say v_3, then $(H + v_9) - \{v_3, v_7\}$ is isomorphic to H_7^1, and by Lemmas 22, 32, 33, 34, G contains one of the graphs H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7. □

Lemmas 28, 29, 30, 31, 32, 33, 34, 35, 36, 37 complete the proof of Claim 27. □

Claim 38 *G contains one of the graphs H_{10}^0, H_{10}^1, H_{10}^2, H_{10}^3.*

Proof: By Claim 27, G contains one of the graphs H_9^0, H_9^1, H_9^2, H_9^3, H_9^4, H_9^5, H_9^6, H_9^7, H_9^8. We consider each of these graphs.

Lemma 39 *G does not contain H_9^0 or H_9^1.*

Proof: Let H be an induced subgraph of G isomorphic to either H_9^0 or H_9^1. Let the components of H_R be the edge v_1v_2 and H'. Let $v_3, v_4, v_5, v_6, v_7, v_8, v_9$ be the 7-cycle in H'. The remaining edges in H' are v_3v_7, v_5v_7, v_4v_9 if H is isomorphic to H_9^0 and v_3v_7, v_4v_7, v_5v_9 if H is isomorphic to H_9^1.

Suppose there exists a vertex v_{10} in $G-H$ such that $w(v_{10}, H) \geq 39$. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 8$ then either v_1 or v_2 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $2 \leq |R(v_{10}, H)| \leq 3$. However, $\{v_1, v_4, v_6, v_8\}$ and $\{v_2, v_5\}$ are disjoint deficient sets, implying $|R(v_{10}, H)| \geq 4$, a contradiction.

Suppose that for every vertex v in $G-H$, $w(v, H) \leq 38$. Since $w(H) = 158$, by Lemma 4, with $X = \{v_1, v_2\}$, $r = 2$, and $s = 2$, there is a vertex v_{10} in $G-H$ such that either $w(v_{10}, H) \geq 35$ and $w(v_{10}, \{v_1, v_2\}) > 6$ or $w(v_{10}, H) = 38$ and $w(v_{10}, \{v_1, v_2\}) = 6$.

Suppose any one of the edges v_1v_{10}, v_2v_{10}, say v_1v_{10}, is a double edge. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 6$ then v_1 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. Since $w(v_{10}, H) \geq 35$, this implies $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $4 \leq |R(v_{10}, H) \leq 5$. However, since $\{v_1, v_i\}$, $i \in \{3, 4, 5, 6, 8, 9\}$, are deficient pairs in H and $v_1 \in D(v_{10}, H)$, we must have $|R(v_{10}, H)| \geq 6$, a contradiction.

The only other possibility is that $w(v_{10}, H) = 38$ and $\{v_1, v_2\} \subset R(v_{10}, H)$. Then $R(v_{10}, H) = \{v_1, v_2\}$, $|D(v_{10}, H)| = 6$ and $|B(v_{10}, H)| = 1$. If $v_4 \notin B(v_{10}, H)$ then $(\{v_1, v_2, v_4, v_{10}\}, \{v_6, v_8\})$ is a (4,2)-clique in G, else $(\{v_1, v_2, v_6, v_{10}\}, \{v_4, v_8\})$ is a (4,2)-clique in G, a contradiction. □

Lemma 40 *G does not contain H_9^2.*

Proof: Let H be an induced subgraph of G isomorphic to H_9^2. Let $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$ be the subsets of vertices that induce triangles in H_R. By Lemma 2, there is a vertex v_{10} in $G-H$ such that $w(v_{10}, H) \geq 37$. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 7$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies that $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $3 \leq |R(v_{10}, H)| \leq 4$. We may assume, without loss of generality, $v_1 \in D(v_{10}, H)$. Since $\{v_1, v_i\}$, $i \in \{4, 5, 6, 7, 8, 9\}$, are deficient pairs, we must have $|R(v_{10}, H)| \geq 6$, a contradiction. □

Lemma 41 *If G contains H_9^3 then G contains H_{10}^0.*

Proof: Let H be an induced subgraph of G isomorphic to H_9^3. Let $\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$ be the subsets of vertices that induce triangles in H_R and let v_6v_7 be the remaining red edge. By Lemma 2, there is a vertex v_{10} in $G-H$ such that $w(v_{10}, H) \geq 37$. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 7$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies that $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $3 \leq |R(v_{10}, H)| \leq 4$. If any of the vertices v_4, v_5, v_8, v_9, say v_4, is not in $R(v_{10}, H)$, then since $\{v_4, v_i\}$, $i \in \{1, 2, 3, 8, 9\}$ are deficient pairs, we must have $|R(v_{10}, H)| \geq 5$, a contradiction. Therefore $R(v_{10}, H) = \{v_4, v_5, v_8, v_9\}$ and $H + v_{10}$ is isomorphic to H_{10}^0. □

Lemma 42 *If G contains H_9^4 then G contains H_{10}^1.*

Proof: Let H be an induced subgraph of G isomorphic to H_9^4. Let $\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$ be the subsets of vertices that induce triangles in H_R and let v_3v_4, v_6v_7 be the remaining edges in H_R. By Lemma 2, there is a vertex v_{10} in $G - H$ such that $w(v_{10}, H) \geq 37$. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 7$ then at least one of the vertices v_1, v_2, v_5 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies that $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $3 \leq |R(v_{10}, H)| \leq 4$. Since $\{v_5, v_7\}$ is a deficient pair, at least one of the vertices v_1, v_2, v_8, v_9, say v_1, is not in $R(v_{10}, H)$. Since $\{v_1, v_i\}$, $i \in \{4, 5, 8, 9\}$, are deficient pairs, we must have $R(v_{10}, H) = \{v_4, v_5, v_8, v_9\}$, and $H + v_{10}$ is isomorphic to H_{10}^1. $\Box$

Lemma 43 *If G contains H_9^5 then G contains H_{10}^0 or H_{10}^1.*

Proof: Let H be an induced subgraph of G isomorphic to H_9^5. Let the components of H_R be the edge v_1v_2 and H'. Let $v_3, v_4, v_5, v_6, v_7, v_8, v_9$ be the 7-cycle in H' and let $v_3v_8, v_4v_6, v_5v_7, v_7v_9$ be the remaining edges in H'.

