Coloured matchings in edge-coloured graphs

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Abstract

Erdős and Gallai proved that every graph of order n with more than $f(k,n) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$ edges contains a matching with k edges. We generalize this and show that if R (red) and B (blue) are graphs on the same vertex set of size n, each with more than $f(k,n)$ edges, then the edge-coloured multigraph $R \cup B$ contains any 2-edge-coloured matching with k edges. In general, we prove that for $n \geq 3k-1$, if $G_1, G_2, \ldots, G_t$ are graphs on the same vertex set of size n, such that the edges of G_i are coloured i and G_i has more than $f(k,n)$ edges for all $1 \leq i \leq t$, then the edge-coloured multigraph $G = G_1 \cup G_2 \cup \cdots \cup G_t$ contains any t-edge-coloured matching with k edges. We also generalize the corresponding minimum degree result of Erdős and Pósa. If $n \geq 2k$ and $\delta(G_i) \geq k$ for all $1 \leq i \leq t$, then G contains any t-edge-coloured matching with k edges.

1 Introduction

The fundamental question in *extremal graph theory* is to determine the maximum number of edges that a graph on n vertices can have if it does not contain a specified graph H as a subgraph. This number is denoted by $\mathrm{ex}(n,H)$. The classical Erdős–Simonovits–Stone theorem [5, 6] determines $\mathrm{ex}(n,H)$, at least asymptotically, for all non-bipartite graphs H. In the bipartite case, this number is known for very few families of graphs.

One of the families of bipartite graphs for which $\mathrm{ex}(n,H)$ is known exactly is matchings. Erdős and Gallai [3] proved that

$$\mathrm{ex}(n,M) = f(k,n) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

for a matching M with k edges. Brandt [1] later showed that $\mathrm{ex}(n,F) \leq f(k,n)$ for any forest F with k edges and no isolated vertices, as was conjectured by Erdős and Sós [2].

We extend the fundamental question of extremal graph theory to edge-coloured graphs. Let $G_1, G_2, \ldots, G_t$, $t \geq 1$ be graphs on the same vertex set of size n. We think of G_i as a graph whose edges are coloured i, and the multigraph $G = G_1 \cup G_2 \cup \cdots \cup G_t$ as an edge-coloured multigraph. For any graph H, we denote by $\mathrm{ex}(n, H, t)$ the smallest number m such that if each of the graphs $G_1, G_2, \ldots, G_t$ has more than m edges, then for every t-edge-colouring (not necessarily proper) of H with colours $\{1, 2, \ldots, t\}$, G contains the corresponding coloured copy of H.

Clearly $\mathrm{ex}(n, H) = \mathrm{ex}(n, H, 1)$ and $\mathrm{ex}(n, H, t{+}1) \geq \mathrm{ex}(n, H, t)$. The phenomenon that we investigate is that for certain graphs H, we have $\mathrm{ex}(n, H, t) = \mathrm{ex}(n, H)$ for all $t \geq 1$. We show that $\mathrm{ex}(n, M, 2) = \mathrm{ex}(n, M, 1) = \mathrm{ex}(n, M)$ for any matching M and all n, and if $n \geq 3k-1$ then $\mathrm{ex}(n, M, t) = \mathrm{ex}(n, M)$ for all $t \geq 1$ and a matching M with k edges. We believe that for all n, $\mathrm{ex}(n, M, t) = \mathrm{ex}(n, M)$ for all $t \geq 1$ and any matching M.

A similar generalization can be considered for the extremal values of other graph parameters, in particular, the minimum degree. Erdős and Pósa [4] showed that every graph of order at least $2k$ and minimum degree at least k contains a matching with k edges. We show that if $G_1, G_2, \ldots, G_t$ are graphs on the same vertex set of size at least $2k$, such that the edges of G_i are coloured i and $\delta(G_i) \geq k$ for all $1 \leq i \leq t$, then the edge-coloured multigraph $G = G_1 \cup G_2 \cup \cdots \cup G_t$ contains every t-edge-coloured matching with k edges.

Brandt [1] generalized the theorem of Erdős and Pósa and showed that if F is any forest with n vertices and k edges then every graph of order at least n and minimum degree at least k contains F. We believe that a similar generalization holds for edge-coloured forests.

