Degree condition for vertex-disjoint paths.

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Abstract

Let G be a connected graph of order $\geq n_1 + \ldots + n_k$, where $n_i \geq 2$ are integers, for all $1 \leq i \leq k$, and suppose the minimum degree of G is $\geq \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_k}{2} \rfloor$. We prove that G contains k vertex-disjoint paths of orders $n_1, \ldots, n_k$, unless it is an exceptional graph. The exceptions occur only when either $k = 2$ and $n_1 = n_2 = 2l + 1$, $l \geq 1$, or $n_1 = \ldots = n_k = 3$, $k \geq 2$. We also characterize all the exceptional graphs in these cases. This generalizes a result of R. Johansson [J. Graph Theory, 28(1998), 39-42], who showed that such paths always exist if $|G| = n_1 + \ldots + n_k$.

Keywords: degree, vertex-disjoint, paths.

A basic result in graph theory is Dirac's theorem [3]; every graph of order $n \geq 3$, and minimum degree $\geq n/2$ is Hamiltonian. An immediate consequence of this is that every graph of order $n \geq 2$, and minimum degree $\geq (n-1)/2$ contains a Hamiltonian path. Dirac also showed that every 2-connected graph of order $\geq 2d$, and minimum degree $\geq d$, contains a cycle of length $\geq 2d$. Similarly, every connected graph of order $\geq 2d + 1$, and minimum degree $\geq d$, contains a path of order $\geq 2d + 1$.

Dirac's original theorem has been generalized in many ways. El-Zahár [4] showed that every graph of order $n_1 + n_2$, and minimum degree $\geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$, contains two vertex-disjoint cycles of lengths n_1 and n_2, where n_1, n_2 are integers ≥ 3. He also conjectured that every graph of order $n_1 + \ldots + n_k$, and minimum degree $\geq \lceil \frac{n_1}{2} \rceil + \ldots + \lceil \frac{n_k}{2} \rceil$, contains k vertex-disjoint cycles of lengths $n_1, \ldots, n_k$, where n_i is an integer ≥ 3, for all $1 \leq i \leq k$.

R. Johansson [5] proved a result similar to El-Zahár's conjecture, in the case of paths. He showed that every connected graph of order $n_1 + \ldots + n_k$, $n_i \geq 2$ for all $1 \leq i \leq k$, and minimum degree $\geq \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_k}{2} \rfloor$, contains k vertex-disjoint paths of orders $n_1, \ldots, n_k$.

In this article we consider analogous generalizations of the 'long cycle' or 'long path' versions of Dirac's theorem. We prove the following generalization of Johansson's result.

Theorem. Let G be a connected graph, with $|G| \geq n_1 + n_2 + \ldots + n_k$, and $\delta(G) \geq \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_k}{2} \rfloor$, where $n_i \geq 2$ are integers, for all $1 \leq i \leq k$. G contains k vertex-disjoint paths of orders $n_1, n_2 \ldots n_k$, unless one of the following holds.

a) $k = 2$, $n_1 = n_2 = 2t + 1$ for some $t \geq 1$, and $G = pK_{2t} + K_1$, for some $p \geq 3$.

b) $n_i = 3$, $1 \leq i \leq k$, $k \geq 2$, and $G = G_1 + pK_2$, for some graph G_1 with $|G_1| = k - 1$ and $p > k$.

c) $n_i = 3$, $1 \leq i \leq k$, $k \equiv 2 \bmod 3$, and $G = 3K_k + K_1$.

We also propose a conjecture generalizing El-Zahár's conjecture in a similar way.

We consider only simple graphs throughout. The notation and terminology used is mostly standard and follows, for example, [1].

Proof. If $k = 1$, or if all the n_i are even, the result follows from Dirac's theorem. We may therefore assume $k \geq 2$, and at least one of the n_i is odd. Moreover, as in [5], it is sufficient to prove the result for the case when all n_i are odd. If say, $n_1, \ldots, n_p$ are even, and n_{p+1} is the largest odd number, then let $n_1' = n_1 + \ldots + n_p + n_{p+1}$ and let $n_i' = n_{p+i}$, for all $2 \leq i \leq k - p$. The n_i' are all odd, and there are no exceptional graphs in this case, since either $p = k - 1$ or the n_i' are not all equal.