Suppose there is a vertex v_{10} in $G-H$ such that $w(v_{10}, H) \geq 39$. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 8$ then either v_1 or v_2 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $2 \leq |R(v_{10}, H)| \leq 3$. Since $\{v_1, v_i\}, \{v_2, v_i\}$, $i \in \{3, 4, 5, 6, 8, 9\}$, are deficient pairs, we have $\{v_1, v_2\} \subset R(v_{10}, H)$. If $|R(v_{10}, H)| = 2$ then $H + v_{10}$ is isomorphic to H_{10}^0. If any of the vertices v_5, v_6, v_8, v_9 is in $R(v_{10}, H)$ then $H + v_{10}$ is isomorphic to H_{10}^1. If $v_7 \in R(v_{10}, H)$ then $(H + v_{10}) - v_7$ is isomorphic to H_9^3, and by Lemma 41, G contains H_{10}^0. If any of the vertices v_3, v_4, say v_3 is in $R(v_9, H)$, then $(H + v_{10}) - \{v_3, v_7\}$ is isomorphic to H_8^4. Lemmas 32, 40, 41 imply that G contains H_{10}^0.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 38$. Since $w(H) = 156$, by Lemma 3, with $X = \{v_1, v_2\}$ and $r = 2$, there is a vertex v_{10} in $G - H$ such that $w(v_{10}, H) \geq 35$ and $w(v_{10}, \{v_1, v_2\}) > 6$. We may assume that the edge v_1v_{10} is a double edge. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 6$ then v_1 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $4 \leq |R(v_{10}, H)| \leq 5$. However, since $v_1 \in D(v_{10}, H)$ and $\{v_1, v_i\}$, $i \in \{3, 4, 5, 6, 8, 9\}$, are deficient pairs, we must have $|R(v_{10}, H)| \geq 6$, a contradiction. $\Box$

Lemma 44 *If G contains H_9^6 then G contains H_{10}^0 or H_{10}^1 or H_{10}^2.*

Proof: Let H be an induced subgraph of G isomorphic to H_9^7. Let the triangle v_1, v_2, v_3 and H' be the components of H_R and let $v_4, v_5, v_6, v_7, v_8, v_9$ be the 6-cycle in H'. Let v_7v_9, v_4v_8 be the remaining edges in H'. By Lemma 2, there is a vertex v_{10} in $G - H$ such that $w(v_{10}, H) \geq 37$. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 7$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $3 \leq |R(v_{10}, H)| \leq 4$. Since $\{v_5, v_i\}, \{v_6, v_i\}$, $i \in \{1, 2, 3, 8, 9\}$, are deficient pairs, $\{v_5, v_6\} \subset R(v_{10}, H)$. Since $\{v_4, v_7\}$ is a deficient pair at least one of v_4, v_7, say v_4, is in $R(v_{10}, H)$. If $|R(v_{10}, H)| = 3$, then $H + v_{10}$ is isomorphic to H_{10}^0. If any one of v_1, v_2, v_3 is in $R(v_{10}, H)$, then $H + v_{10}$ is isomorphic to H_{10}^1. If either v_8 or v_9 is in $R(v_{10}, H)$ then $H + v_{10}$ is isomorphic to H_{10}^2. If $v_7 \in R(v_{10}, H)$ then $(H + v_{10}) - \{v_4, v_7\}$ is isomorphic to H_8^4. Lemmas 32, 40, 41 imply that G contains H_{10}^0. $\Box$

Lemma 45 *If G contains H_9^7 then G contains H_{10}^2 or H_{10}^3.*

Proof: Let H be an induced subgraph of G isomorphic to H_9^7. Let the triangle v_1, v_2, v_3 and H' be the components of H_R and let $v_4, v_5, v_6, v_7, v_8, v_9$ be the 6-cycle in H'. Let v_7v_9, v_4v_6 be the remaining edges in H'. By Lemma 2, there is a vertex v_{10} in $G - H$ such that $w(v_{10}, H) \geq 37$. If $|B(v_{10}, H) \cup D(v_{10}, H)| \geq 7$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $3 \leq |R(v_{10}, H)| \leq 4$. Since $\{v_5, v_7\}, \{v_5, v_8\}, \{v_6, v_8\}, \{v_5, v_i\}, \{v_8, v_i\}$, $i \in \{1, 2, 3\}$, are deficient pairs, $\{v_5, v_8\} \subset R(v_{10}, H)$. Since $\{v_6, v_9\}, \{v_4, v_7\}$ are deficient pairs, one of $\{v_4, v_6\}, \{v_6, v_7\}, \{v_7, v_9\}, \{v_4, v_9\}$ is a subset of $R(v_{10}, H)$. If either $\{v_6, v_7\}$ or $\{v_4, v_9\}$ is contained in $R(v_{10}, H)$ then $H + v_{10}$ is isomorphic to H_{10}^2, otherwise it is isomorphic to H_{10}^3. $\square$

Lemma 46 *If G contains H_9^8 then G contains one of the graphs H_{10}^0, H_{10}^1, H_{10}^2, H_{10}^3.*

Proof: Let H be an induced subgraph of G isomorphic to H_9^8. Let the triangle v_1, v_2, v_3 and H' be the components of H_R and let $v_4, v_5, v_6, v_7, v_8, v_9$ be the 6-cycle in H'. Let v_4v_7, v_5v_9, v_6v_8 be the remaining edges in H'. By Lemma 2, there is a vertex v_{10} in $G - H$ such that $w(v_{10}, H) \geq 37$. If $|B(v_{10}, H) \cup D(v_{10}, H) \geq 7$, at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{10}, H) \cup D(v_{10}, H)$. This implies $V(H) = R(v_{10}, H) \cup D(v_{10}, H)$ and $3 \leq |R(v_{10}, H| \leq 4$.

Assume, without loss of generality, that $v_4 \notin R(v_{10}, H)$. Since $\{v_4, v_6\}, \{v_4, v_8\}$ are deficient pairs, we have $\{v_6, v_8\} \subset R(v_{10}, H)$. Since $\{v_5, v_7\}, \{v_7, v_9\}$ are deficient pairs either $v_7 \in R(v_{10}, H)$ or $R(v_{10}, H) = \{v_5, v_6, v_8, v_9\}$.

