$\vec{P}_3$-decomposition of Directed Graphs

Ajit A. Diwan

Department of Computer Science and Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India.

Abstract

A $\vec{P}_3$-decomposition of a directed graph D is a partition of the arcs of D into directed paths of length 2. We characterize symmetric digraphs that do not admit a $\vec{P}_3$-decomposition. We show that the only 2-regular, connected directed graphs that do not admit a $\vec{P}_3$-decomposition are obtained from undirected odd cycles by replacing each edge by two oppositely directed arcs. In both cases, we give a linear-time algorithm to find a $\vec{P}_3$-decomposition, if it exists.

1. Introduction.

Let G be a graph and $G_1, G_2, \ldots, G_k$ be subgraphs of G. We say that the collection of subgraphs $G_1, G_2, \ldots, G_k$ is a decomposition of the graph G, if every edge in G is an edge in exactly one of the subgraphs. In other words, the subgraphs G_i are pairwise edge-disjoint, and their union is the graph G. If $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ is a family of graphs, an $\mathcal{F}$-decomposition of G is a decomposition of G into subgraphs, each of which is isomorphic to some graph in $\mathcal{F}$. If $\mathcal{F}$ contains a single graph H, an $\mathcal{F}$-decomposition is called an H-decomposition. In particular, a P_3-decomposition of a graph G is a partition of the edge set of G into paths of length 2.

The same notion of decomposition applies to directed graphs D as well, where we require each arc in D to be contained in exactly one of the subgraphs in the decomposition. A $\vec{P}_3$-decomposition of a directed graph is a partition of the arc set of D into directed paths of length 2.

A classical result on graph decomposition, originally noted by Kotzig [3], but now considered a simple exercise, is the following.

Email address: `aad@cse.iitb.ac.in` (Ajit A. Diwan)

Theorem 1. *A graph G has a P_3-decomposition iff every connected component of G has an even number of edges.*

However, no such characterization of directed graphs that admit a $\vec{P_3}$-decomposition is known, and the problem does not appear to have been studied much. This question is more difficult as even a simple directed graph may contain cycles of length 2, and two adjacent arcs do not necessarily form a $\vec{P_3}$. The problem of characterizing multigraphs that admit a P_3-decomposition was raised in [5], and some partial results were obtained in [1, 2, 4]. However, even this does not appear to have been solved in general, and we do not know of any such work for directed graphs.

It may be noted that all these problems can be solved in polynomial-time, by a reduction to the perfect matching problem. Given a graph (multigraph, directed graph), we construct a new graph whose vertices are the edges (arcs) in the given graph, and two vertices are adjacent in the new graph iff the corresponding edges (arcs) induce a path (directed path) of length 2 in the original graph. A $P_3(\vec{P_3})$-decomposition of the given graph (directed graph) then corresponds to a perfect matching in the new graph, and vice-versa.

While Tutte's 1-factor theorem [6] gives a necessary and sufficient condition for a graph to have a perfect matching, we would like a simple condition on the original graph (multigraph, directed graph) that guarantees a $P_3(\vec{P_3})$-decomposition. Theorem 1 gives such a condition for graphs. Further, while a perfect matching can be found in polynomial-time, we would like a simpler algorithm to find a $P_3(\vec{P_3})$-decomposition. The proof of Theorem 1 also gives a simple linear-time algorithm to find a P_3-decomposition of graphs.

A directed graph is said to be *symmetric* if for every pair of distinct vertices u, v, there is an arc from u to v iff there is an arc from v to u. In other words, the directed graph is obtained by replacing each edge in an undirected graph by two oppositely directed arcs. Our main result is a characterization of symmetric directed graphs that do not admit a $\vec{P_3}$-decomposition. The characterization is an explicit constructive characterization that describes the structure of symmetric directed graphs that do not admit a $\vec{P_3}$-decomposition. This also leads to a simple linear-time algorithm to find a $\vec{P_3}$-decomposition of a symmetric directed graph, if it exists.

A directed graph is 2-regular if every vertex has indegree and outdegree 2. We show that the only 2-regular, connected directed graphs, not necessarily symmetric, that do not admit a $\vec{P_3}$-decomposition, are obtained from undirected odd cycles by replacing each edge by two oppositely directed arcs.

Again, we give a linear-time algorithm to find a $\vec{P}_3$-decomposition of all other 2-regular, connected directed graphs.

In this paper, for ease of using induction, we will consider graphs that may have loops as well as multiple edges, and we will call them pseudo-graphs to emphasize the fact. A simple graph will be referred to as a graph. We assume that each edge in an undirected pseudo-graph has two ends, which could be the same vertex, if the edge is a loop. An edge with ends u, v will be denoted $\{u, v\}$. Similarly, every arc in a directed pseudo-graph will have two ends, one of which is called the head, and the other the tail. An arc with tail u and head v will be denoted (u, v).

We will consider decompositions of directed pseudo-graphs into subgraphs with two arcs. Any subgraph in such a decomposition will be denoted simply by the two arcs it contains. The family $\mathcal{F}$ of these directed pseudo-graphs is defined below.