Let G be a connected graph with $|G| \geq n_1 + \ldots + n_k$, and minimum degree

$\geq \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_k}{2} \rfloor$, where $n_1, \ldots, n_k$, $k \geq 2$, are odd integers ≥ 3. Let $n = n_1 + \ldots + n_k$ and $d = \lfloor \frac{n_1}{2} \rfloor + \ldots + \lfloor \frac{n_k}{2} \rfloor$, then $|G| \geq n$ and $\delta(G) \geq d = (n-k)/2$.

Let $P = v_1, v_2, \ldots, v_l$ be a longest path in G and let $R = G - V(P)$. If $l \geq n$, there are k vertex-disjoint subpaths of P of orders $n_1, \ldots, n_k$. We may therefore assume $l < n$ and hence R is not empty. Moreover, by Dirac's theorem, $l \geq 2\delta(G) + 1 \geq n - k + 1$. We now consider two cases.

Case 1. R contains at least one nontrivial connected component.

Let C be a nontrivial component in R and let $Q = u_1, u_2, \ldots, u_m$, $m \geq 2$ be a longest path in C. We say that Q is *good* if there exist vertices $v_i, v_j \in V(P)$, $i < j$, such that v_i is adjacent to one of $\{u_1, u_m\}$ and v_j to the other. C is said to be *good* if there exists a longest path Q in C which is good. We again consider two subcases.

Case 1.1. There exists a good component C in R.

Let $Q = u_1, u_2, \ldots, u_m$, $m \geq 2$ be a good longest path in C. Let $v_i, v_j \in V(P)$, $i < j$, be vertices such that v_i is adjacent to one of $\{u_1, u_m\}$ and v_j to the other, and $j - i$ is minimum amongst all possible such pairs of vertices. $j - i \geq m + 1$, otherwise we get a path longer than P in G.

Let A and B be the set of neighbours in $V(P)$ of u_1 and u_m respectively. Let $A^+ = \{v_i \mid v_i \in V(P) \text{ and } v_{i-1} \in A\}$ and $A^- = \{v_i \mid v_i \in V(P) \text{ and } v_{i+1} \in A\}$. The sets B^+ and B^- are defined similarly. u_1 can have at most $m-1$ neighbours in C and $d(u_1) \geq d$, hence $|A| \geq d - (m-1)$. Similarly, $|B| \geq d - (m-1)$. The sets $\{v_1, \ldots, v_{m-1}\}$, $\{v_{i+2}, \ldots, v_{j-2}\}$, $\{v_{l-m+2}, \ldots, v_l\}$, A, A^+ and B^- are all mutually disjoint, otherwise we get a path longer than P in G. Therefore, $l \geq 3(d - (m-1)) + 2(m-1) + (m-2) = 3d - 1 = 3(n-k)/2 - 1$. However, $l \leq n - 1$ and thus $3(n-k)/2 \leq n$, which implies $n \leq 3k$. Since all $n_i \geq 3$, we must have $n = 3k$, $d = k$, $n_i = 3$, for all $1 \leq i \leq k$, and $l = 3k - 1$. There are $k - 1$ vertex-disjoint paths of order 3 in P, and if any component in R contains ≥ 3 vertices, we can find the remaining path in that component. We may therefore assume every component in R contains at most 2 vertices and $m = 2$.

Suppose a vertex $v \in V(R)$ is adjacent to a vertex $v_i \in V(P)$ such that $i \not\equiv 0 \bmod 3$. Let v_j be the neighbour of v_i in P such that $j \not\equiv 0 \bmod 3$. There are $k-1$ vertex-disjoint paths of order 3 in $P - \{v_i, v_j\}$, and together with the path v, v_i, v_j, we get the required k paths.

Thus a vertex in $V(R)$ can only be adjacent to the vertices v_{3i}, $1 \leq i < k$. This means that R cannot contain any isolated vertex. Thus every connected component of R must be K_2, and every vertex in $V(R)$ is adjacent to the vertices v_{3i}, $1 \leq i < k$. Considering the longest path $P' = v_1, \ldots, v_{3i}, u_1, u_2, v_{3i+3}, \ldots, v_l$, $0 \leq i < k$ and arguing similarly, we must have v_{3i+1}, v_{3i+2} adjacent to all the vertices v_{3j}, $1 \leq j < k$ and no others apart from each other. Therefore $G = G_1 + pK_2$, where G_1 is the subgraph induced by the vertices v_{3i}, $1 \leq i < k$ and $p > k$.

Case 1.2. There is no good component in R.