Suppose $v_7 \in R(v_{10}, H)$. If any one of the vertices v_5, v_9, say v_5, is in $R(v_{10}, H)$, then $(H + v_{10}) - \{v_5, v_7, v_8\}$ is isomorphic to H_7^1. Lemmas 22, 32, 33, 34, 41, 42, 43, 44, 45 imply that G contains one of the graphs H_{10}^0, H_{10}^1, H_{10}^2, H_{10}^3. If either $|R(v_{10}, H)| = 3$ or one of the vertices v_1, v_2, v_3, say v_3, is in $R(v_{10}, H)$ then $(H + v_{10}) - \{v_3, v_5, v_7, v_8\}$ is isomorphic to H_6^1. Lemmas 17, 21, 22, 28, 29, 30, 31, 32, 33, 34, 39, 40, 41, 42, 43, 44, 45 imply that G contains one of the graphs H_{10}^0, H_{10}^1, H_{10}^2, H_{10}^3.

If $R(v_{10}, H) = \{v_5, v_6, v_8, v_9\}$, then $(H + v_{10}) - \{v_5, v_8\}$ is isomorphic to H_8^7, and by Lemmas 35, 40, 41, 42, 43, 44, 45, G contains one of the graphs H_{10}^0, H_{10}^1, H_{10}^2, H_{10}^3. $\square$

Lemmas 39, 40, 41, 42, 43, 44, 45, 46 complete the proof of Claim 38. $\square$

Claim 47 *G contains one of the graphs H_{11}^0, H_{11}^1, H_{11}^2, H_{11}^3, H_{11}^4, H_{11}^5, H_{11}^6.*

Proof: By Claim 38, G contains one of the graphs H_{10}^0, H_{10}^1, H_{10}^2, H_{10}^3. We consider each of these graphs.

Lemma 48 *G does not contain H_{10}^0.*

Proof: Let H be an induced subgraph of G isomorphic to H_{10}^0. Let the components of H_R be the triangle v_1, v_2, v_3 and H' and let $v_4, v_5, v_6, v_7, v_8, v_9, v_{10}$ be the 7-cycle in H'. Let $v_4v_6, v_5v_{10}, v_7v_9, v_8v_{10}$ be the remaining edges in H'.

Suppose there is a vertex v_{11} in $G-H$ such that $w(v_{11}, H) \geq 42$. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 8$, then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $3 \leq |R(v_{11}, H)| \leq 4$.

Since $\{v_1, v_i\}, \{v_2, v_i\}$, $i \in \{4, 5, 6, 7, 8, 9\}$, are deficient pairs of vertices, we have $\{v_1, v_2\} \subset R(v_{11}, H)$. Since $\{v_5, v_7\}, \{v_5, v_8\}, \{v_5, v_9\}$ and $\{v_6, v_8\}, \{v_6, v_9\}, \{v_6, v_{10}\}$ are deficient pairs, we have $\{v_5, v_6\} \subset R(v_{11}, H)$. Then $\{v_7, v_{10}\}$ is a deficient pair not including any vertex in $R(v_{11}, H)$, a contradiction.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 41$. Since $w(H) = 197$, by Lemma 3, with $X = \{v_1, v_2, v_3\}$ and $r = 1$, there is a vertex v_{11} in $G - H$ such that $w(v_{11}, H) \geq 40$ and $w(v_{11}, \{v_1, v_2, v_3\}) > 9$. Assume, without loss of generality, that the edge v_1v_{11} is double. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 7$, then v_1 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $4 \leq R(v_{11}, H) \leq 5$. However, since $\{v_1, v_i\}$, $i \in \{4, 5, 6, 7, 8, 9\}$, are deficient pairs and $v_1 \in D(v_{11}, H)$, we must have $|R(v_{11}, H)| \geq 6$, a contradiction. $\square$

Lemma 49 *G does not contain H^1_{10}.*

Proof: Let H be an induced subgraph of G isomorphic to H^1_{10}. Let v_3v_4 be the bridge in H_R and let the triangle v_1, v_2, v_3 and H' be the components of $H_R - v_3v_4$. Let $v_4, v_5, v_6, v_7, v_8, v_9, v_{10}$ be the 7-cycle in H' and let $v_4v_6, v_5v_{10}, v_7v_9, v_8v_{10}$ be the remaining edges in H'.

Suppose there is a vertex v_{11} in $G-H$ such that $w(v_{11}, H) \geq 42$. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 8$, then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $3 \leq |R(v_{11}, H)| \leq 4$.

Since $\{v_1, v_i\}, \{v_2, v_i\}$, $i \in \{4, 5, 6, 7, 8, 9\}$, are deficient pairs of vertices, we have $\{v_1, v_2\} \subset R(v_{11}, H)$. Since $\{v_5, v_7\}, \{v_5, v_8\}, \{v_5, v_9\}$ and $\{v_6, v_8\}, \{v_6, v_9\}, \{v_6, v_{10}\}$ are deficient pairs, we have $\{v_5, v_6\} \subset R(v_{11}, H)$. Then $\{v_7, v_{10}\}$ is a deficient pair not including any vertex in $R(v_{11}, H)$, a contradiction.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 41$. Since $w(H) = 195$, by Lemma 3, with $X = \{v_1, v_2\}$ and $r = 1$, there is a vertex v_{11} in $G - H$ such that $w(v_{11}, H) \geq 40$ and $w(v_{11}, \{v_1, v_2\}) > 6$. Assume, without loss of generality, that the edge v_1v_{11} is double. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 7$, then v_1 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $4 \leq R(v_{11}, H) \leq 5$. However, since $\{v_1, v_i\}$, $i \in \{4, 5, 6, 7, 8, 9\}$, are deficient pairs and $v_1 \in D(v_{11}, H)$, we must have $|R(v_{11}, H)| \geq 6$, a contradiction. $\square$

Lemma 50 *If G contains H^2_{10} then G contains one of the graphs H^0_{11}, H^1_{11}, H^2_{11}, H^3_{11}.*

Proof: Let H be an induced subgraph of G isomorphic to H^2_{10}. Let the components of H_R be the triangle v_1, v_2, v_3 and H' and let $v_4, v_5, v_6, v_7, v_8, v_9, v_{10}$ be the 7-cycle in H'. Let $v_4v_6, v_4v_9, v_5v_{10}, v_7v_9, v_8v_{10}$ be the remaining edges in H'.

