Cycles of even lengths modulo k

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Abstract

Thomassen [9] conjectured that for all natural numbers $k > 0$ and m, every graph of minimum degree $k+1$ contains a cycle of length congruent to $2m$ modulo k. We prove that this is true for $k \geq 2$ if the minimum degree is $2k-1$, which improves the previously known bound of $3k-2$. We also show that Thomassen's conjecture is true for $m = 2$.
Keywords: cycle lengths; minimum degree

1 Introduction

The study of cycle lengths in a graph modulo a positive integer k was initiated by Erdős and Burr [6], who conjectured that for every odd positive integer k, there exists a constant c_k such that for all natural numbers m, every graph of average degree at least c_k contains a cycle of length congruent to m modulo k. They proved this for $m = 2$ and the full conjecture was proved by Bollobás [1], who showed that $c_k = 2((k+1)^k - 1)/k$ suffices. Thomassen [9, 10] showed a more general result, that an average degree of $4k(k+1)$ implies that the graph contains a cycle of length congruent to $2m$ modulo k for all natural numbers $k > 0$ and m. He also conjectured that a minimum degree of $k+1$ suffices.

Verstraëte [11] showed that if $k \geq 2$ and the average degree is at least $8k$, then there exist k cycles of consecutive even lengths, which implies that for all natural numbers m, there exists a cycle of length congruent to $2m$ modulo k. Fan [7] showed that if $k \geq 2$ and the minimum degree is $3k-2$, then there exist k cycles with consecutive even lengths, or consecutive odd lengths. This implies that for odd $k \geq 2$, if the minimum degree is $3k-2$, then the graph contains a cycle of length congruent to $2m$ modulo k for all m. We give a simple proof that for $k \geq 2$, a minimum degree of $2k-1$ suffices to ensure that the graph contains a cycle of length congruent to $2m$ modulo k for all m. The proof is an application of a technique used by Mader in [8].

Thomassen's conjecture is known to be true for $k \leq 4$, [3, 4, 5] and for claw-free graphs [2]. For $m = 1$, it follows easily by considering a longest path and the edges incident with an endpoint of the path. We prove that it is true for $m = 2$.

All graphs considered are simple and finite. We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$ or simply $N(v)$ when G is understood from the context. If $V \subset V(G)$ then $G - V$ is the subgraph of G obtained by deleting all vertices in V and edges incident with them. The graph $G - (V(G) \setminus V)$ is the subgraph of G induced by V.

A path with endpoints $\{u, v\}$ is called a u–v path, and a u–u path is the trivial path containing only the vertex u. The length of a path or cycle P is denoted by $l(P)$. A set of paths (or cycles) $\{P_1, P_2, \ldots, P_r\}$ is said to have distinct lengths modulo k if $l(P_i) \not\equiv l(P_j) \bmod k$ for all $1 \leq i < j \leq r$. A graph G is said to contain all cycles of even lengths modulo k, if for all natural numbers m, G contains a cycle of length congruent to $2m$ modulo k. If P and Q are graphs, we denote by $P \cup Q$ the graph with vertex set $V(P) \cup V(Q)$ and edge set $E(P) \cup E(Q)$. To simplify the notation, we will consider an edge to be a graph with two vertices, where necessary.

2 Main Result

Theorem 1 *Let $k \geq 2$ be an integer and G a graph with minimum degree $2k-1$. For all natural numbers m, G contains a cycle of length congruent to $2m$ modulo k.*

We will assume throughout that $k \geq 2$ is a fixed integer. We will be considering ordered pairs of the form (G, K), where K is a proper complete subgraph of G, possibly empty.

Definition 1 *The ordered pair (G, K) contains a configuration of type **A** if $V(G) \setminus V(K)$ contains a vertex u such that $|N(u) \cap V(K)| \geq 2k-1$.*

Definition 2 *The ordered pair (G, K) contains a configuration of type **B** if $V(G) \setminus V(K)$ contains two vertices u, v such that $|N(u) \cap V(K)| \geq 2k-2$, $|N(v) \cap V(K)| \geq 2k-2$, and there exists a u–v path P in $G - V(K)$.*

Definition 3 *The ordered pair* (G, K) *contains a configuration of type* $\boldsymbol{C}$ *and* rank r*, if* $V(G) \setminus V(K)$ *contains two vertices* u, v *such that* $|N(u) \cap V(K)| \geq 2(k-r)+1$, $|N(v) \cap V(K)| \geq 2(k-r)+1$*, and there exists a set* $\{P_1, P_2, \ldots, P_r\}$ *of* r u*–*v *paths in* $G - V(K)$ *having distinct lengths modulo* k*.*