Definition 1. Let $\mathcal{F}$ be the family of all directed pseudo-graphs with 2 arcs and no isolated vertices, such that the head of one of the arcs is the tail of the other. Thus $\mathcal{F}$ contains the five directed pseudo-graphs with arcs $(u, u), (u, u)$ and $(u, u), (u, v)$ and $(u, v), (v, v)$ and $(u, v), (v, u)$ and $(u, v), (v, w)$.

All other definitions and notations are standard.

2. Symmetric Directed Graphs.

If G is a pseudo-graph, let $D(G)$ denote the directed pseudo-graph obtained from G by replacing each edge $e = \{u, v\}$ in G, by two oppositely directed arcs, (u, v) and (v, u). We will call one of these arcs e^+ and the other e^- arbitrarily. Note that a loop in G is replaced by two loops in $D(G)$.

Let G be any pseudo-graph. An $\mathcal{F}$-decomposition of $D(G)$ is said to be *compatible*, if for every edge e in G, the arcs e^+ and e^- are contained in different subgraphs in the decomposition of $D(G)$.

Definition 2. Let $\mathcal{G}$ be the minimal set of pseudo-graphs that satisfies the following properties.

1. The two connected pseudo-graphs with one edge, $\{u, u\}$ and $\{u, v\}$, are in $\mathcal{G}$.

2. If G is a pseudo-graph in $\mathcal{G}$ and e an edge in G, then the pseudo-graph obtained by subdividing twice the edge e, is in $\mathcal{G}$. In other words, if $e = \{u, v\}$, then the pseudo-graph obtained from G by deleting the edge e, adding two new vertices x, y, and the edges $\{u, x\}, \{x, y\}, \{y, v\}$, is in $\mathcal{G}$.
3. If G is a pseudo-graph in $\mathcal{G}$ and v a vertex in G, then the pseudo-graph obtained from G by adding a new vertex x, and edges $\{x, v\}, \{x, x\}$ is in $\mathcal{G}$.
4. If G is a pseudo-graph in $\mathcal{G}$ and v a vertex in G, then the pseudo-graph obtained from G by adding two new vertices x, y, and edges $\{x, v\}, \{x, y\}$ is in $\mathcal{G}$.

We are now ready to state the main result.

Theorem 2. *For any connected pseudo-graph G, $D(G)$ has a compatible $\mathcal{F}$-decomposition iff $G \notin \mathcal{G}$.*

We note the following corollary.

Corollary 1. *For any connected graph G, $D(G)$ has a $\vec{P}_3$-decomposition iff $G \notin \mathcal{G}$.*

Proof. This follows from Theorem 2, since if G is a graph, then $D(G)$ does not have any loops, and any 2-cycle in $D(G)$ contains arcs e^+, e^- for some edge e in G. Therefore, any compatible $\mathcal{F}$-decomposition of $D(G)$ is in fact a $\vec{P}_3$-decomposition, and vice-versa. □

Before proving Theorem 2, we prove a few Lemmas that are used in the proof.

Lemma 1. *If G is a connected pseudo-graph with an even number of edges, then $D(G)$ has a compatible $\mathcal{F}$-decomposition.*

Proof. The proof is by induction on the number of edges. If G has only two edges, e and f, since G is connected, we may assume that in $D(G)$, head(e^+) = tail(f^-). Then the subgraphs of $D(G)$ with arcs e^+, f^- and e^-, f^+, form a compatible $\mathcal{F}$-decomposition of $D(G)$.

Suppose G has more than two edges. We will show that there exist two edges e, f in G, incident with a vertex v, such that $G - \{e, f\}$ has exactly one non-trivial component C. We may assume, without loss of generality, that $\text{head}(e^+) = \text{tail}(f^-) = v$. By induction, $D(C)$ has a compatible $\mathcal{F}$-decomposition, which together with the subgraphs having arcs e^+, f^- and e^-, f^+, gives a compatible $\mathcal{F}$-decomposition of $D(G)$.

Consider a depth-first-search tree T in G, and let v be the deepest vertex with degree at most one in T, and subject to this condition, has the largest degree in G. If there are two edges e, f not in T incident with v, then $G - \{e, f\}$ is connected, and we get the two required edges. If there is exactly one edge e not in T that is incident with v in G, then G must have at least two vertices and there exists an edge f in T incident with v. Again $G - \{e, f\}$ has two components, one of which is the isolated vertex v.

Suppose there is only an edge e that is incident with v. Then e must be in T and let u be the other end of e. If there exists an edge f not in T that is incident with u, then $G - \{e, f\}$ has two components, one of which is the isolated vertex v, and we get the two required edges. If u has a child $w \neq v$ in T, the choice of v implies that w also has degree one in G. Then $G - \{\{u, v\}, \{u, w\}\}$ has exactly one non-trivial component, and two isolated vertices v, w. Finally, if u has no child other than v in T, and no edge not in T incident with it, then it must have a parent w in T. Now $G - \{\{u, v\}, \{u, w\}\}$ has exactly one non-trivial component, and two isolated vertices u, v. □

Lemma 2. *Let G' be a pseudo-graph obtained from a pseudo-graph G by subdividing an edge twice. Then $D(G')$ has a compatible $\mathcal{F}$-decomposition iff $D(G)$ has one.*

Proof. Let $e = \{u, v\}$ be the subdivided edge in G. Without loss of generality, we can assume that $\text{head}(e^+) = v$ in $D(G)$. Let x, y be the new vertices added in G'. Suppose $D(G')$ has a compatible $\mathcal{F}$-decomposition. The subgraph containing the arc (x, y) in this decomposition must contain either the arc (u, x) or (y, v). Similarly, the subgraph containing (y, x) must contain either (x, u) or (v, y).