Suppose there is a vertex v_{11} in $G-H$ such that $w(v_{11}, H) \geq 42$. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 8$, then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $3 \leq |R(v_{11}, H)| \leq 4$.

Since $\{v_5, v_7\}, \{v_6, v_8\}$ are disjoint deficient pairs of vertices, at least one of the vertices v_1, v_2, v_3, say v_3, is not in $R(v_{11}, H)$. Since $\{v_3, v_i\}$, $i \in \{5, 6, 7, 8\}$, are deficient pairs, we have $R(v_{11}, H) = \{v_5, v_6, v_7, v_8\}$ and $H + v_{11}$ is isomorphic to H^0_{11}.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 41$. Since $w(H) = 195$, by Lemma 3, with $X = \{v_1, v_2\}$ and $r = 1$, there is a vertex v_{11} in $G - H$ such that $w(v_{11}, H) \geq 40$ and $w(v_{11}, \{v_1, v_2\}) > 6$. Assume that the edge v_1v_{11} is double. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 7$, then v_1 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $4 \leq R(v_{11}, H) \leq 5$. However, since $\{v_1, v_i\}$, $i \in \{5, 6, 7, 8\}$, are deficient pairs and $v_1 \in D(v_{11}, H)$, we must have $\{v_5, v_6, v_7, v_8\} \subset R(v_{11}, H)$. If $|R(v_{11}, H)| = 4$ then $H + v_{11}$ is isomorphic to H^0_{11}. If any of the vertices v_2, v_3 is in $R(v_{11}, H)$ then $H + v_{11}$ is isomorphic to H^1_{11}. If $v_{10} \in R(v_{11}, H)$ then $H + v_{11}$ is isomorphic to H^2_{11} else $H + v_{11}$ is isomorphic to H^3_{11}. □

Lemma 51 *If G contains H^3_{10} then G contains one of the graphs H^4_{11}, H^5_{11}, H^6_{11}.*

Proof: Let H be an induced subgraph of G isomorphic to H^3_{10}. Let the components of H_R be the triangle v_1, v_2, v_3 and H' and let $v_4, v_5, v_6, v_7, v_8, v_9, v_{10}$ be the 7-cycle in H'. Let $v_4v_6, v_4v_8, v_5v_{10}, v_6v_{10}, v_7v_9$ be the remaining edges in H'.

Suppose there is a vertex v_{11} in $G - H$ such that $w(v_{11}, H) \geq 42$. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 8$, then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $3 \leq |R(v_{11}, H)| \leq 4$.

Since $\{v_5, v_7\}, \{v_6, v_8\}$ are disjoint deficient pairs of vertices, at least one of the vertices v_1, v_2, v_3, say v_3, is not in $R(v_{11}, H)$. Since $\{v_3, v_i\}$, $i \in \{5, 7, 8, 9\}$, are deficient pairs, we have $R(v_{11}, H) = \{v_5, v_7, v_8, v_9\}$ and $H + v_{11}$ is isomorphic to H^4_{11}.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 41$. Since $w(H) = 195$, by Lemma 3, with $X = \{v_1, v_2\}$ and $r = 1$, there is a vertex v_{11} in $G - H$ such that $w(v_{11}, H) \geq 40$ and $w(v_{11}, \{v_1, v_2\}) > 6$. Assume that the edge v_1v_{11} is double. If $|B(v_{11}, H) \cup D(v_{11}, H)| \geq 7$, then v_1 is a deficient vertex in $B(v_{11}, H) \cup D(v_{11}, H)$. This implies $V(H) = R(v_{11}, H) \cup D(v_{11}, H)$ and $4 \leq R(v_{11}, H) \leq 5$. Since $\{v_1, v_i\}$, $i \in \{5, 7, 8, 9\}$, are deficient pairs and $v_1 \in D(v_{11}, H)$, we must have $\{v_5, v_7, v_8, v_9\} \subset R(v_{11}, H)$. If $|R(v_{11}, H)| = 4$ then $H + v_{11}$ is isomorphic to H^4_{11}. If any of the vertices v_1, v_2, v_3 is in $R(v_{11}, H)$ then $H + v_{11}$ is isomorphic to H^5_{11} else $H + v_{11}$ is isomorphic to H^6_{11}. □

Lemmas 49, 50, 51 complete the proof of Claim 47. □

Claim 52 *G contains one of the graphs $H^0_{12}, H^1_{12}, H^2_{12}$.*

Proof: By Claim 47, G contains one of the graphs H^0_{11}, H^1_{11}, H^2_{11}, H^3_{11}, H^4_{11}, H^5_{11}, H^6_{11}. We consider each of these graphs.

Lemma 53 *G does not contain H^0_{11} or H^4_{11}.*

Proof: Let H be an induced subgraph of G isomorphic to either H^0_{11} or H^4_{11}. Suppose the triangle v_1, v_2, v_3 and H' are the two components of H_R. Note that H' has 8 vertices and is 4-regular.

Suppose there is a vertex v_{12} in $G-H$ such that $w(v_{12}, H) \geq 47$. If $|B(v_{12}, H) \cup D(v_{12}, H)| \geq 9$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{12}, H) \cup D(v_{12}, H)$. This implies $V(H) = R(v_{12}, H) \cup D(v_{12}, H)$ and $3 \leq R(v_{12}, H) \leq 4$. Since $\{v_1, v_i\}, \{v_2, v_i\}, \{v_3, v_i\}$, $v_i \in V(H')$, are deficient pairs, we must have $\{v_1, v_2, v_3\} \subset R(v_{12}, H)$. Since any non-adjacent pair of vertices in H' is a deficient pair, and there are at least two such disjoint deficient pairs, we must have $|R(v_{12}, H)| \geq 5$, a contradiction.

