On 2-factors in powers of connected graphs

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Abstract

It is well-known that the cube of a connected graph and the square of a 2-connected graph are Hamiltonian. In this paper, we investigate the existence of more general 2-factors in powers of connected graphs. We show that the fifth power of a connected graph contains all 2-factors, while the fourth power contains all 2-factors with at most three cycles. These results cannot be improved in general.

1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G, denoted $|G|$, is the number of vertices in G. A graph H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning* subgraph of G is a subgraph of the same order as G. A *2-factor* is a 2-regular spanning subgraph. In general, a 2-factor is a collection of vertex-disjoint cycles. A 2-factor containing a single cycle is called a *Hamilton cycle* and a graph containing such a 2-factor is said to be *Hamiltonian*. We say that a graph G contains all 2-factors if for any 2-regular graph H of order $|G|$, G contains a subgraph isomorphic to H. Similarly, G contains all 2-factors with k cycles if for any 2-regular graph H of order $|G|$ containing k disjoint cycles, G contains a subgraph isomorphic to H.

The distance between two vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G, and is denoted by $d_G(u,v)$. The graph G^k, for any integer $k \geq 1$, has $V(G^k) = V(G)$, $E(G^k) = \{\{u,v\} \mid d_G(u,v) \leq k\}$ and is called the kth power of the graph G. G^2 is called the square of G and G^3 is the cube of the graph G. A path P in a graph G from vertex u to v is denoted $P(u,v)$. For convenience, we consider a path to have two endpoints, not necessarily distinct, and $P(u,u)$ will denote a path of order one containing only the vertex u. If T is any tree and uv an edge in T, we denote by T_u and T_v the components of $T - uv$ containing u and v respectively. We note that the distance between two vertices in any subtree of a tree T is the same as the distance between them in T. All other terminology may be found in standard books such as [1].

It is well-known that the cube of any connected graph and the square of any 2-connected graph are Hamiltonian [1, 2, 5, 7]. Recently, many of the classical conditions, especially degree conditions, which imply that a graph is Hamiltonian, have been shown to also imply the existence of 2-factors with more than one component. A survey of results on more general 2-factors may be found in [3]. Similar results have also been obtained for claw-free graphs [4]. In this paper we attempt a similar generalization for powers of connected graphs.

In general, the cube of a connected graph does not necessarily contain all 2-factors. In fact, it may not even contain all 2-factors with two cycles. On the other hand, it is easy to see that the sixth power of a connected graph contains the square of a Hamilton cycle, and hence contains all 2-factors.

In this paper, we show that in fact the fifth power of a connected graph contains the square of a Hamilton cycle and hence all possible 2-factors. Also, the fourth power of a connected graph contains all 2-factors with at most three cycles. These results cannot be improved in general.

2 Results

In this section we prove the main results. Note that it is sufficient to prove the results for trees, since the kth power of a subgraph of a graph G is a subgraph of G^k, and any connected graph contains a spanning tree.

Lemma 1 *Any graph containing the square of a Hamilton path contains all 2-factors.*

Proof: Let $P = v_1, v_2, \ldots, v_n$ be a Hamilton path in a graph G such that $P^2 \subseteq G$. Let H be any 2-regular graph of order n containing k cycles of lengths

$n_1,\ n_2,\ldots,n_k$. Note that the subgraph of G induced by a consecutive set of vertices $v_i, v_{i+1}, \ldots, v_{i+j}$ is Hamiltonian, for all $1 \le i \le n-2, 2 \le j \le n-i$. If j is even, $v_i, v_{i+2}, \ldots, v_{i+j}, v_{i+j-1}, \ldots, v_{i+3}, v_{i+1}, v_i$ is the required cycle. If j is odd just interchange v_{i+j} and v_{i+j-1} to get the cycle. We can now find a subgraph isomorphic to H by partitioning the vertices into k consecutive sets of sizes $n_1,\ n_2,\ldots,n_k$ and taking the Hamilton cycle in the subgraph induced by each set. □

Theorem 1 *Let T be any tree and u a specified vertex in T. Then T^3 contains a Hamilton path $P(u,v) = u, v_2, v_3, \ldots, v_{n-1}, v$ such that $P^2 \subseteq T^5$, $d_T(u,v) \le 1$ and $d_T(u,v_2)$, $d_T(v_{n-1},v) \le 2$.*