Suppose the subgraph containing (x, y) contains (y, v) and the subgraph containing (y, x) contains (v, y). Then by replacing the arc (u, x) by the arc e^+ and the arc (x, u) by e^- in the subgraphs containing them, and deleting the subgraphs $(x, y), (y, v)$ and $(y, x), (v, y)$, we get a compatible $\mathcal{F}$-decomposition of $D(G)$. Note that (u, x) and (x, u) must be in different subgraphs, due to the compatibility of the decomposition. A symmetrical argument holds if the subgraph containing (x, y) contains (u, x) and the subgraph containing (y, x) contains (x, u).

Suppose the subgraph containing (x, y) contains (y, v) and that containing (y, x) contains (x, u). Then replacing the arcs (u, x) by e^+ and (v, y) by e^- in the subgraphs containing them, and deleting the subgraphs $(x, y), (y, v)$ and $(y, x), (x, u)$, gives a compatible $\mathcal{F}$-decomposition of G. Note that (u, x) and (v, y) cannot be contained in the same subgraph as $\text{head}((u, x)) \neq \text{tail}((v, y))$ and $\text{head}((v, y)) \neq \text{tail}((u, x))$. A symmetrical argument holds if the subgraph containing (x, y) contains (u, x) and the subgraph containing (y, x) contains (v, y). Thus in all cases, we get a compatible $\mathcal{F}$-decomposition of $D(G)$.

Conversely suppose $D(G)$ has a compatible $\mathcal{F}$-decomposition. Then the subgraph containing the arc e^+ must contain an arc f such that either $\text{head}(f) = u$ or $\text{tail}(f) = v$. Note that $f \neq e^-$ due to compatibility of the decomposition. Suppose $\text{head}(f) = u$. Replace the arc e^+ in this subgraph by the arc (u, x) and add the subgraph containing the arcs $(x, y), (y, v)$. If $\text{tail}(f) = v$, replace e^+ by (y, v) and add the subgraph with arcs $(u, x), (x, y)$. A similar replacement for the arc e^-, gives a compatible $\mathcal{F}$-decomposition of $D(G')$. □

Lemma 3. *Let G be a pseudo-graph and v a vertex in G. Let G' be obtained from G by adding a new vertex x and edges $\{v, x\}, \{x, x\}$. Then $D(G')$ has a compatible $\mathcal{F}$-decomposition iff $D(G)$ has one.*

Proof. Suppose $D(G')$ has a compatible $\mathcal{F}$-decomposition. There are two loops in $D(G')$ at the vertex x, and they cannot be contained in the same subgraph in a compatible $\mathcal{F}$-decomposition, since they come from the same edge $\{x, x\}$ in G'. Therefore the subgraph containing one of the loops must contain the arc (v, x) and the subgraph containing the other loop must contain the arc (x, v). The remaining subgraphs form a compatible $\mathcal{F}$-decomposition of $D(G)$.

Conversely, if $D(G)$ has a compatible $\mathcal{F}$-decomposition, adding the subgraphs with arcs $(v, x), (x, x)$ and $(x, v), (x, x)$ gives the required decomposition of $D(G')$. □

Lemma 4. *Let G be a pseudo-graph and v a vertex in G. Let G' be obtained from G by adding two new vertices x, y and edges $\{v, x\}, \{x, y\}$. Then $D(G')$ has a compatible $\mathcal{F}$-decomposition iff $D(G)$ has one.*

Proof. Any compatible $\mathcal{F}$-decomposition of $D(G')$ must include the subgraphs with arcs $(v, x), (x, y)$ and $(y, x), (x, v)$. The remaining subgraphs form a compatible $\mathcal{F}$-decomposition of $D(G)$. Conversely, adding these subgraphs to a compatible $\mathcal{F}$-decomposition of $D(G)$ gives the required decomposition of $D(G')$. $\square$

Proof (Theorem 1). We first show that if $G \in \mathcal{G}$ then $D(G)$ does not have a compatible $\mathcal{F}$-decomposition. The proof is by induction on the number of edges in G. This is clear if G contains only one edge. Otherwise, G is obtained from some pseudo-graph $G' \in \mathcal{G}$ by either subdividing an edge twice, or by adding one or two new vertices and two edges. By induction, G' does not have a compatible $\mathcal{F}$-decomposition, and Lemmas 2, 3, 4 imply that G also does not have a compatible $\mathcal{F}$-decomposition.