Definition 4 *The ordered pair* (G, K) *contains a configuration of type* $\boldsymbol{D}$ *and* rank r*, if* $V(G) \setminus V(K)$ *contains three vertices* u, w, v *satisfying the following properties.*

1. uw *and* wv *are edges in* $G - V(K)$*.*
2. *There exist vertices* $\{u', w', v'\} \subseteq V(G) \setminus V(K)$ *and vertex-disjoint paths* P_u, P_w, P_v *in* $G - V(K)$ *such that* $|N(u') \cap V(K)| \geq 2(k-r)-1$, $|N(w') \cap V(K)| \geq 2(k-r)$, $|N(v') \cap V(K)| \geq 2(k-r)$*, and* P_z *is a* z*–*z' *path for all* $z \in \{u, v, w\}$*. Note that the vertex* z' *may be the same as the vertex* z*, for* $z \in \{u, v, w\}$*.*
3. *There is a set* $\{P_1, P_2, \ldots, P_r\}$ *of* r u*–*v *paths in* $G - V(K)$ *having distinct lengths modulo* k*, such that* P_i *is internally vertex-disjoint from* P_u*,* P_w *and* P_v *for* $1 \leq i \leq r$*. Also, one of the paths in* $\{P_1, P_2, \ldots, P_r\}$ *has length congruent to 0 modulo* k*.*

Lemma 1 *Let* G *be a graph and* K *a proper complete subgraph of* G*. If every vertex in* $V(G) \setminus V(K)$ *has degree at least* $2k-1$ *in* G*, then either* $G - V(K)$ *contains cycles of all even lengths modulo* k*, or the ordered pair* (G, K) *contains a configuration of one of the types* $\boldsymbol{A}$*,* $\boldsymbol{B}$*,* $\boldsymbol{C}$*,* $\boldsymbol{D}$*. If the configuration is of type* $\boldsymbol{C}$*, then the rank is at least two and at most* k*. If the configuration is of type* $\boldsymbol{D}$*, then the rank is at least two and less than* k*.*

Proof: Let (G, K) be a counterexample that minimizes $|V(G) \setminus V(K)|$ and subject to this condition, minimizes $|V(K)|$. If $|V(G) \setminus V(K)| = 1$, the only vertex $u \in V(G) \setminus V(K)$ has degree at least $2k-1$, hence $|N(u) \cap V(K)| \geq 2k-1$. This implies (G, K) contains a configuration of type **A**, a contradiction.

Suppose $|V(G) \setminus V(K)| > 1$. We consider two cases.

Case 1.

Suppose there exists a vertex $x \in V(G) \setminus V(K)$ that is adjacent to all vertices in $V(K)$. Let K' be the complete subgraph of G induced by $V(K) \cup \{x\}$. Then the ordered pair (G, K') satisfies the hypothesis of Lemma 1, and by the minimality of (G, K), either $G - V(K')$ contains cycles of all even lengths modulo k or (G, K') contains a configuration of one of the four types. Since $G - V(K')$ is a subgraph of $G - V(K)$, we may assume that the latter holds. Now we show that in each case, the configuration in (G, K') can be modified to either find cycles of all even lengths modulo k in $G - V(K)$, or get a configuration of one of the four types in (G, K), contradicting the fact that (G, K) is a counterexample.

Case 1.1

Suppose (G, K') contains a configuration of type **A**. Let u be a vertex in $V(G) \setminus V(K')$ such that $|N(u) \cap V(K')| \geq 2k - 1$. If u is not adjacent to x, then $|N(u) \cap V(K)| \geq 2k - 1$ and (G, K) contains a configuration of type **A**. If u is adjacent to x, then $|N(u) \cap V(K)| \geq 2k - 2$ and since x is adjacent to every vertex in $V(K)$, $|N(x) \cap V(K)| = |V(K)| \geq 2k - 2$. The edge ux implies that (G, K) contains a configuration of type **B**.

Case 1.2

Suppose (G, K') contains a configuration of type **B**. Let u, v be vertices in $V(G) \setminus V(K')$ such that $|N(u) \cap V(K')| \geq 2k - 2$, $|N(v) \cap V(K')| \geq 2k - 2$ and there is a u–v path P in $G - V(K')$.

If x is not adjacent to any of the vertices $\{u, v\}$, then u, v satisfy the same properties with K' replaced by K, and (G, K) contains the same configuration of type **B**.