Suppose that for every vertex v in $G-H$, $w(v, H) \leq 46$. Since $w(H) = 237$, by Lemma 4, with $X = \{v_1, v_2, v_3\}$, $r = 2$ and $s = 1$, there is a vertex v_{12} in $G-H$ such that either $w(v_{12}, H) \geq 43$ and $w(v_{12}, \{v_1, v_2, v_3\}) > 9$ or $w(v_{12}, H) = 46$ and $w(v_{12}, \{v_1, v_2, v_3\}) = 9$.

Suppose at least one of the vertices v_1, v_2, v_3, say v_1, is in $D(v_{12}, H)$. If $|B(v_{12}, H) \cup D(v_{12}, H)| \geq 7$ then v_1 is a deficient vertex in $B(v_{12}, H) \cup D(v_{12}, H)$. Since $w(v_{12}, H) \geq 43$, this implies $V(H) = R(v_{12}, H) \cup D(v_{12}, H)$ and $5 \leq |R(v_{12}, H| \leq 6$. However, since $\{v_1, v_i\}$, $v_i \in V(H')$, are deficient pairs and $v_1 \in D(v_{12}, H)$, we must have $|R(v_{12}, H)| \geq 8$, a contradiction. The only other possibility is that $w(v_{12}, H) = 46$ and $\{v_1, v_2, v_3\} \subset R(v_{12}, H)$. This implies that $|D(v_{12}, H)| = 7$ and $|B(v_{12}, H)| = 1$ and $(\{v_1, v_2, v_3, v_{12}\}, \{v_i, v_j\})$ is a (4,2)-clique in G for any non-adjacent pair of vertices v_i, v_j in H'. □

Lemma 54 *G does not contain H^1_{11} or H^5_{11}.*

Proof: Let H be an induced subgraph of G isomorphic to either H^1_{11} or H^5_{11}. Let $v_3 v_4$ be the bridge in H_R and let the triangle v_1, v_2, v_3 and H' be the components of $H_R - v_3 v_4$. Note that H' has 8 vertices and is 4-regular.

Suppose there is a vertex v_{12} in $G-H$ such that $w(v_{12}, H) \geq 47$. If $|B(v_{12}, H) \cup D(v_{12}, H)| \geq 9$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{12}, H) \cup D(v_{12}, H)$. This implies $V(H) = R(v_{12}, H) \cup D(v_{12}, H)$ and $3 \leq R(v_{12}, H) \leq 4$. Since $\{v_1, v_i\}, \{v_2, v_i\}$, $v_i \in V(H')$, are deficient pairs, we must have $\{v_1, v_2\} \subset R(v_{12}, H)$. Since any pair of non-adjacent vertices in $H' - v_4$ is a deficient pair, and there are at least 3 such disjoint deficient pairs, we must have $|R(v_{12}, H)| \geq 5$, a contradiction.

Suppose there is a vertex v_{12} such that $w(v_{12}, H) \geq 43$ and $w(v_{12}, \{v_1, v_2\}) > 6$. We may assume that the edge $v_1 v_{12}$ is double. If $|B(v_{12}, H) \cup D(v_{12}, H)| \geq 7$ then v_1 is a deficient vertex in $B(v_{12}, H) \cup D(v_{12}, H)$. This implies $V(H) = R(v_{12}, H) \cup D(v_{12}, H)$ and $5 \leq |R(v_{12}, H| \leq 6$. Since $\{v_1, v_i\}$, $v_i \in V(H')$, are deficient pairs, we must have $|R(v_{12}, H)| \geq 8$, a contradiction.

Suppose that for every vertex v in $G-H$, $w(v, H) \leq 46$ and if $w(v, H) \geq 43$ then $w(v, \{v_1, v_2\}) \leq 6$. Since $w(H) = 235$, by Lemma 2, $w(H, G-H) > 44n - 477$. Let $A = \{v \in G-H \mid w(v, H) \geq 43\}$ and let $|A| = k$. Then $46k + 42(n-k-11) \geq w(H, G-H) > 44n - 477$, which implies $4k > 2n - 15$. Also, since $v_1 v_3$ and $v_2 v_3$ are red edges, $6k + 10(n-k-3) + 12 \geq w(v_1) + w(v_2) \geq 8n - 2$, which implies $4k \leq 2n - 16$, a contradiction. □

Lemma 55 *If G contains one of the graphs H^2_{11}, H^3_{11}, H^6_{11} then G contains one of the graphs H^0_{12}, H^1_{12}, H^2_{12}.*

Proof: Let H be an induced subgraph of G isomorphic to H^2_{11}, H^3_{11}, or H^6_{11}. Let the triangle

v_1, v_2, v_3 be one of the components of H_R and let H' be the other component. Note that H' has 6 vertices of degree 4 and 2 of degree 5. Let A denote the set of vertices of degree 4 in H'.

Suppose there is a vertex v_{12} in $G-H$ such that $w(v_{12}, H) \geq 47$. If $|B(v_{12}, H) \cup D(v_{12}, H)| \geq 9$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{12}, H) \cup D(v_{12}, H)$. This implies $V(H) = R(v_{12}, H) \cup D(v_{12}, H)$ and $3 \leq |R(v_{12}, H)| \leq 4$. Since $\{v_1, v_i\}, \{v_2, v_i\}, \{v_3, v_i\}$, $v_i \in A$, are deficient pairs, we must have $\{v_1, v_2, v_3\} \subset R(v_{12}, H)$. Since any non-adjacent pair of vertices in H' is a deficient pair, and there are at least two such disjoint deficient pairs, we must have $|R(v_{12}, H)| \geq 5$, a contradiction.