To prove the converse, suppose there exists a pseudo-graph $G \notin \mathcal{G}$ that does not have a compatible $\mathcal{F}$-decomposition. Consider such a counterexample G with minimum number of edges. Lemma 1 implies that G has an odd number of edges greater than one.

Suppose there exist 3 edges e, f, g that are incident with a vertex v, and $G - \{e, f, g\}$ has no component with odd number of edges. Then for every component C of $G - \{e, f, g\}$, Lemma 1 implies that $D(C)$ has a compatible $\mathcal{F}$-decomposition. We may assume, without loss of generality, that v is the tail of the arcs e^+, f^+, and g^+. Then the subgraphs with arcs e^+, f^- and f^+, g^- and g^+, e^-, together with the decompositions of the components of $G - \{e, f, g\}$, give a compatible $\mathcal{F}$-decomposition of $D(G)$, a contradiction.

Suppose there are no such edges. We will show that G is obtained from a pseudo-graph G' using one of the operations in Definition 2. Then Lemmas 2, 3, 4 imply that G' does not have a compatible $\mathcal{F}$-decomposition, and by Definition 2, $G' \notin \mathcal{G}$. This contradicts the choice of G.

Again, consider a depth-first-search tree T in G, and let v be the deepest vertex in T that has degree at most one in T, and subject to this condition, has the largest degree in G. If there are 3 or more edges incident with v in G, we can find 3 edges e, f, g such that $G - \{e, f, g\}$ has at most one non-trivial component, a contradiction.

Suppose there are two edges e, f incident with v. We may assume e is not in T and f is in T. Let u be the other end of f.

If e is a loop, then G is obtained from $G - v$ by adding the vertex v and the edges $\{v, u\}, \{v, v\}$. Then $G - v$ is the required graph G'. So we may assume e is not a loop.

Suppose the other end of e is also u. If there is an edge g not in T incident with u, then $G - \{e, f, g\}$ has at most one non-trivial component. If u has another child w in T, then the choice of v implies that w also has degree one in T, and degree at most two in G. By the same argument as applied to v, we may assume there is no loop incident with w. Then e, f and $\{u, w\}$ are 3 edges such that deleting them gives only one non-trivial component, a contradiction. If u has no other child, and no edge not in T incident with it, then u must have a parent w in T, and again deleting the edges $e, f, \{u, w\}$ gives only one non-trivial component.

Suppose the other end of e is a vertex different from u, v. If u has an edge not in T incident with it, then deleting it leaves G connected. If u has another child w, we may assume that w has degree one in T, and degree at most two in G. Further, if there is any edge not in T that is incident with w, its other end is different from u, w. Thus deleting the edge $\{u, w\}$ can give at most one non-trivial component. Note that u must have a parent in T. Thus if u has 4 or more edges incident with it, deleting any 3 of them, other than the edge joining u to its parent in T, will result in a pseudo-graph with at most one non-trivial component. If there are 3 edges incident with u, then deleting all the 3 of them will give at most one non-trivial component. The only other possibility is that u has exactly 2 edges incident with it, f and the edge joining u to its parent in T. In this case, let G' be the graph obtained from $G - \{u, v\}$, by adding an edge with ends the parent of u in T, and the end of e other than v. Then G is obtained from G' by subdividing this added edge twice.

Finally, suppose v has degree one in G. If u has any other child in T, the choice of v implies it has degree one in G. Thus if u has 3 or more edges incident with it, we can find 3 whose deletion gives at most one non-trivial component. The only other possibility is that u has 2 edges incident with it in G, and must have degree 2 in G. Let G' be the graph $G - \{u, v\}$. Then G is obtained from G' by adding two vertices and two edges. This gives the required graph G', and completes the proof of Theorem 2. $\square$

Note that it is straightforward to modify the proof of Theorem 2 to get a linear-time algorithm to find a compatible $\mathcal{F}$-decomposition of $D(G)$, if it exists. This simply requires a post-order traversal on the depth-first-search tree, and applying a suitable reduction to the pseudo-graph, if necessary. The depth-first-search tree is also reduced to find a depth-first-search tree in the reduced pseudo-graph. Each reduction step can be done in constant time. This process will stop, when we either reach a pseudo-graph with one edge, in which case no compatible $\mathcal{F}$-decomposition exists, or we get a pseudo-graph with an even number of edges. The proof of Lemma 1 gives a linear-time algorithm for finding a compatible $\mathcal{F}$-decomposition of $D(G)$ when G has an even number of edges. The decomposition of the reduced pseudo-graph can then be extended to get a decomposition of the original pseudo-graph in linear-time.

While Theorem 2 gives a constructive characterization of pseudo-graphs G for which $D(G)$ does not have a compatible $\mathcal{F}$-decomposition, we can also give a direct structural characterization of pseudo-graphs G for which $D(G)$ has such a decomposition.