Proof: The result is trivial if $|T| = 1$, so let uv be any edge in T incident to u. By induction, T_u^3 contains a Hamilton path $P_u(u,u_1)$, such that $P_u^2 \subseteq T^5$, $d_T(u,u_1) \le 1$ and the neighbours of u and u_1 in P_u are at distance at most two in T from u and u_1 respectively. Similarly, T_v^3 contains a Hamilton path $P_v(v_1,v)$ such that $P_v^2 \subseteq T^5$, $d_T(v,v_1) \le 1$ and the neighbours of v_1 and v in P_v are at distance at most two in T from v_1 and v respectively. Since $d_T(u_1,v_1) \le 3$ and $d_T(u,v_1)$, $d_T(u_1,v) \le 2$, adding edge u_1v_1 to $P_u \cup P_v$ gives the required path P in T^3. Note that the induction hypothesis is satisfied even if either T_u or T_v has order one. □

Theorem 1 implies that the fifth power of any tree contains the square of a Hamilton path and by lemma 1 it contains all 2-factors.

We next show that the fourth power of a connected graph contains all 2-factors with at most three cycles. In the next section we give an example of a tree whose fourth power does not contain a 2-factor with four cycles of equal length.

Lemma 2 *Let T be any tree of order $n > 1$ and let u be any specified vertex in T. For any two positive integers n_1, n_2 such that $n_1 + n_2 = n$, T^4 contains two vertex-disjoint paths $P_1(u,x)$ and $P_2(v,y)$ of orders n_1 and n_2 respectively, such that $d_T(u,x) \le 2$, v is adjacent to u and $d_T(u,y) \le 3$.*

Proof: Again, the proof is by induction on n. The result is trivial if $n = 2$ so let T be any tree of order $n > 2$ and let u be any specified vertex.

Suppose there is an edge uv_1 incident to u such that $|T_{v_1}| < n_2$. There is a Hamilton path $P'(v_1,y)$ in $T_{v_1}^4$ such that $d_T(v_1,y) \le 1$, by theorem 1. By induction, T_u^4 contains vertex-disjoint paths $P_1(u,x)$ and $P''(v,v_2)$ of orders n_1 and $n_2 - |T_{v_1}|$ such that $d_T(u,x) \le 2$, v is adjacent to u and $d_T(u,v_2) \le 3$.

Since v_2 is adjacent to v_1 in T^4, adding edge v_2v_1 to $P'' \cup P'$ gives the required path P_2 of order n_2 from v to y.

Similarly, suppose there is an edge uv_1 incident to u such that $|T_{v_1}| < n_1$. There is a Hamilton path $P'(v_1, x)$ in $T^4_{v_1}$ such that $d_T(v_1, x) \leq 1$. By induction, T^4_u contains two vertex-disjoint paths $P''(u, v_2)$ and $P_2(v, y)$ of orders $n_1 - |T_{v_1}|$ and n_2 respectively, such that $d_T(u, v_2) \leq 2$, v is adjacent to u and $d_T(u, y) \leq 3$. Since v_2 is adjacent to v_1 in T^4, adding edge v_2v_1 to $P'' \cup P'$ gives the required path P_1 of order n_1 from u to x.

If neither of the above two cases holds then u must have degree one in T. Let uv be the edge incident to u in T. If $n_1 = 1$, let P_1 be the path of order one containing u, and let $P_2(v, y)$ be a Hamilton path in T^4_v such that $d_T(v, y) \leq 1$. If $n_1 > 1$, then by induction, T^4_v contains two vertex-disjoint paths $P_2(v, y)$ and $P'(x, u_1)$ of orders n_2 and $n_1 - 1$ respectively, such that $d_T(v, y) \leq 2$, x is adjacent to v and $d_T(v, u_1) \leq 3$. Since u is adjacent to u_1 in T^4, adding vertex u and edge uu_1 to P' gives the required path P_1 of order n_1 from x to u. □

Lemma 2 implies that the fourth power of any tree contains all 2-factors with two cycles, since the endpoints of the paths in lemma 2 are adjacent in the fourth power of the tree. We next show that the fourth power of a tree contains all 2-factors with three cycles.