Let C be any nontrivial component in R and let $Q = u_1, \ldots, u_m$, $m \geq 2$, be a longest path in C. At least one of $\{u_1, u_m\}$ is adjacent to at most one vertex in $V(P)$ else Q and C are good. Since $\delta(G) \geq d$, $m \geq d = (n-k)/2$.

Assume, without loss of generality, that $n_1 \geq n_2 \geq \ldots \geq n_k$, and suppose $l + m \geq n + n_2 - 1$. We claim that $P \cup Q$ contains k vertex-disjoint paths of orders $n_1, \ldots, n_k$. Suppose it is not true, and let $i < k$ be the largest number such that $P \cup Q$ contains i vertex-disjoint paths of orders $n_1, \ldots, n_i$. Since $n_i \geq 3$, $n_1 \leq n - 3(k-1) \leq n - (k-1) \leq l$, and hence $i \geq 1$. Now $l + m \leq n_1 + \ldots + n_i + 2(n_{i+1} - 1)$, otherwise we can find vertex-disjoint paths of orders $n_1, \ldots, n_{i+1}$ in $P \cup Q$, contradicting the definition of i. Hence $l + m \leq n - n_{i+1} + 2(n_{i+1} - 1) = n + n_{i+1} - 2 \leq n + n_2 - 2$, a contradiction.

Therefore we may assume $l + m \leq n + n_2 - 2$. Since $n_i \geq 3$ and $n_1 \geq n_2$, $2n_2 + 3(k-2) \leq n$ and $n_2 \leq (n - 3(k-2))/2$. Thus $l + m \leq 3(n-k)/2 + 1$, and since $l \geq n - k + 1$, we have $m \leq (n-k)/2$. We thus conclude that $m = (n-k)/2$, $l = n - k + 1$ and $n_2 = (n - 3(k-2))/2$. This implies that $n_1 = n_2$ and $n_i = 3$, for all $3 \leq i \leq k$. We have $l = 2n_2 + 2k - 5$ and $m = n_2 + k - 3$.

Suppose $k \geq 3$, then there are vertex-disjoint paths in P, such that two paths are of order n_2, and $\lfloor (2k-5)/3 \rfloor$ paths are of order 3. Similarly, there are $\lfloor (n_2 + k - 3)/3 \rfloor$ vertex-disjoint paths of order 3 in Q. If either $n_2 > 3$ or $k \not\equiv 2$

mod 3, this gives the required k paths. We may therefore assume that either $n_1 = n_2$ and $k = 2$, or $n_i = 3$, for all $1 \le i \le k$, $k > 2$ and $k \equiv 2 \bmod 3$.

First consider the case $k = 2$, and let $n_1 = n_2 = 2t + 1$, $t \ge 1$. Then $m = 2t$ and $l = 4t + 1$. The vertices u_1 and u_m cannot be adjacent to any vertex in $V(P)$ other than v_{2t+1}, otherwise there is a path longer than P in G. Since $\delta(G) \ge d = 2t$, u_1 and u_m must be adjacent. Since Q is a longest path in C, C cannot contain any vertex $\notin V(Q)$. Further, no vertex in C can be adjacent to any vertex in $V(P)$ other than v_{2t+1}, and thus C must be K_{2t}. If R contains an isolated vertex v, it must be adjacent to the vertices v_{2i}, $1 \le i \le 2t$ in $V(P)$. We then get two vertex-disjoint paths of order $2t+1$, $v_1, \ldots, v_{2t}, v$ and $v_{2t+1}, \ldots, v_{4t+1}$. Therefore every connected component in R must be K_{2t}. Considering the longest path $P' = u_1, \ldots, u_m, v_{2t+1}, \ldots, v_{4t+1}$ and arguing similarly, we can show that the vertices $\{v_1, \ldots, v_{2t}\}$ induce a complete graph, and are adjacent only to the vertex v_{2t+1} in $V(P')$. Similarly the vertices $\{v_{2t+2}, \ldots, v_{4t+1}\}$ induce a complete graph, and are all adjacent to v_{2t+1}. Thus every component of $G - v_{2t+1}$ is K_{2t} and $G = pK_{2t} + K_1$ for some $p \ge 3$.