Theorem 3. *Let G be a connected pseudo-graph. Then $D(G)$ has a compatible $\mathcal{F}$-decomposition iff either G has an even number of edges, or there exist 3 edges incident with a common vertex, such that deleting them gives a pseudo-graph with no component having an odd number of edges.*

PROOF. The sufficiency follows from Lemma 1. To prove the necessity, it is sufficient to observe that if G is a pseudo-graph obtained from G' using one of the operations in Definition 2, then G has 3 edges incident with a common vertex, such that deleting them gives no component with odd number of edges, iff G' has such edges. $\square$

3. 2-regular Directed Graphs

In this section, we characterize 2-regular directed graphs that admit a $\vec{P}_3$-decomposition. We show that the only connected, 2-regular directed graphs that do not have a $\vec{P}_3$-decomposition are the symmetric directed graphs $D(C_{2n+1})$, for $n \geq 1$, where C_l denotes the undirected cycle of length l.

As in section 2, we need to extend the definitions to directed pseudo-graphs. Let D be a directed pseudo-graph and let $\mathcal{C}$ be a collection of arc-disjoint 2-cycles in D. Let $\mathcal{F}$ be the family of directed pseudo-graphs defined in Definition 1. An $\mathcal{F}$-decomposition of D is said to be $\mathcal{C}$-compatible if no 2-cycle in $\mathcal{C}$ is a subgraph in the decomposition. Note that a compatible $\mathcal{F}$-decomposition of a symmetric directed pseudo-graph $D(G)$ is a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition, where $\mathcal{C}$ is the collection of 2-cycles with arcs e^+, e^- for each edge e in G.

Theorem 4. *Let D be a connected, 2-regular directed pseudo-graph and let $\mathcal{C}$ be a collection of arc-disjoint 2-cycles in D. Then D has a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition unless D is $D(C_{2n+1})$ for some $n \geq 1$, and $\mathcal{C}$ contains all 2-cycles in D.*

We note the following corollary of Theorem 4.

Corollary 2. *Let D be a connected, 2-regular directed graph. Then D has a $\vec{P}_3$-decomposition unless D is $D(C_{2n+1})$ for some $n \geq 1$.*

Proof. This follows from Theorem 4 by taking $\mathcal{C}$ to be the set of all 2-cycles in D. If D is a directed graph, then the 2-cycles in D are arc-disjoint, and since D has no loops, any $\mathcal{C}$-compatible $\mathcal{F}$-decomposition must be a $\vec{P}_3$-decomposition of D, and vive versa. □

Proof (Theorem 4). The proof is by induction on the number of vertices in D. If D has only one vertex, then there are no 2-cycles in D and $\mathcal{C}$ is empty, and D itself forms a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D. If D has 2 vertices u and v, then the arcs in D are either $(u,u),(u,v),(v,u),(v,v)$ or $(u,v),(v,u),(u,v),(v,u)$. In the first case, the subgraphs with arcs $(u,u),(u,v)$ and $(v,u),(v,v)$ give a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D. In the second case, since the 2-cycles in $\mathcal{C}$ are arc-disjoint, there are at most two 2-cycles in $\mathcal{C}$. We can then choose two different 2-cycles to partition the arc set of D, and get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

Suppose D has 3 or more vertices. If $\mathcal{C}$ is empty, we find an arbitrary Euler tour in D, and partition the arc set into consecutive pairs of arcs in the tour. Since any two consecutive arcs in the tour form a subgraph in $\mathcal{F}$, this gives a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

Suppose $\mathcal{C}$ contains a 2-cycle with arcs $e = (u,v)$ and $f = (v,u)$. We consider all possible configurations of the other arcs incident with u and v.

Case 1. Suppose there is a loop incident with either u or v. Note that since D is 2-regular, connected and has at least 3 vertices, there cannot be a loop at both u and v. Without loss of generality, assume there is a loop at v. Let $(a,u),(u,b)$ be the other arcs incident with u, where $a,b \notin \{u,v\}$ are not necessarily distinct vertices.

Let D' be the pseudo-graph obtained from $D - \{u,v\}$ by adding an arc $g = (a,b)$. Let $\mathcal{C}'$ be the collection of 2-cycles in $\mathcal{C}$ that do not contain either u or v. Since D' is a 2-regular, connected pseudo-graph, and the arc g in D' is not contained in any 2-cycle in $\mathcal{C}'$, by induction, D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. The subgraph containing the arc g in this decomposition must contain another arc g', such that either $\text{head}(g') = a$, or $\text{tail}(g') = b$. If $\text{head}(g') = a$, replace this subgraph by the subgraph with arcs $g',(a,u)$ and add the subgraphs with arcs $f,(u,b)$ and $e,(v,v)$ to get a $\mathcal{C}$-compatible

$\mathcal{F}$-decomposition of D. If $\text{tail}(g') = b$, replace the subgraph with arcs g, g' by one with arcs $g', (u, b)$ and add the subgraphs with arcs $(a, u), e$ and $f, (v, v)$ to get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

Case 2. Suppose there is another arc g in D with ends u and v. Without loss of generality, $g = (u, v)$. Then D must contain arcs (a, u) and (v, b) where $a, b \notin \{u, v\}$ are not necessarily distinct vertices. Note that since the 2-cycles in $\mathcal{C}$ are arc-disjoint, g cannot be contained in any 2-cycle in $\mathcal{C}$.