If x is adjacent to u but not adjacent to v, then $|N(v) \cap V(K)| \geq 2k - 2$ and hence $|N(x) \cap V(K)| \geq 2k - 2$. Also, $P \cup xu$ is an x–v path in $G - V(K)$. Hence (G, K) contains a configuration of type **B**. A symmetrical argument holds if x is adjacent to v but not adjacent to u.

Suppose x is adjacent to both u and v. Then $|N(u) \cap V(K)| \geq 2k - 3$, $|N(v) \cap V(K)| \geq 2k - 3$ and hence $|N(x) \cap V(K)| \geq 2k - 3$. If $l(P) \equiv 0 \bmod k$ and $k > 2$, then P and $Q = ux \cup xv$ are two u–v paths in $G - V(K)$ of distinct lengths modulo k, hence (G, K) contains a configuration of type **C** and rank two. If $l(P) \equiv 0 \bmod k$ and $k = 2$, then $P \cup Q$ is an even cycle in $G - V(K)$, hence $G - V(K)$ contains cycles of all even lengths modulo 2. If $l(P) \not\equiv 0 \bmod k$, then the edge ux and the path $P \cup vx$ are two u–x paths in $G - V(K)$ of distinct lengths modulo k, hence (G, K) contains a configuration of type **C** and rank two.

Case 1.3

Suppose (G, K') contains a configuration of type **C** and rank r, for some $2 \leq r \leq k$. Let u, v be vertices in $V(G) \setminus V(K')$ such that $|N(u) \cap V(K')| \geq 2(k-r)+1$, $|N(v) \cap V(K')| \geq 2(k - r) + 1$ and let $\{P_1, P_2, \ldots, P_r\}$ be the set of r u–v paths in $G - V(K')$ having distinct lengths modulo k.

If x is not adjacent to any of the vertices $\{u, v\}$, then (G, K) also contains the same configuration of type **C** and rank r.

If x is adjacent to u but not to v, then $|N(v) \cap V(K)| \geq 2(k - r) + 1$ and hence $|N(x) \cap V(K)| \geq 2(k - r) + 1$. The paths $P_i \cup xu$ for $1 \leq i \leq r$ are x–v paths in

$G-V(K)$ having distinct lengths modulo k. Hence (G,K) contains a configuration of type **C** and rank r. A symmetrical argument holds if x is adjacent to v but not adjacent to u.

Suppose x is adjacent to both u and v. If $r=k$, then the cycles $ux \cup xv \cup P_i$ for $1 \leq i \leq k$ have distinct lengths modulo k, hence $G-V(K)$ contains cycles of all even lengths modulo k. If $r<k$, then $|N(u) \cap V(K)| \geq 2(k-r)$, $|N(v) \cap V(K)| \geq 2(k-r)$ and hence $|N(x) \cap V(K)| \geq 2(k-r)$. If none of the paths $P_1, P_2, \ldots, P_r$ has length congruent to 0 modulo k, then the edge ux and the paths $P_i \cup vx$ for $1 \leq i \leq r$ are $r+1$ u–x paths in $G-V(K)$ with distinct lengths modulo k, and (G,K) contains a configuration of type **C** and rank $r+1$.

Suppose one of the paths $P_1, P_2, \ldots, P_r$ has length congruent to 0 modulo k. Then by relabeling the vertex x as w, choosing the vertices u', w', v' to be the vertices u, w, v respectively, and the paths P_u, P_w, P_v to be trivial, we get a configuration of type **D** and rank r in (G,K).

Case 1.4

Finally, suppose (G,K') contains a configuration of type **D** and rank r, for some $2 \leq r < k$. Let u, w, v be the three vertices in $V(G) \setminus V(K')$ such that uw, wv are edges in G. Let u', w', v' be the vertices in $V(G) \setminus V(K')$ such that $|N(u') \cap V(K')| \geq 2(k-r)-1$, $|N(w') \cap V(K')| \geq 2(k-r)$ and $|N(v') \cap V(K')| \geq 2(k-r)$ and let P_u, P_w, P_v be the vertex-disjoint u–u', w–w' and v–v' paths in $G-V(K')$, respectively. Let $\{P_1, P_2, \ldots, P_r\}$ be the set of r u–v paths in $G-V(K')$ that are internally vertex-disjoint from the paths P_u, P_w, P_v and have distinct lengths modulo k.