2 Edge Bound

Theorem 1 *Let R (red) and B (blue) be graphs on the same set of vertices of size n and let k be a positive integer. If both R and B have more than*

$$f(k, n) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

edges, then the edge-coloured multigraph $G = R \cup B$ contains a matching with i red edges and $k - i$ blue edges for all $0 \leq i \leq k$.

Proof: We first note that if $i = 0$ or $i = k$, the result follows by applying the Erdős–Gallai theorem to the graphs B or R respectively. We may assume by symmetry that $i \leq k/2$, otherwise the same argument can be applied by interchanging the graphs R and B. We will prove by induction that for all $1 \leq i \leq k/2$, G contains a matching with i red and $k - i$ blue edges. Note that

$$f(k, n) = \begin{cases} \binom{2k-1}{2} & \text{if } n \leq \frac{5k}{2} - 1. \\ \binom{k-1}{2} + (k-1)(n-k+1) & \text{if } n \geq \frac{5k}{2} - 1. \end{cases}$$

Basis

The Erdős–Gallai theorem implies that B and hence G contains a matching M with k blue edges. Let X be the set of vertices that are matched by M and let $Y = V(G) \setminus X$. Suppose there is a vertex in Y and a red edge incident with it. We can replace an edge in M by this red edge to get a matching with one red and $k-1$ blue edges. Similarly, if a red edge is parallel to an edge in M we can swap them to get the required matching.

We may assume that every red edge is adjacent to two edges in M. Since the number of red edges is more than $f(k,n) \geq \binom{2k-1}{2}$, and $|X| = 2k$, there is a red edge incident with every vertex in X.

Suppose there is a vertex in Y and a blue edge incident with it. We can replace an edge in M by this edge to get a new matching M' with k blue edges. However, since there is a vertex in X not matched by M', and every vertex in X is incident with a red edge, there is red edge in G adjacent to at most one edge in M'. Replacing an edge in M' by this edge gives a matching with one red and $k-1$ blue edges.

We may therefore assume that the vertices in Y, if any, are isolated vertices and every edge in G is either in M or adjacent to two edges in M. The total number of edges in G is

$$|E(G)| > 2f(k,n) \geq 2(2k-1)(k-1) \geq 2k(k-1) + k = 4\binom{k}{2} + k.$$

This implies that there are two edges e, f in M such that there are at least 5 edges in G that are adjacent to both e and f. We can find two disjoint edges e', f' in G that are adjacent to both e and f, such that e' is red and f' is blue. Replacing the edges e, f in M by the edges e', f' gives a matching in G with one red and $k-1$ blue edges.

Induction

Suppose for some i, $1 \leq i \leq k/2 - 1$, G contains a matching M with i red edges and $k-i$ blue edges. We will show that G contains a matching with $i+1$ red and $k-i-1$ blue edges. Let X be the set of vertices matched by the red edges, Y the set of vertices matched by the blue edges, and $Z = V(G) \setminus (X \cup Y)$.

Case 1: $n \leq \frac{5k}{2} - 1$

In this case $f(k,n) = \binom{2k-1}{2}$. Adding isolated vertices to G if necessary, we may assume without loss of generality that $n = \left\lfloor \frac{5k}{2} \right\rfloor - 1$. Thus $|X| = 2i$, $|Y| = 2(k-i)$ and $|Z| = \left\lfloor \frac{k}{2} \right\rfloor - 1$.

If there is a red edge in G joining two vertices in Z, or a red edge joining a vertex in Z to a vertex in Y, we can replace a blue edge in M by this edge to get a matching with $i+1$ red and $k-i-1$ blue edges. If $R[X \cup Z]$ has more than $f(i+1, 2i + \left\lfloor \frac{k}{2} \right\rfloor - 1)$ edges, then $R[X \cup Z]$ contains a matching with $i+1$ red edges. The edges in this matching, together with $k-i-1$ blue edges in M gives the required matching in G.