Let D' be the pseudo-graph obtained from $D - \{u, v\}$ by adding an arc $h = (a, b)$. Let $\mathcal{C}'$ be the collection of 2-cycles in $\mathcal{C}$ that do not contain either u or v. Since D' is a 2-regular, connected pseudo-graph, and the arc h in D' is not contained in any 2-cycle in $\mathcal{C}'$, by induction, D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. The subgraph containing the arc h in this decomposition must contain another arc h', such that either $\text{head}(h') = a$, or $\text{tail}(h') = b$. If $\text{head}(h') = a$, replace this subgraph by the subgraph with arcs $h', (a, u)$ and add the subgraphs with arcs $e, (v, b)$ and f, g to get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D. If $\text{tail}(h') = b$, replace the subgraph with arcs h, h' by one with arcs $h', (v, b)$ and add the subgraphs with arcs $(a, u), e$ and f, g to get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

Case 3. Suppose there are 4 distinct arcs apart from e, f incident with $\{u, v\}$. Let these arcs be $(a, u), (u, b), (c, v), (v, d)$ for some, not necessarily distinct, vertices a, b, c, d. The distinctness of the arcs implies that $a, b, c, d \notin \{u, v\}$. We consider all possibilities for the vertices a, b, c, d, taking into account the symmetry between vertices u and v, and also, if necessary, reversing the directions of all arcs in D.

Case 3.1. Suppose $a = b = c = d$. In this case, D must be the graph $D(C_3)$. If $\mathcal{C}$ contains all 2-cycles in $D(C_3)$, no $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of $D(C_3)$ is possible. If at least one 2-cycle is not in $\mathcal{C}$, we can choose it as one of the subgraphs along with two $\vec{P}_3$ subgraphs, to get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of $D(C_3)$.

Case 3.2. Suppose $a = b = c$ and $a \neq d$. Consider the directed pseudograph D' obtained from $D - \{u, v\}$ by adding a loop g at a, and an arc $h = (a, d)$. Again, let $\mathcal{C}'$ be the set of 2-cycles in $\mathcal{C}$ that do not contain u or v. Since D' is 2-regular, connected and not isomorphic to $D(C_{2n+1})$ for $n \geq 1$, by induction, D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. If g, h is a subgraph in this decomposition, we delete it, and add the subgraphs $(a,u),e$ and $f,(u,a)$ and $(a,v),(v,d)$ to get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D. Otherwise, the subgraph containing the arc g must contain another arc g' such that either the head or tail of g' is a. Similarly, the subgraph containing h must contain an arc h' such that either $\mathrm{head}(h')=a$ or $\mathrm{tail}(h')=d$.

Suppose $\mathrm{head}(g')=\mathrm{head}(h')=a$. Replace the subgraphs g,g' and h,h' by the subgraphs $g',(a,u)$ and $h',(a,v)$ and add the subgraphs $f,(u,a)$ and $e,(v,d)$ to get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

If $\mathrm{head}(g')=a$ and $\mathrm{tail}(h')=d$, replace g,g' by $g',(a,v)$ and h,h' by $(v,d),h'$ and add the subgraphs $(a,u),e$ and $f,(u,a)$.

If $\mathrm{tail}(g')=a$ and $\mathrm{head}(h')=a$, replace g,g' by $(u,a),g'$ and h,h' by $(a,u),h'$, and add the subgraphs $(a,v),f$ and $e,(v,d)$.

If $\mathrm{tail}(g')=a$ and $\mathrm{tail}(h')=d$, replace g,g' by $(u,a),g'$ and h,h' by $(v,d),h'$ and add the subgraphs $(a,u),e$ and $(a,v),f$.

In all cases, these replacements give a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

Case 3.3. $a=b$ and $c=d$, but $a\neq c$. Let D' be the 2-regular, connected directed pseudo-graph obtained from $D-\{u,v\}$ by adding an arc $g=(a,c)$ and an arc $h=(c,a)$. Let $\mathcal{C}'$ contain all 2-cycles in $\mathcal{C}$ that do not contain u or v, along with the 2-cycle g,h. If D' is isomorphic to $D(C_{2n+1})$ for some $n\geq 1$, then D is isomorphic to $D(C_{2(n+1)+1})$. If $\mathcal{C}$ contains all 2-cycles in D, then no $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of $D(C_{2n+3})$ is possible. If at least one 2-cycle in $D(C_{2n+3})$ is not in $\mathcal{C}$, we can choose that as one of the subgraphs, and decompose the rest of the graph into directed paths of length 2.

We may therefore assume that D' is not isomorphic to $D(C_{2n+1})$ for any $n\geq 1$. By induction, D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. The subgraph containing the edge g in this decomposition must contain an edge g' such that $\mathrm{head}(g')=a$ or $\mathrm{tail}(g')=c$. Note that since the 2-cycle g,h is in $\mathcal{C}'$, the arc g' must be distinct from h. Similarly, the subgraph containing h contains an arc h' such that $\mathrm{head}(h')=c$ or $\mathrm{tail}(h')=a$, and $h'\neq g$.