Suppose that for every vertex v in $G - H$, $w(v, H) \leq 46$. Since $w(H) = 235$, by Lemma 3, with $X = \{v_1, v_2, v_3\}$ and $r = 2$, there is a vertex v_{12} in $G - H$ such that $w(v_{12}, H) \geq 43$ and $w(v_{12}, \{v_1, v_2, v_3)\}) > 9$. Assume, without loss of generality, that $v_1 v_{12}$ is a double edge. If $|B(v_{12}, H) \cup D(v_{12}, H)| \geq 7$ then v_1 is a deficient vertex in $B(v_{12}, H) \cup D(v_{12}, H)$. This implies $V(H) = R(v_{12}, H) \cup D(v_{12}, H)$ and $5 \leq |R(v_{12}, H)| \leq 6$. Since $\{v_1, v_i\}$, $v_i \in A$, are deficient pairs and $v_1 \in D(v_{12}, H)$, we must have $R(v_{12}, H) = A$ and $H + v_{12}$ is isomorphic to H^0_{12}, H^1_{12} or H^2_{12}, if H is isomorphic to H^2_{11}, H^3_{11} or H^6_{11} respectively. □

Lemmas 53, 54, 55 complete the proof of Claim 52.

Claim 56 *G does not contain any of the graphs H^0_{12}, H^1_{12}, H^2_{12}.*

Proof: Let H be an induced subgraph of G isomorphic to one of the graphs H^0_{12}, H^1_{12}, H^2_{12}. Let the triangle v_1, v_2, v_3 and H' be the components of H_R. Note that H' has 9 vertices, one of which has degree 6 and all others have degree 5.

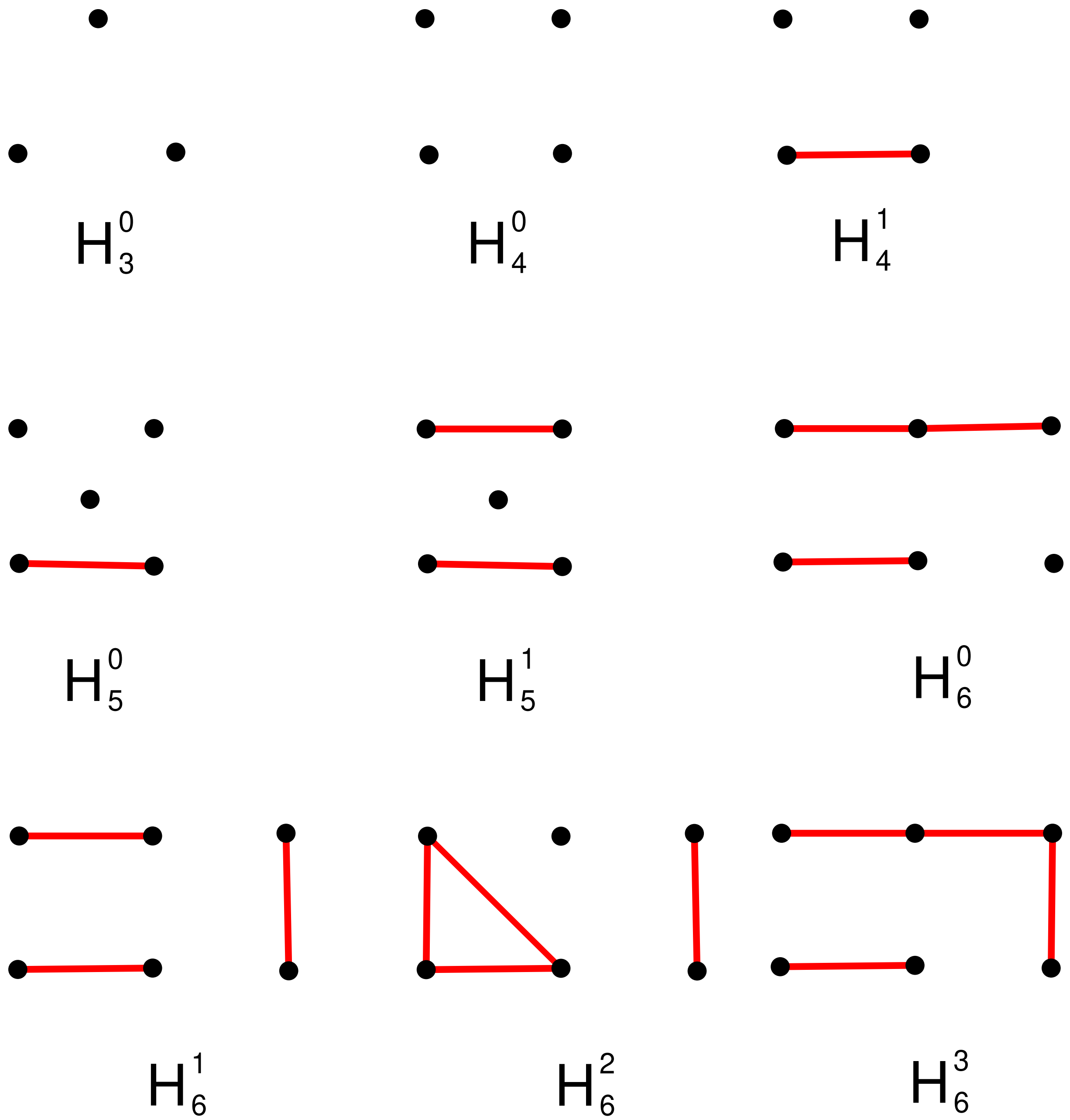
Suppose there is a vertex v_{13} in $G-H$ such that $w(v_{13}, H) \geq 52$. If $|B(v_{13}, H) \cup D(v_{13}, H)| \geq 10$ then at least one of the vertices v_1, v_2, v_3 is a deficient vertex in $B(v_{13}, H) \cup D(v_{13}, H)$. This implies $V(H) = R(v_{13}, H) \cup D(v_{13}, H)$ and $3 \leq |R(v_{13}, H)| \leq 4$. Since $\{v_1, v_i\}, \{v_2, v_i\}, \{v_3, v_i\}$ are deficient pairs for all 8 vertices v_i of degree 5 in H', we must have $\{v_1, v_2, v_3\} \subset R(v_{13}, H)$. Since every non-adjacent pair of vertices in H' is a deficient pair, and there are at least two such disjoint deficient pairs, at least two vertices from H' must be in $R(v_{13}, H)$, implying $|R(v_{13}, H)| \geq 5$, a contradiction.

