Balanced Group Labeled Graphs

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Outline

1. Introduction
   - Group Labeled Graphs
   - Balanced Labellings
   - Characterization

2. Results
   - Counting Number of Balanced labellings
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Oriented Group Labeled Graphs

- Oriented graphs
  - Edges labeled by elements of a group
  - Label of a path in underlying undirected graph
  - Add labels of edges in sequence
  - Labels of oppositely oriented edges are subtracted
  - Cycle has (non)-zero label independent of starting vertex
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- Undirected graphs with loops and multiple edges
- Edges / vertices labeled by elements of an abelian group
- Labels are also called weights
- Weight of a subgraph
- Sum of weights of vertices and edges in the subgraph

Consider only undirected group labeled graphs
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Signed and Marked Graphs

- Signed graphs
  - Undirected graphs with edges labeled ‘+’ or ‘−’

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- Special cases of $\mathbb{Z}_2$-labeled graphs

- Well-studied in the literature

- Extend some results to general group labeled graphs
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Labellings with Specified Subgraphs of Weight Zero

- \( \mathcal{F} \) is a family of graphs
- \( \mathcal{F} \)-balanced labellings of a graph \( G \)
- Every subgraph of \( G \) in \( \mathcal{F} \) has weight zero
- Labellings form a group
- What is its order?
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Balanced Labellings

- $\mathcal{F}$ is the set of all cycles
  - Labellings in which every cycle has weight zero — Balanced labellings
  - Balanced signed graphs
  - Consistent marked graphs
  - Characterizations of such $\mathbb{Z}_2$–labellings known
  - Extend these to labellings by arbitrary abelian groups
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Theorem (Harary, 1954)

A signed graph is balanced iff the vertex set can be partitioned into two parts such that an edge has a ‘−’ sign if and only if it has an endvertex in each part.
Theorem (Hoede, 1992)

A marked graph is consistent iff

1. Every fundamental cycle with respect to any fixed spanning tree \( T \) is balanced.

2. Any path in \( T \) that is the intersection of two fundamental cycles has endvertices with the same signs.

Earlier characterizations, more complicated, given by Rao and Acharya.
Alternative Characterizations

Theorem (Roberts and Xu, 2003)

A marked graph is consistent iff

1. Every cycle in some basis for the cycle space is balanced.
2. Every 3-connected pair of vertices has the same sign.

Theorem (Roberts and Xu, 2003)

A marked graph is consistent iff

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2. Every cycle that is the symmetric difference of two fundamental cycles is balanced.

Same statement holds for general group labeled graphs
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3-Connected Pairs of Vertices

Theorem (Roberts and Xu, 2003)

The following statements are equivalent for any marked graph

1. Every 3-connected pair of vertices have the same sign.
2. Every 3-edge-connected pair of vertices have the same sign.
3. For any spanning tree $T$, the endvertices of any path in $T$ that is the intersection of two fundamental cycles, have the same sign.
Balanced Group Labeled Graphs

**Theorem**

Let \( w : V(G) \cup E(G) \rightarrow \Gamma \) be a labeling of a graph \( G \) by an arbitrary abelian group \( \Gamma \). Then \( w \) is a balanced labeling iff

1. Every cycle in some basis has weight zero.
2. For every 3-connected pair of vertices \( u, v \) and any path \( P \) between \( u \) and \( v \), \( 2w(P) = w(u) + w(v) \).

Other characterizations extend similarly

- Replace the condition “have the same sign” by “\( 2w(P) = w(u) + w(v) \)”
- Holds if labels are assigned to vertices and edges
- Linear-time algorithm to test balance
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Counting Number of Balanced Labellings

- Define a relation \( \sim \) on \( V(G) \)
  - \( u \sim v \) iff \( u = v \) or there exist three edge-disjoint paths between \( u \) and \( v \) in \( G \)
- \( \sim \) is an equivalence relation on \( V(G) \)
- \( \sigma(G) \) is the number of equivalence classes of \( \sim \)
- \( c(G) \) is the number of connected components of \( G \)
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Counting Number of Balanced Labellings

Define a relation ~ on V(G)

- u ~ v iff u = v or there exist three edge-disjoint paths between u and v in G
- ~ is an equivalence relation on V(G)
- σ(G) is the number of equivalence classes of ~
- c(G) is the number of connected components of G
Theorem

The number of distinct balanced labellings of a graph $G$ by a finite abelian group $\Gamma$ is

$$|\Gamma| |G| + \sigma(G) - c(G)$$

- Depends only on the order and not the structure of $\Gamma$
- If $\Gamma$ is $\mathbb{Z}_2$ this follows from the characterization
- Sufficient to prove it for cyclic groups $\mathbb{Z}_k$ and 2-edge-connected graphs
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Lemma

Let \( w \) be a labeling of a 3-edge-connected graph \( G \). Then \( w \) is balanced iff for any two vertices \( u, v \) and path \( P \) between \( u \) and \( v \), \( 2w(P) = w(u) + w(v) \).