If $\mathrm{head}(g')=a$ and $\mathrm{head}(h')=c$, replace g,g' by $g',(a,u)$ and h,h' by $h',(c,v)$ and add the subgraphs $f,(u,a)$ and $e,(v,c)$.

If $\mathrm{head}(g')=a$ and $\mathrm{tail}(h')=a$, replace g,g' by $g',(a,u)$ and h,h' by $(u,a),h'$ and add the subgraphs $e,(v,c)$ and $(c,v),f$.

The other cases can be argued symmetrically. This gives a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D, unless D is isomorphic to $D(C_{2n+3})$ and all 2-cycles in D are contained in $\mathcal{C}$.

Case 3.4. $a = c$ and $b = d$ but $a \neq b$. Let D' be the directed pseudo-graph obtained from $D - \{u, v\}$ by adding two arcs $g = (a, b)$ and $h = (a, b)$. Let $\mathcal{C}'$ be the collection of 2-cycles in $\mathcal{C}$ that do not contain u or v. Since D' is 2-regular, connected and not isomorphic to $D(C_{2n+1})$ for any $n \geq 1$, D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. The subgraph containing the arc g in this decomposition contains an arc g' such that head$(g') = a$ or tail$(g') = b$. Similarly, h, h' is a subgraph for an arc h' such that head$(h') = a$ or tail(h') $= b$.

If head$(g') = a$ and head$(h') = a$, replace g, g' by $g', (a, u)$ and h, h' by $h', (a, v)$ and add the subgraphs $e, (v, b)$ and $f, (u, b)$.

If head$(g') = a$ and tail$(h') = b$, replace g, g' by $g', (a, u)$ and h, h' by $(u, b), h'$ and add the subgraphs $(a, v), f$ and $e, (v, b)$.

The other cases can be handled symmetrically. Thus in all cases, we get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

Case 3.5. $a = d$ and $b = c$ but $a \neq b$. Let D' be the pseudo-graph obtained from $D - \{u, v\}$ by adding a loop $g = (a, a)$ and a loop $h = (b, b)$. Let $\mathcal{C}'$ be the set of 2-cycles in $\mathcal{C}$ that do not contain u or v. Note that in this case, D' may not be connected, but each connected component of D' is 2-regular, and not isomorphic to $D(C_{2n+1})$ for any $n \geq 1$. Applying induction to each component of D', we get a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition of D'. If g, g' is a subgraph in this decomposition then head$(g') = a$ or tail$(g') = a$. Similarly, if h, h' is a subgraph in the decomposition, then head$(h') = b$ or tail$(h') = b$.

If head$(g') = a$ and head$(h') = b$, replace g, g' by $g', (a, u)$ and h, h' by $h', (b, v)$ and add the subgraphs $e, (v, a)$ and $f, (u, b)$.

If head$(g') = a$ and tail$(h') = b$, replace g, g' by $g', (a, u)$ and h, h' by $(u, b), h'$ and add the subgraphs $e, (v, a)$ and $f, (b, v)$.

The other cases can be argued symmetrically. Thus in all cases, we get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D.

Case 3.6. $a = b$ but all other vertices are distinct. Let D' be the pseudo-graph obtained from $D - \{u, v\}$ by adding arcs $g = (c, a)$ and $h = (a, d)$. Let $\mathcal{C}'$ be the set of 2-cycles in $\mathcal{C}$ that do not contain u or v. Then D' is a 2-regular, connected pseudo-graph and at least one arc in D' is not contained in any 2-cycle in $\mathcal{C}'$. By induction, D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. If g, h is a subgraph in this decomposition, we delete it, and add the subgraphs

$(a,u),e$ and $f,(u,a)$ and $(c,v),(v,d)$. Otherwise, if g,g' is a subgraph in this decomposition, then head$(g') = c$ or tail$(g') = a$. If h,h' is a subgraph in the decomposition of D', then head$(h') = a$ or tail$(h') = d$.

If head$(g') = c$ and head$(h') = a$, replace g,g' by $g',(c,v)$ and h,h' by $h',(a,u)$ and add the subgraphs $e,(v,d)$ and $f,(u,a)$.

If head$(g') = c$ and tail$(h') = d$, replace g,g' by $g',(c,v)$ and h,h' by $h',(v,d)$ and add the subgraphs $(a,u),e$ and $f,(u,a)$.

If tail$(g') = a$ and head$(h') = a$, replace g,g' by $g',(u,a)$ and h,h' by $h',(a,u)$ and add the subgraphs $e,(v,d)$ and $(c,v),f$.

The remaining case is symmetrical. Again, we get a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D in all cases.

Case 3.7. $a = c$ and all other vertices are distinct. Let D' be the pseudograph obtained from $D - \{u,v\}$ by adding arcs $g = (a,b)$ and $h = (a,d)$. Let $\mathcal{C}'$ be the collection of 2-cycles in $\mathcal{C}$ that do not contain u or v. By similar arguments as in previous cases, D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. If g,g' is a subgraph in this decomposition, then head$(g') = a$ or tail$(g') = b$. If h,h' is a subgraph in the decomposition of D', then head$(h') = a$ or tail(h') $= d$.