If x is not adjacent to any of the vertices $\{u', w', v'\}$, it is clear that (G,K) contains the same configuration of type **D** and rank r. If x is adjacent to exactly one of the vertices $\{u', w', v'\}$, then $|N(x) \cap V(K)| \geq 2(k-r)$. If x is adjacent only to z' for some $z \in \{u, w, v\}$, replace the vertex z' by the vertex x and the path P_z by the path $P_z \cup xz'$. This gives a configuration of type **D** and rank r in (G,K).

Suppose x is adjacent to u' and v' but not to w'. Then $|N(w') \cap V(K)| \geq 2(k-r)$, $|N(v') \cap V(K)| \geq 2(k-r)-1$ and hence $|N(x) \cap V(K)| \geq 2(k-r)$. Replace the vertex u' by x and the path P_u by the path $P_u \cup xu'$. Now interchanging the labels of the vertices $\{u, v\}$, labeling x as v' and v' as u', gives a configuration of type **D** and rank r in (G,K).

Suppose x is adjacent to w' and u' and may or may not be adjacent to v'. Then $|N(w') \cap V(K)| \geq 2(k-r)-1$, $|N(v') \cap V(K)| \geq 2(k-r)-1$ and hence $|N(x) \cap V(K)| \geq 2(k-r)-1$. Let $Q_i = xu' \cup P_u \cup P_i \cup P_v$ and $Q'_i = xw' \cup P_w \cup wu \cup P_i \cup P_v$ for $1 \leq i \leq r$, be $2r$ x–v' paths in $G-V(K)$. If amongst the $2r$ paths $\{Q_1, Q'_1, \ldots, Q_r, Q'_r\}$, there are $r+1$ paths of distinct lengths modulo k, then (G, K) contains a configuration of type **C** and rank $r+1$, with x and v' as the two required vertices. Similarly, let $S_i = w'x \cup xu' \cup P_u \cup P_i \cup P_v$ and $S'_i = P_w \cup wu \cup P_i \cup P_v$ for $1 \leq i \leq r$, be $2r$ w'–v' paths in $G - V(K)$. If amongst the $2r$ paths $\{S_1, S'_1, \ldots, S_r, S'_r\}$ there are $r+1$ paths of distinct lengths modulo k, then (G, K) contains a configuration of type **C** and rank $r+1$, with w' and v' as the required vertices.

Suppose both sets of paths $\{Q_1, Q'_1, \ldots, Q_r, Q'_r\}$ and $\{S_1, S'_1, \ldots, S_r, S'_r\}$ contain at most r paths of distinct lengths modulo k. Note that $l(Q_i) \equiv l(P_i) + C_1 \mod k$ for some constant C_1 and all $1 \leq i \leq r$, which implies that $\{Q_1, \ldots, Q_r\}$ have distinct lengths modulo k. Similarly, $l(Q'_i) \equiv l(P_i) + C_2 \mod k$ for some constant C_2. Also $l(S_i) = l(Q_i) + 1$ and $l(S'_i) = l(Q'_i) - 1$. Suppose $l(P_i) \equiv a \mod k$ for some $1 \leq i \leq r$ and natural number a. Then $l(Q'_i) \equiv a + C_2 \mod k$ and there exists an index j such that $l(Q_j) \equiv a + C_2 \mod k$. Therefore $l(S_j) \equiv a + C_2 + 1 \mod k$. Hence, there is an index m such that $l(S'_m) \equiv a + C_2 + 1 \mod k$, which implies $l(P_m) \equiv a + 2 \mod k$. Since this holds for all paths P_i, and there exists a path of length congruent to 0 modulo k, there must be paths of all even lengths modulo k in $\{P_1, P_2, \ldots, P_r\}$. Then $uw \cup wv \cup P_i$ for $1 \leq i \leq r$ are cycles of all even lengths modulo k in $G - V(K)$. Note that this case can occur only when k is even and $r = k/2$.

If x is adjacent to w' and v' but not to u', then $|N(w') \cap V(K)| \geq 2(k-r) - 1$, $|N(u') \cap V(K)| \geq 2(k-r) - 1$ and hence $|N(x) \cap V(K)| \geq 2(k-r) - 1$. Now, we can use the same argument as before, by interchanging the vertices u, v and u', v'.
Case 2.