- Sufficient to prove it for edges
- To prove for edges, sufficient to prove for 3 edge-disjoint paths with same endvertices
- Balance implies this for 3 internally vertex-disjoint paths
- Use induction on the sum of path lengths
Lemma

Let \( w \) be a labeling of a 3-edge-connected graph \( G \). Then \( w \) is balanced iff for any two vertices \( u, v \) and path \( P \) between \( u \) and \( v \),
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3-Edge-Connected Graphs
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- If $k$ is odd, in any balanced labeling by $Z_k$, labels of vertices uniquely determine the labels of edges.
- Labels of vertices may be arbitrary.
- Number of labellings is $k^{|G|}$. 
3-Edge-Connected Graphs

- If $k$ is even, all vertex labels must have same parity
- Two possible choices of $w(uv)$ such that
  \[2w(uv) + w(u) + w(v) = 0\]
- Choices cannot be made arbitrarily
- There is a partition of the vertex set into two parts such that
  \[w(uv) = -\frac{w(u) + w(v) + k}{2}\]
  iff $u, v$ are in different parts
- Number of labellings is
  \[2(k/2)^{|G|} \times 2^{|G| - 1} = k^{|G|}\]
2-Edge Connected Graphs

Simple inductive argument

- Consider 2-edge cut $X$ such that size of smaller component of $G - X$ is minimum
- Apply Lemma for 3-edge connected graphs to this component if it is non-trivial
- Apply induction to the other component
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Let $G$ be a 3-edge-connected graph and $F$ a subset of edges of $G$. Let $c(F)$ denote the number of connected components of the spanning subgraph of $G$ with edge set $F$, and let $b(F)$ be the number of these components that are bipartite. The number of balanced labellings of $G$ by $Z_k$, with edges in $F$ having label 0 is

1. $k^{b(F)}$ if $k$ is odd or $k$ is even and $c(F) = b(F)$.
2. $k^{b(F)}2^{c(F) - b(F) - 1}$ if $k$ is even and $c(F) > b(F)$.

- Argument extends to 2-edge-connected graphs
- Efficient algorithm for counting such labellings
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Which graphs have a non-trivial balanced labeling by $\mathbb{Z}_2$ with all edge labels zero? (Beineke and Harary, 1978)

Roberts (1995) characterized all such 2-connected graphs with longest cycle of length at most 5.

Constructive characterization of all such graphs

Follows from the inductive characterization of balanced labellings
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Theorem

A 2-edge-connected graph $G$ is markable if and only if it satisfies one of the following properties.

(a) $G$ is bipartite.
Characterization of Markable Graphs

(b) There is a 3-edge-connected graph $G'$ and a non-empty proper subset $\emptyset \subset A \subset V(G')$, such that $G$ is obtained by subdividing exactly once every edge in the cut $(A, \overline{A})$. 
Characterization of Markable Graphs

(c) $G$ is obtained from the disjoint union of a 2-edge-connected markable graph $G_1$ and an arbitrary 2-edge-connected graph $G_2$, by replacing edges $p_i q_i \in E(G_i)$ by edges $p_1 p_2$ and $q_1 q_2$. 
Characterization of Markable Graphs

(d) $G$ is obtained from the disjoint union of two 2-edge-connected markable graphs $G_1$ and $G_2$ by deleting vertices $p_i \in V(G_i)$ of degree 2 and adding edges $q_1q_2$, $r_1r_2$, where $q_i, r_i$ are the neighbors of $p_i$ in $G_i$. 

![Diagram of Characterization of Markable Graphs]
Characterizations of balanced signed graphs and consistent marked graphs extend to arbitrary group labeled graphs with edge and vertex weights.

Count the number of balanced labellings with specified elements labeled 0.

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Conclusions

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Some Questions

- Extend to balanced labellings of other subgraphs (perhaps disjoint cycles, \( r \)-regular graphs)
- Nowhere-zero balanced labellings (similar to nowhere-zero flows)
- A deletion-contraction recurrence for the number of nowhere-zero balanced labellings
- Measures of imbalance
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