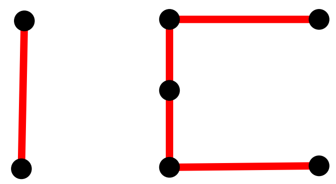
Suppose that for every vertex v in $G - H$, $w(v, H) \leq 51$. Since $w(H) = 278$, by Lemma 4, with $X = \{v_1, v_2, v_3\}$, $r = 3$ and $s = 2$, there is a vertex v_{13} in $G - H$ such that either $w(v_{13}, H) \geq 46$ and $w(v_{13}, \{v_1, v_2, v_3\}) > 9$ or $w(v_{13}, H) = 51$ and $w(v_{13}, \{v_1, v_2, v_3\}) = 9$.

Suppose one of the vertices v_1, v_2, v_3, say v_1, is in $D(v_{13}, H)$. If $|B(v_{13}, H) \cup D(v_{13}, H)| \geq 7$, then v_1 is a deficient vertex in $B(v_{13}, H) \cup D(v_{13}, H)$. Since $w(v_{13}, H) \geq 46$, this implies $V(H) = R(v_{13}, H) \cup D(v_{13}, H)$ and $6 \leq |R(v_{13}, H)| \leq 7$. However, there are 8 vertices v_i of degree 5 in H', and $\{v_1, v_i\}$ is a deficient pair for any such vertex v_i. Since $v_1 \in D(v_{13}, H)$, we must have $|R(v_{13}, H)| \geq 8$, a contradiction.

The only other possibility is that $w(v_{13}, H) = 51$ and $\{v_1, v_2, v_3\} \subset R(v_{13}, H)$. Then $|D(v_{13}, H)| = 8$, $|B(v_{13}, H)| = 1$ and $(\{v_1, v_2, v_3, v_{13}\}, \{v_i, v_j\})$ is a (4,2)-clique in G, for any non-adjacent pair of vertices v_i, v_j in H'. □

Claims 52, 56 now give the final contradiction and complete the proof of Theorem 1. □


H_3^0
H_4^0
H_4^1
H_5^0
H_5^1
H_6^0
H_6^1
H_6^2
H_6^3



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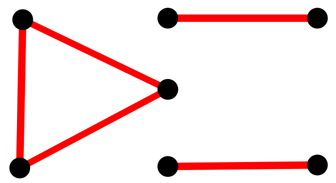
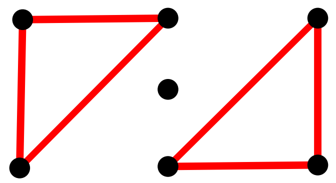


H_7^1



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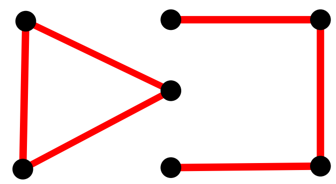

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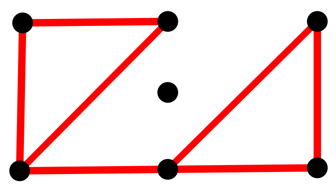
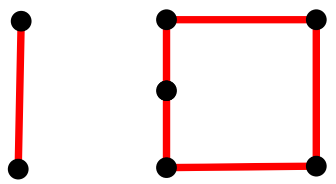


H_7^4



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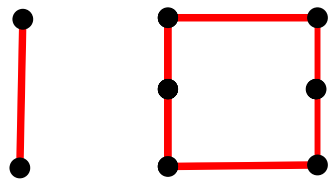

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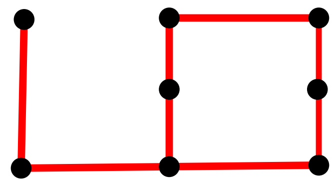

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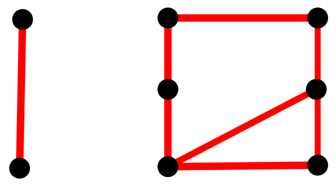

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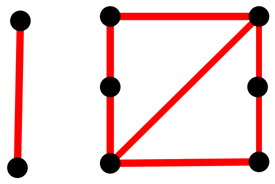

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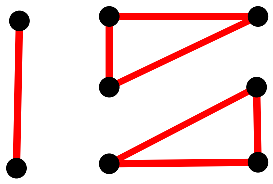

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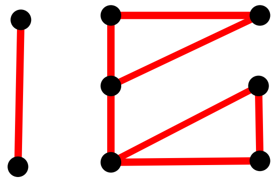

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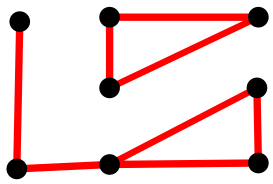

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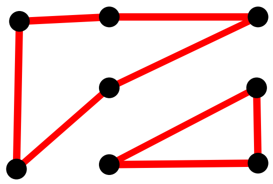

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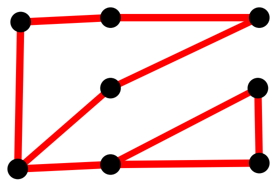

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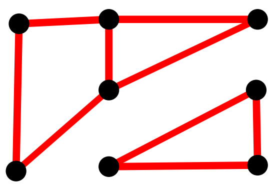

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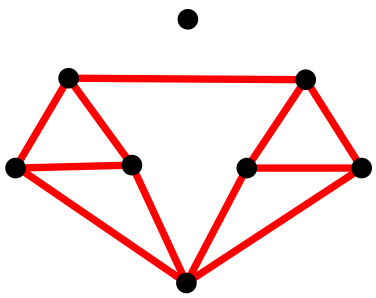

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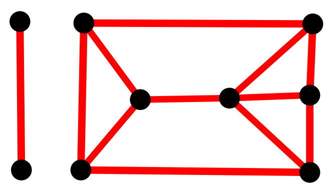

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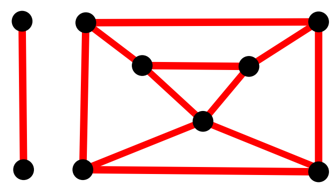

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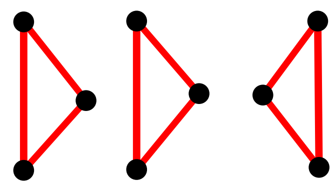