We note that

$$f(i+1, 2i + \left\lfloor \frac{k}{2} \right\rfloor - 1) = \binom{i}{2} + i\left(i + \left\lfloor \frac{k}{2} \right\rfloor - 1\right) \tag{1}$$

unless

$$2i + \left\lfloor \frac{k}{2} \right\rfloor - 1 < \frac{5(i+1)}{2} - 1.$$

Since $k \geq 2i+2$, (1) holds if $i \geq 3$ or $k \geq 2i+4$.

Consider a red edge e and a blue edge f in M. If there are 3 red edges in G that are adjacent to both e and f, then there are two disjoint red edges e', f' in G that are adjacent to both e and f. Replacing the edges e, f in M by e', f' gives a matching in G with $i+1$ red and $k-i-1$ blue edges. Similarly if a red edge is parallel to a blue edge in M, swapping the two edges gives the required matching.

Therefore, if $i \geq 3$ or $k \geq 2i+4$, the total number of red edges in G is at most

$$\begin{aligned} |E(R)| &\leq \binom{i}{2} + i\left(i + \left\lfloor \frac{k}{2} \right\rfloor - 1\right) + 2i(k-i) + 4\binom{k-i}{2} \\ &= \frac{3i^2}{2} - i\left(\left\lceil \frac{3k}{2} \right\rceil - \frac{1}{2}\right) + 2k(k-1) \end{aligned} \tag{2}$$

The expression (2) is a monotonically decreasing function of i, for any fixed n, k and $1 \leq i \leq k/2 - 1$ and hence is maximum when $i = 1$. Therefore, substituting $i = 1$ in expression (2), we get

$$|E(R)| \leq 2k^2 - \left\lceil \frac{7k}{2} \right\rceil + 2.$$

However, $|E(R)| > f(k,n) = 2k^2 - 3k + 1$, which gives a contradiction.

If $i = 1$ and $4 \leq k \leq 5$, then Z contains only one vertex and $R[X \cup Z]$ has at most 3 edges. If $k = 4$, then $|E(R)| \leq 3 + 2\cdot 3 + 4 \cdot 3 = 21 = f(4,9)$, while if $k = 5$, $|E(R)| \leq 3 + 2 \cdot 4 + 4 \cdot 6 = 35 < f(5,11)$. In both cases we get a contradiction.

Similarly, if $i = 2$ and $6 \leq k \leq 7$, then Z contains two vertices and $R[X \cup Z]$ has at most $f(3,6) = 10$ edges. If $k = 6$, then $|E(R)| \leq 10+2\cdot 2\cdot 4+4\cdot 6 = 50 < f(6,14)$, while if $k = 7$, $|E(R)| \leq 10 + 2 \cdot 2 \cdot 5 + 4 \cdot 10 = 70 < f(7,16)$. Again, in both cases we get a contradiction.

Case 2: $n > \frac{5k}{2} - 1$

The argument is essentially the same as in the previous case. The graph $R[X \cup Z]$ has at most $f(i+1, n-2k+2i)$ edges and there is no red edge joining a vertex in Z to a vertex in $Z \cup Y$. It is easy to check that for all $1 \leq i \leq k/2 - 1$,

$$f(i+1, n-2k+2i) = \binom{i}{2} + i(n-2k+i).$$

Therefore,

$$\begin{aligned} |E(R)| &\leq \binom{i}{2} + i(n-2k+i) + 2i(k-i) + 4\binom{k-i}{2} \\ &= \frac{3i^2}{2} + i\left(n - 4k + \frac{3}{2}\right) + 2k(k-1). \end{aligned} \tag{3}$$

The expression (3) is a quadratic function of i with positive second derivative, and hence for any fixed n and k, (3) is maximum if either $i = 1$ or $i = k/2 - 1$. Substituting $i = 1$ and $i = k/2 - 1$ in (3), we get

$$|E(R)| \leq \max\left\{n + 2k^2 - 6k + 3\ ,\left(\frac{k}{2} - 1\right)n + \frac{3k^2 + 10k}{8}\right\}. \tag{4}$$

However,

$$\begin{aligned} |E(R)| > f(k,n) &= \binom{k-1}{2} + (k-1)(n-k+1) \\ &= (k-1)n - k(k-1)/2. \end{aligned} \tag{5}$$