Suppose there is no vertex $x \in V(G) \setminus V(K)$ that is adjacent to all vertices in $V(K)$. Let v be any vertex in $V(K)$. For every vertex $u \in N_G(v) \setminus V(K)$, let $f(u)$ denote any vertex in $V(K)$ that is not adjacent to u. Let G' be the graph obtained from $G - \{v\}$ by adding edges $uf(u)$ for all vertices $u \in N_G(v) \setminus V(K)$. Let $K' = K - \{v\}$.

Now, $|V(G') \setminus V(K')| = |V(G) \setminus V(K)|$ but $|V(K')| < |V(K)|$, and every vertex in $V(G') \setminus V(K')$ has degree at least $2k-1$ in G'. Hence by the minimality of (G, K), either $G' - V(K')$ contains cycles of all even lengths modulo k, or (G', K') contains one of the four types of configurations. Since $G' - V(K') = G - V(K)$ and $|N_{G'}(u) \cap V(K')| = |N_G(u) \cap V(K)|$ for every vertex $u \in V(G) \setminus V(K)$, it follows that either $G - V(K)$ contains cycles of all even lengths modulo k, or (G, K) contains the same configuration as (G', K'). □

The proof of Theorem 1 follows immediately from Lemma 1. If G is a graph with minimum degree $2k-1$, then the ordered pair $(G, \emptyset)$ satisfies the hypothesis of

Lemma 1 and hence either G contains cycles of all even lengths modulo k, or $(G, \emptyset)$ contains a configuration of one of the four types. However, since K is empty, the latter is not possible, and the theorem follows. □

Note that the proof shows that the cycles can be chosen such that two adjacent edges are included in all cycles.

3 Remarks

As mentioned in the introduction, it is straightforward to verify Thomassen's conjecture for $m = 1$. However, we do not know any other natural numbers m for which it is known to be true. We show that it is true for $m = 2$.

Theorem 2 *Every graph with minimum degree $k + 1$ contains a cycle of length congruent to 4 modulo k, for all integers $k \geq 2$.*

We use the same technique as in the previous section. Here, we need only the configurations of type **A** and **C**, with some modifications. In a configuration of type **A** we require $|N(u) \cap V(K)| \geq k + 1$. In a configuration of type **C** and rank r, we require $|N(u) \cap V(K)| \geq k - r + 1$ and $|N(v) \cap V(K)| \geq k - r + 1$, where the rank r is at least one and at most k. The only difference in the proof is that when considering a configuration of type **C** and rank r, in the case when x is adjacent to both u and v, if any of the u–v paths in the configuration has length congruent to 2 modulo k, then $G - V(K)$ contains a cycle of length congruent to 4 modulo k. If no path has length congruent to 2 modulo k, then there are $r + 1$ u–v paths in $G - V(K)$ of distinct lengths modulo k, which gives a configuration of type **C** and rank $r + 1$ in (G, K). □

References

[1] B. Bollobás, Cycles modulo k, Bull. London Math. Soc. 9 (1977), 97-98.

[2] X. Cai and W. Shreve, Pancyclicity mod k of claw-free graphs and $K_{1,4}$-free graphs, Discrete Math. 230 (2001), 113-118.

[3] G. T. Chen and A. Saito, Graphs with a cycle of length divisible by three, J. Combin. Theory Ser. B 60 (1994), 277-292.

[4] N. Dean, A. Kaneko, K. Ota and B. Toft, Cycles modulo 3, DIMACS Technical Report 91-32 (1991).

[5] N. Dean, L. Lesniak and A. Saito, Cycles of length 0 modulo 4 in graphs, Discrete Math. 121 (1993), 37-49.

[6] P. Erdős, Some recent problems and results in graph theory, combinatorics, and number theory, Proc. Seventh S—E Conf. Combinatorics, Graph Theory and Computing, Utilitas Math., Winnipeg, (1976) 3-14.

[7] G. Fan, Distribution of cycle lengths in graphs, J. Combin. Theory Ser B 84 (2002), 187-202.

[8] W. Mader, Existenz gewisser Konfigurationen in n-gesättigten Graphen und in Graphen genügend großer Kantendichte, Math. Ann. 194 (1971), 295-312.

[9] C. Thomassen, Graph decomposition with applications to subdivisions and path systems modulo k, J. Graph Theory 7 (1983), 261-271.

[10] C. Thomassen, 'Paths, Circuits and Subdivisions', Selected Topics in Graph Theory 3 L.W. Beineke and R.J. Wilson, (Editors), Academic Press, New York (1988), pp. 97-132.

[11] J. Verstraëte, On arithmetic progressions of cycle lengths in graphs, Combin. Probab. and Comput., 9 (2000), 369-373.