Now suppose $n_i = 3$, for all $1 \le i \le k$, $k > 2$, and $k \equiv 2 \bmod 3$. Then $n = 3k$, $l = 2k + 1$, and $m = d = k$. The vertices u_1 and u_m can only be adjacent to the vertex v_{k+1} in $V(P)$, and since $\delta(G) \ge k$, u_1 must be adjacent to u_m. Arguing as above, C must be K_k and every vertex in C is adjacent to the vertex v_{k+1}. If there is any other nontrivial component in R, it must also be K_k, and we can find $(2k - 1)/3$ vertex-disjoint paths of order 3 in P, and $2(k - 2)/3$ vertex-disjoint paths of order 3 in R. Since $k \ge 5$, this gives the required k vertex-disjoint paths of order 3. Similarly, if there is any isolated vertex v in R, it must be adjacent to the vertices v_{2i}, $1 \le i \le k$ in $V(P)$. We can again find k vertex-disjoint paths of order 3, $(2k - 1)/3$ of them in $P - \{v_{2k}, v_{2k+1}\}$, $(k - 2)/3$ in Q and the path v, v_{2k}, v_{2k+1}. Thus R contains exactly one component which is K_k. Considering the longest path $P' = u_1, \ldots, u_m, v_{k+1}, \ldots, v_{2k+1}$ and arguing similarly, we can show that the vertices $\{v_1, \ldots, v_k\}$ induce a complete graph, and v_{k+1} is the only other vertex to which they are adjacent. Similarly, $\{v_{k+2}, \ldots, v_{2k+1}\}$ induce a complete graph, and are all adjacent to v_{k+1}. Thus $G = 3K_k + K_1$.

This completes the proof for Case 1.

Case 2. R contains only isolated vertices.

This case is essentially similar to the proof in [5]; we include it here for completeness. We prove a slightly stronger statement.

Lemma. Let G be any graph with $|G| \geq n = n_1 + \ldots + n_k$, where n_i are odd integers ≥ 3. Suppose there is a path P in G such that $R = G - V(P)$ contains only isolated vertices. If every vertex in $V(R)$ has degree $\geq d = (n-k)/2$, then G contains k vertex-disjoint paths of orders $n_1, \ldots, n_k$.

Proof. The proof is by induction on k. Suppose $k = 1$ and length of P is $< n$. Since every vertex in R has degree $\geq (n-1)/2$, there must be a vertex in R adjacent to an endpoint of P, or to two consecutive vertices in P. We then extend P by including such a vertex in it. Repeating this process we get the required path of length n.

Suppose $k > 1$. We again extend P, by including any vertex in R adjacent to an endpoint of P, or to two consecutive vertices in P, until there is no such vertex. Let $P = v_1, \ldots, v_l$ with $l \geq n-k+1$. Since n_1 is odd, a vertex in R can be adjacent to at most $(n_1 - 1)/2$ of the vertices $\{v_1, \ldots, v_{n_1}\}$. Hence, the graph $G^{'}$ $= G - \{v_1, \ldots, v_{n_1}\}$ contains a path $P^{'} = v_{n_1+1}, \ldots, v_l$, such that $R^{'} = G^{'} - V(P^{'})$ $= R$ contains only isolated vertices. Moreover, the degree of every vertex in $R^{'}$ is $\geq (n-k)/2 - (n_1-1)/2 = ((n-n_1) - (k-1))/2$. Hence, by induction, there are $k-1$ vertex-disjoint paths of orders $n_2, \ldots, n_k$ in $G^{'}$. These paths, together with the path $v_1, \ldots, v_{n_1}$, give the required k vertex-disjoint paths in G.

The proof of Case 2 follows immediately from the lemma.

Remarks. It would be interesting to see if a similar generalization is possible for cycles. We suggest a generalization of El-Zahár's conjecture.

Conjecture. Let G be a 2-connected graph of order $\geq n_1 + \ldots + n_k$, and minimum degree $\geq \lceil \frac{n_1}{2} \rceil + \ldots + \lceil \frac{n_k}{2} \rceil$. G contains k vertex-disjoint cycles $C_1, \ldots, C_k$, such that C_i has length $\geq n_i$, for all $1 \leq i \leq k$, unless one of the following holds.

a) $k = 2$, $n_1 = n_2 = 2l$, for some $l \geq 2$, and $G = 3K_1 + pK_{2l-2}$, for some $p \geq 3$.

b) $n_1 = n_2 = \ldots = n_k = 4$, $k \geq 2$, $G = G_1 + pK_2$, where G_1 is an arbitrary graph of order $2k-1$, and $p > k$.

It may be noted that the theorem of Corradi and Hajnal [2], shows that this conjecture is true in the special case when $n_i = 3$, for all $1 \leq i \leq k$. It must be emphasized that we consider cycles of length $\geq n_i$ and not exactly n_i. In the case of paths, the two are equivalent.

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