Combining inequalities (4) and (5), we must have either

$$n + 2k^2 - 6k + 3 > (k-1)n - k(k-1)/2 \tag{6}$$

or

$$\left(\frac{k}{2} - 1\right)n + \frac{3k^2 + 10k}{8} > (k-1)n - k(k-1)/2. \tag{7}$$

However, inequality (6) implies $(k-2)n < (5k^2 - 13k + 6)/2$ and hence $n < (5k-3)/2$, contradicting $n > (5k-2)/2$. Similarly, inequality (7) implies $kn/2 < (7k^2 + 6k)/8$ and hence $n < 7k/4 + 3/2$. Since $n > 5k/2 - 1$, this implies that $k < 4$, contradicting the fact that $i \geq 1$ and $k \geq 2i + 2$.

This completes the proof of Theorem 1. □

We now prove that if n is sufficiently large compared to the number of edges in a matching M, then $\mathrm{ex}(n, M, t) = \mathrm{ex}(n, M)$.

Theorem 2 *Let $k, t \geq 1$ and $n \geq 3k-1$ be integers. Then ex$(n, M, t) = \mathrm{ex}(n, M) = f(k, n)$ for a matching M with k edges.*

Proof: It is sufficient to show that if $G_1, G_2, \ldots, G_k$ are graphs on the same vertex set of size $n \geq 3k - 1$ with more than $f(k, n)$ edges each, then the edge-coloured multigraph $G = G_1 \cup G_2 \cdots G_k$ contains a multicoloured matching with k edges of distinct colours. Any specified colouring with $t < k$ colours can be obtained from this by considering some of the G_i to be the same graphs.

We will prove this by induction on k. It clearly holds for $k = 1$. Since $n \geq 3k-1$, we have $f(k, n) = \binom{k-1}{2} + (k-1)(n-k+1)$.

Suppose there is a vertex v such that the degree of v is at least $2k - 1$ in some G_i, without loss of generality, say G_1. Then $G_i - v$ has more $f(k, n) - (n-1) = f(k-1, n-1)$ edges for all $1 \leq i \leq k$. Since $(n-1) \geq 3(k-1) - 1$, by induction, the edge-coloured multigraph $(G_2 \cup G_3 \cup \cdots \cup G_k) - v$ contains a multicoloured matching M' with $k-1$ edges of colours $2, 3, \ldots, k$. Since the degree of v in G_1 is at least $2k - 1$, there is a neighbour u of v in G_1 such that u is not incident with any edge in M'. Then $M' \cup uv$ is a multicoloured matching in G with k edges of distinct colour.

Suppose $\Delta(G_i) \leq 2k-2$ for all $1 \leq i \leq k$. Let uv be any edge in G_1. Then $G_i - \{u,v\}$ contains more than $f(k,n) - 4k + 4$ edges. Since $n \geq 3k-1$, we have

$$\begin{aligned} f(k,n) - 4k + 4 &= \binom{k-1}{2} + (k-1)(n-k+1) - 4k + 4 \\ &= \binom{k-2}{2} + (k-2)(n-k) + n - 3k + 1 \\ &\geq f(k-1, n-2). \end{aligned}$$

Since $n - 2 \geq 3(k-1) - 1$, by induction, $(G_2 \cup G_3 \cup \cdots \cup G_k) - \{u,v\}$ contains a multicoloured matching M' with $k-1$ edges of colours $2, 3, \ldots, k$. Then $M' \cup uv$ is a multicoloured matching in G with k edges of distinct colours. □

3 Degree Bound

Theorem 3 *Let $G_1, G_2, \ldots, G_t$ be graphs on the same vertex set of size at least $2k$, where k is a positive integer. The edges of G_i are assumed to be coloured i for all $1 \leq i \leq t$. Let $k_1, k_2, \ldots, k_t$ be non-negative integers such that $k = \sum_{i=1}^{t} k_i$. If $\delta(G_i) \geq k$ for all $1 \leq i \leq t$, then the edge-coloured multigraph $G = G_1 \cup G_2 \cup \cdots \cup G_t$ contains a matching with exactly k_i edges of colour i for all $1 \leq i \leq t$.*

Proof: We prove the theorem by induction on k. Clearly it is true for $k = 1$. Assume that $k > 1$ and the theorem is true for $k-1$. We may assume without loss of generality that $k_1 > 0$, and by the induction hypothesis, G contains a matching M with $k_1 - 1$ edges of colour 1, and k_i edges of colour i for all $2 \leq i \leq t$. Let X be the subset of vertices matched by M and let $Y = V(G) \setminus X$. Since G has at least $2k$ vertices and $|X| = 2k-2$, $|Y| \geq 2$. Let u_0, v_0 be any two vertices in Y and let $w_0 = v_0$, $c_0 = 1$.