If head$(g') = a$ and head$(h') = a$, replace g,g' by $g',(a,u)$ and h,h' by $h',(a,v)$ and add the subgraphs $e,(v,d)$ and $f,(u,b)$.

If head$(g') = a$ and tail$(h') = d$, replace g,g' by $g',(a,v)$ and h,h' by $h',(v,d)$ and add the subgraphs $(a,u),e$ and $f,(u,b)$.

If tail$(g') = b$ and tail$(h') = d$, then replace g,g' by $(u,b),g'$ and h,h' by $(v,d),h'$ and add the subgraphs $(a,u),e$ and $(a,v),f$.

The remaining case can be handled by symmetrical arguments. This gives a $\mathcal{C}$-compatible $\mathcal{F}$-decomposition of D in all cases.

Case 3.8. $a = d$ and all other vertices are distinct. Let D' be the pseudograph obtained from $D - \{u,v\}$ by adding a loop $g = (a,a)$ and an arc $h = (c,b)$. Let $\mathcal{C}'$ be the collection of 2-cycles in $\mathcal{C}$ that do not contain u or v. Again, in this case, D' may not be connected, but each component of D' is 2-regular. By similar arguments as in previous cases, each component of D', and hence D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. If g,g' is a subgraph in this decomposition, then head$(g') = a$ or tail$(g') = a$. If h,h' is a subgraph in the decomposition of D', then head$(h') = c$ or tail$(h') = b$.

If head$(g') = a$ and head$(h') = c$, replace g,g' by $g',(a,u)$ and h,h' by $h',(c,v)$ and add the subgraphs $e,(v,a)$ and $f,(u,b)$.

If head$(g') = a$ and tail$(h') = b$, replace g, g' by $g', (a, u)$ and h, h' by $h', (u, b)$ and add the subgraphs $e, (v, a)$ and $(c, v), f$.

The other cases can be argued by symmetrical arguments.

Case 3.9. All 4 vertices a, b, c, d are distinct. Let D' be the pseudo-graph obtained from $D - \{u, v\}$ by adding arcs $g = (a, d)$ and $h = (c, b)$. Let $\mathcal{C}'$ be the collection of 2-cycles in $\mathcal{C}$ that do not contain u or v. Again, in this case, D' may not be connected, but each component of D' is 2-regular. By similar arguments as in previous cases, each component of D', and hence D' has a $\mathcal{C}'$-compatible $\mathcal{F}$-decomposition. If g, g' is a subgraph in this decomposition, then head$(g') = a$ or tail$(g') = d$. If h, h' is a subgraph in the decomposition of D', then head$(h') = c$ or tail$(h') = b$.

If head$(g') = a$ and head$(h') = c$, replace g, g' by $g', (a, u)$ and h, h' by $h', (c, v)$ and add the subgraphs $e, (v, d)$ and $f, (u, b)$.

If head$(g') = a$ and tail$(h') = b$, replace g, g' by $g', (a, u)$ and h, h' by $h', (u, b)$ and add the subgraphs $e, (v, d)$ and $(c, v), f$.

The other cases can be argued by symmetrical arguments.

This completes all possible cases and hence the proof of Theorem 4. $\square$

We note briefly that the proof of Theorem 4 also gives a linear-time algorithm to find a $\vec{P}_3$-decomposition of a 2-regular directed graph, if it exists. The reduction at each step can be applied in constant time, until $\mathcal{C}$ becomes empty. Then a decomposition can be found for each component of the reduced graph, by finding an Euler tour in each component. Finally, the decomposition of the reduced graph can be modified to get the decomposition of the original graph. Again, this modification can be done in constant time for each reduction.

4. Remarks

It would be interesting to see if a characterization of $\vec{P}_3$-decomposable directed graphs can be obtained for general directed graphs, or other classes of directed graphs, such as tournaments. In particular, it would be interesting to find a faster algorithm to decide the existence of such a decomposition, without resorting to perfect matchings.

References

[1] J. Ivančo, M. Meszka, Z. Skupień, Decomposition of multigraphs into isomorphic graphs with two edges, Ars Combin. 51 (1999) 105-112.

[2] J. Ivančo, M. Meszka, Z. Skupień, Decomposition of multigraphs into parts with two edges, Discuss. Math. Graph Theory 22 (2002) 113-121.

[3] A. Kotzig, From the theory of finite regular graphs of degree three and four, Časopis Pěst Mat 82 (1957) 76-92.

[4] M. Kouider, M. Mahéo, K. Bryś, Z. Lonc, Decomposition of multigraphs, Discuss. Math. Graph Theory 18 (1998) 225-232.

[5] Z. Skupień, Problem 270 [On 2-edge decomposable multigraphs], in: A. Rycerz, A.P. Wojda, and M. Woźniak, eds., The Second Kraków Conf. on Graph Theory (Proc. Conf. Zgorzelisko 1994), Discrete Math. 164 (1997) 320-321.

[6] W.T. Tutte, The factorisation of linear graphs, J. London Math. Society 22 (1947) 107-111.