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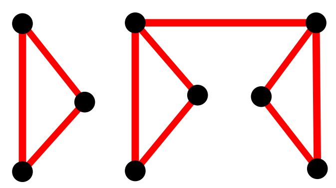

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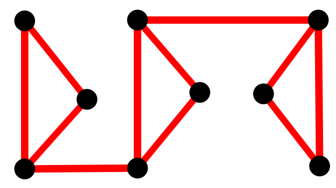

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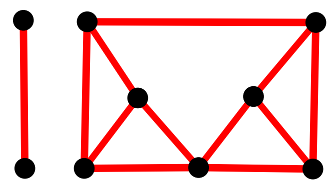

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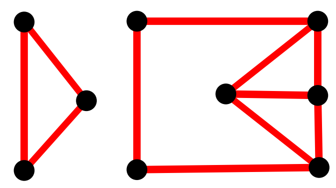

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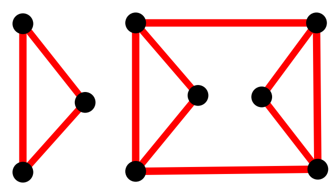

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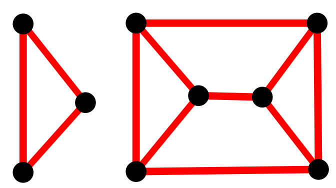

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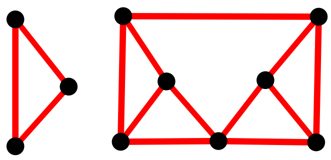

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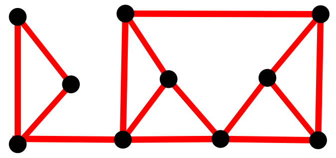

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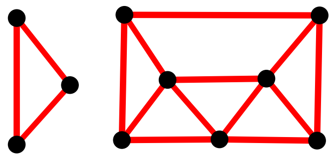

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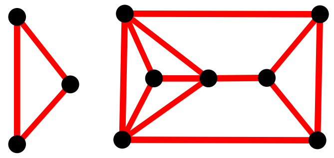

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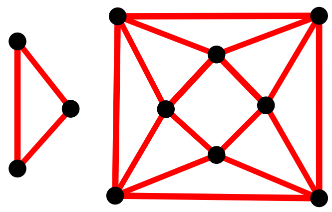

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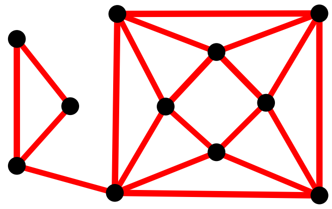

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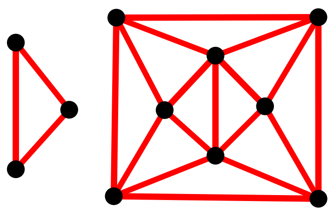

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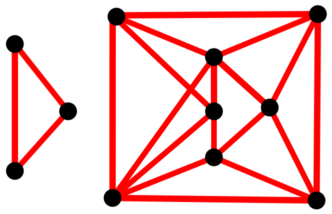

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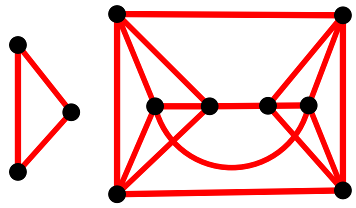

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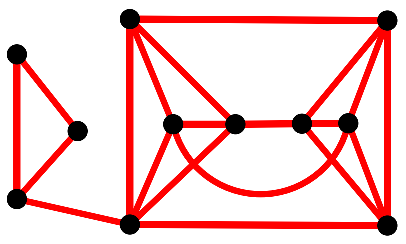
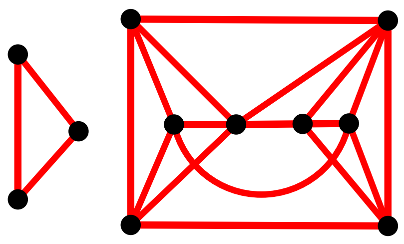
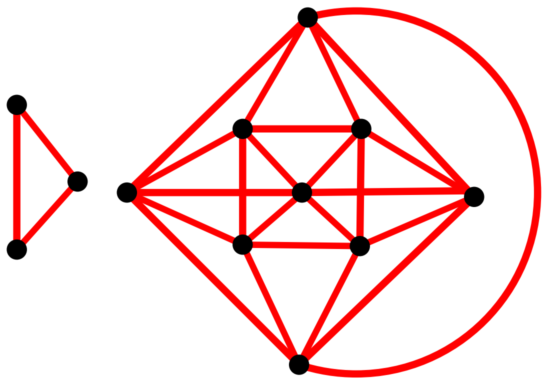


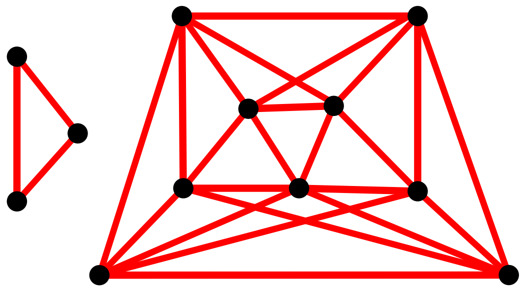
H^{5}_{11}



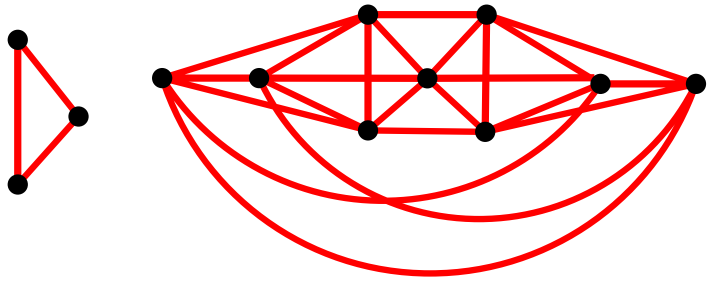
H^{6}_{11}



H^{0}_{12}



H^{1}_{12}



H^{2}_{12}