Suppose that G does not contain the required matching having k_i edges of colour i for all $1 \leq i \leq t$. We now prove the following claim that immediately gives a contradiction.

Claim 4 *For every positive integer l, there exists a sequence of $l+1$ 4-tuples (u_i, v_i, w_i, c_i), $0 \leq i \leq l$ satisfying the following properties:*

1. *There is an edge $u_i v_i$ that is contained in M and has colour c_i for all $1 \leq i \leq l$.*
2. *$\{u_i, v_i\} \cap \{u_j, v_j\} = \emptyset$, for all $0 \leq i < j \leq l$.*
3. *The vertex $w_i \in \{u_{i-1}, w_{i-1}\}$ for all $1 \leq i \leq l$ and $w_1 = u_0$.*
4. *There are edges $u_{i-1} v_i$ and $w_{i-1} v_i$ in G with colour c_{i-1} for all $1 \leq i \leq l$.*
5. *If $w_i = u_{i-1}$ then $w_{i-1} u_i$ is an edge in G with color c_{i-1} otherwise $w_i = w_{i-1}$ and $u_{i-1} u_i$ is an edge in G with colour c_{i-1} for all $1 \leq i \leq l$.*

Note that since M contains $k-1$ edges, any sequence satisfying the properties 1 and 2 in Claim 4 has at most k 4-tuples, contradicting the claim. We prove Claim 4 by induction on l.

Basis

If the subgraph $G_1[Y]$ is not empty then any edge of colour 1 in $G_1[Y]$, together with M, gives the required matching in G, contradicting the fact that G does not contain the required matching.

Since $\delta(G_1) \geq k$ and $G_1[Y]$ is empty, there exists an edge u_1v_1 in M such that

$$|E_1(\{u_0, v_0\}, \{u_1, v_1\})| \geq 3.$$

We may assume without loss of generality that u_0v_1, v_0v_1 and v_0u_1 are edges in G coloured $c_0 = 1$. Let c_1 be the colour of the edge u_1v_1. Then (u_0, v_0, v_0, c_0), (u_1, v_1, u_0, c_1) is a sequence satisfying all the properties in Claim 4. □

Induction

Let (u_i, v_i, w_i, c_i), $0 \leq i \leq l$ be a sequence of $l + 1$ 4-tuples satisfying the properties stated in Claim 4. Let

$$Y_i = \bigcup_{j=0}^{i} \{u_j, v_j\} \quad \text{for all } 0 \leq i \leq l.$$

Lemma 5 *For all $0 \leq i < l$, $G[(Y_i \cup \{v_{i+1}\}) \setminus \{u_i\}]$ contains a matching with $i + 1$ edges that are coloured $c_0, c_1, \ldots, c_i$.*

Proof: For $i = 0$, the Lemma holds since the edge v_0v_1 has colour c_0. Suppose that for some $i > 0$, the Lemma holds for all $0 \leq j < i$. By property 4 in Claim 4, w_iv_{i+1} is an edge coloured c_i and by property 3 we may assume that $w_i = u_j$ for some $0 \leq j < i$. Replace the edge u_iv_i by the edge w_iv_{i+1} and the edges u_pv_p by the edges u_pv_{p+1} for all $j < p < i$. These edges, together with the matching in $G[(Y_j \cup \{v_{j+1}\}) \setminus \{u_j\}]$ containing $j + 1$ edges of colours $c_0, c_1, \ldots, c_j$ gives the required matching in $G[(Y_i \cup \{v_{i+1}\}) \setminus \{u_i\}]$. Hence, by induction, the Lemma is true for all $0 \leq i < l$. □

Lemma 6 *For all $0 \leq i < l$, $G[(Y_i \cup \{v_{i+1}\}) \setminus \{v_i\}]$ contains a matching with $i + 1$ edges that are coloured $c_0, c_1, \ldots, c_i$.*

Proof: For $i = 0$, the Lemma holds since the edge u_0v_1 has colour c_0. Similarly for $i = 1$, $\{v_0u_1, u_0v_2\}$ is the required matching. Suppose for some $i > 1$, the Lemma holds for all $0 \leq j < i$. By property 4 in Claim 4, w_iv_{i+1} is an edge coloured c_i and by property 3 w_i is either u_{i-1} or w_{i-1}.

Suppose $w_i = u_{i-1}$. Since $i > 1$, by property 3 $w_{i-1} = u_j$ for some $0 \leq j < i - 1$ and by property 5 $w_{i-1}u_i$ is an edge coloured c_{i-1}. Replace the edge u_iv_i by the edge w_iv_{i+1} and the edge $u_{i-1}v_{i-1}$ by the edge $w_{i-1}u_i$. For all $j < p < i - 1$, replace the edges u_pv_p by the edges u_pv_{p+1}. By Lemma 5, $G[(Y_j \cup \{v_{j+1}\}) \setminus \{u_j\}]$ has a matching with $j + 1$ edges of colours $c_0, c_1, \ldots, c_j$. This gives a matching in $G[(Y_i \cup \{v_{i+1}\}) \setminus \{v_i\}]$ with $i + 1$ edges coloured $c_0, c_1, \ldots, c_i$.

Suppose $w_i = w_{i-1} = u_j$, for some $0 \leq j < i - 1$. In this case, $u_{i-1}u_i$ is an edge coloured c_{i-1}. Replace the edge u_iv_i by the edge w_iv_{i+1} and the edge $u_{i-1}v_{i-1}$

by the edge $u_{i-1}u_i$. For all $j < p < i-1$, replace the edges u_pv_p by the edges u_pv_{p+1}. By Lemma 5, $G[(Y_j \cup \{v_{j+1}\}) \setminus \{u_j\}]$ has a matching with $j+1$ edges of colours $c_0, c_1, \ldots, c_j$. This gives a matching in $G[(Y_i \cup \{v_{i+1}\}) \setminus \{v_i\}]$ with $i+1$ edges coloured $c_0, c_1, \ldots, c_i$. This proves the Lemma for all $0 \le i < l$. □

Lemma 7 *The degree of u_l in the graph $G_{c_l}[Y \cup Y_l]$ is one and u_lv_l is the only edge of colour c_l incident with u_l in $G[Y \cup Y_l]$.*

Proof: Suppose there is an edge coloured c_l joining u_l to a vertex $x \in Y \setminus \{u_0\}$. Replace the edges u_iv_i in M by the edges $u_{i-1}v_i$ for all $1 \le i \le l$ and add the edge u_lx. This gives a matching in G containing exactly k_i edges of colour i for all $1 \le i \le t$.

Suppose there is an edge coloured c_l joining u_l to a vertex u_i, for some $0 \le i < l$. By Lemma 5, $G[(Y_i \cup \{v_{i+1}\}) \setminus \{u_i\}]$ contains a matching with $i+1$ edges of colours $c_0, c_1, \ldots, c_i$. This matching, together with the edges u_jv_{j+1} for all $i < j < l$, the edge u_iu_l and the edges of M in $G - Y_l$ gives the required matching in G.

Finally, suppose there is an edge coloured c_l joining u_l to a vertex v_i for some $0 < i < l$. Replace the edge u_lv_l by the edge u_lv_i and the edges u_jv_j by the edges u_jv_{j+1} for all $i < j < l$. By Lemma 6, $G[(Y_i \cup \{v_{i+1}\}) \setminus \{v_i\}]$ has a matching with $i+1$ edges of colours $c_0, c_1, \ldots, c_i$. This matching, along with the edges of M in $G - Y_l$ gives the required matching in G. □

Lemma 8 *The degree of w_l in $G_{c_l}[Y \cup Y_l]$ is at most $2l-1$.*

Proof: Let $w_l = u_i$ for some $0 \le i < l$. Suppose there is an edge in G coloured c_l joining w_l to a vertex $x \in (Y \cup \{u_l\}) \setminus \{u_0, v_0\}$. Replace the edge u_lv_l by the edge w_lx and the edges u_jv_j by the edges u_jv_{j+1} for all $i < j < l$. By Lemma 5, $G[(Y_i \cup \{v_{i+1}\}) \setminus \{u_i\}]$ contains a matching with $i+1$ edges coloured $c_0, c_1, \ldots, c_i$. This matching together with the edges of M contained in $G - Y_l$ gives the required matching in G.

Suppose there is an edge coloured c_l joining w_l to v_l. If $l = 1$, replacing the edge u_1v_1 in M by the edges u_0v_1 and v_0u_1 with colours c_1 and c_0 respectively, gives the required matching in G. Suppose $l > 1$ and $w_{l-1} = u_j$ for some $0 \le j < l-1$.

If $w_l = u_{l-1}$ then $w_{l-1}u_l$ is an edge in G with colour c_{l-1}. Replace the edge u_lv_l by the edge w_lv_l (coloured c_l) and the edge $u_{l-1}v_{l-1}$ by the edge $w_{l-1}u_l$. Replace the edges u_pv_p by the edges u_pv_{p+1} for all $j < p < l-1$. By Lemma 5, there is a matching in $G[(Y_j \cup \{v_{j+1}\}) \setminus \{u_j\}]$ with $j+1$ edges coloured $c_0, c_1, \ldots, c_j$. This matching, together with the edges in M that are contained in $G - Y_l$ gives the required matching in G.

Similarly, if $w_l = w_{l-1} = u_i$ then $u_{l-1}u_l$ is an edge in G with colour c_{l-1}. Replace the edge u_lv_l by the edge w_lv_l (coloured c_l) and the edge $u_{l-1}v_{l-1}$ by the edge $u_{l-1}u_l$. Replace the edges u_pv_p by the edges u_pv_{p+1} for all $i < p < l-1$. By Lemma 5, $G[(Y_i \cup \{v_{i+1}\}) \setminus \{u_i\}]$ contains a matching with $i+1$ edges coloured $c_0, c_1, \ldots, c_i$. This matching, together with the edges in M that are contained in $G - Y_l$ gives the required matching in G.

Therefore all the neighbours of w_l in $G_{c_l}[Y \cup Y_l]$ are contained in $Y_l \setminus \{u_l, v_l\}$ and hence the degree of w_l in $G_{c_l}[Y \cup Y_l]$ is at most $2l - 1$. $\square$

Proof: (of Claim 4 continued) Since $\delta(G_{c_l}) \geq k$, by Lemmas 7 and 8, there exists an edge $u_{l+1}v_{l+1}$ in $G - (Y \cup Y_l)$ such that $u_{l+1}v_{l+1}$ is contained in M and $|E_{c_l}(\{w_l, u_l\}, \{u_{l+1}, v_{l+1}\})| \geq 3$. We may assume without loss of generality that the edges u_lv_{l+1} and w_lv_{l+1} are coloured c_l. If u_lu_{l+1} is an edge in G coloured c_l then let $w_{l+1} = w_l$ else w_lu_{l+1} is an edge in G coloured c_l and we let $w_{l+1} = u_l$. Let c_{l+1} be the colour of the edge $u_{l+1}v_{l+1}$. Now, (u_0, v_0, w_0, c_0), …, (u_l, v_l, w_l, c_l), $(u_{l+1}, v_{l+1}, w_{l+1}, c_{l+1})$ is the required sequence of $l + 2$ 4-tuples satisfying all the properties stated in Claim 4. This completes the proof of Claim 4 and also Theorem 3. $\square$

We note that the same proof holds under the slightly weaker assumption that for every G_i, the sum of degrees of any two non-adjacent vertices is at least $2k - 1$. Also, the bound is tight for each individual G_i as long as $k_i > 0$. This may be seen by the following example: Let G_1 be the disjoint union of two complete graphs of order k with vertex sets A and B, and let G_i be the complete bipartite graph with bipartition (A, B) for all $2 \leq i \leq t$. This graph does not contain a matching with k edges having an odd number of edges of colour 1.

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