Lemma 3 *Suppose there is a positive integer $k \geq 1$ such that for every edge uv in a tree T, $\min(|T_u|, |T_v|) \leq k$. Then there exists a vertex u such that for every edge uv incident to u, $|T_v| \leq k$.*

Proof: Suppose, for contradiction, that for every vertex u there is an edge uv incident to u such that $|T_v| > k$. Choose a vertex u and the edge uv such that $|T_v| = m > k$ is minimum amongst all possible choices. We must have $|T_u| \leq k$, otherwise uv is an edge with $\min(|T_u|, |T_v|) > k$. Then for any edge vw incident to v, we must have $|T_w| < m$, contradicting the choice of u and v. □

Theorem 2 *The fourth power of any tree T contains all 2-factors with three cycles.*

Proof: Let T be any tree of order $n \geq 9$ and let H be any 2-regular graph of order n containing three cycles of lengths n_1, n_2, n_3 and let n_1 be the smallest of the three numbers.

Suppose there is an edge uv in T such that $\min(|T_u|, |T_v|) > n_1$. We may assume that $|T_u| < n_1 + n_2$, for if not, $|T_v| \leq n_3 < n_1 + n_3$ and we may apply the same argument by interchanging T_u with T_v and n_2 with n_3.

By lemma 2, T_u^4 contains two vertex-disjoint paths $P'(u,u_2)$ and $P_1(u_1,v_1)$ of orders $|T_u|-n_1$ and n_1 respectively, such that $d_T(u,u_2)\le 2$, u_1 is adjacent to u and $d_T(u,v_1)\le 3$. Since the endpoints of P_1 are adjacent in T^4, we get the required cycle of length n_1. Similarly, T_v^4 contains two vertex-disjoint paths $P''(v,v_2)$ and $P_3(u_3,v_3)$ of orders $|T_v|-n_3$ and n_3 respectively, such that $d_T(v,v_2)\le 2$, u_3 is adjacent to v and $d_T(v,v_3)\le 3$. Again, since u_3 is adjacent to v_3 in T^4, we get the cycle of length n_3 in T^4. Also, u_2 is adjacent to v and u is adjacent to v_2 in T^4, so the paths P' and P'' can be combined to get the remaining cycle of length n_2.

If for every edge uv in T, $\min(|T_u|,|T_v|)\le n_1$, then by lemma 3 there is a vertex u such that for every edge uv incident to u, $|T_v|\le n_1$. Let $v_1,\ldots,v_k$ be the vertices adjacent to u. Note that k must be at least three, since n_1 is the smallest of the three numbers. Let i be the smallest index such that $|T_{v_1}|+\cdots+|T_{v_i}|\ge n_1$. Let T' be the subtree of T induced by u and the vertices in $T_{v_1},\ldots,T_{v_i}$, and let T'' be the subtree of T containing u and the subtrees $T_{v_{i+1}},\ldots,T_{v_k}$. Note that $1\le i<k$ and $|T'|\le n_1+n_2$.

By lemma 2, $(T')^4$ contains two vertex-disjoint paths $P_1(u_1,v_1)$ and $P'(u,u_2)$ of orders n_1 and $|T'|-n_1$ respectively, such that $d_T(u,u_2)\le 2$, u_1 is adjacent to u and $d_T(u,v_1)\le 3$. Since the endpoints of P_1 are adjacent in T^4, we get the required cycle of length n_1. Similarly, $(T'')^4$ contains two vertex-disjoint paths $P''(u,v_2)$ and $P_3(u_3,v_3)$ of orders $|T''|-n_3$ and n_3 respectively, such that $d_T(u,v_2)\le 2$, u_3 is adjacent to u and $d_T(u,v_3)\le 3$. Again, since the endpoints of P_3 are adjacent in T^4, we get the required cycle of length n_3. The paths P' and P'' have only the vertex u in common and their union is a path $P_2(u_2,v_2)$ of order n_2 in T^4. Since u_2 is adjacent to v_2 in T^4, we get the remaining cycle of length n_2 and thus a 2-factor isomorphic to H.

This completes the proof of the theorem. $\square$

3 Examples

In this section we give examples that show that the results in the previous section cannot be improved in general. For any graph G, we denote by $S_i(G)$ the graph obtained by replacing every edge uv of G by a path $P(u,v)$ of length i such that the internal vertices of all the paths added are disjoint.

Lemma 4 *Let T be the tree $S_5(K_{1,3})$. Then T^3 does not contain two vertex-disjoint cycles of length eight.*

Proof: Let u be the vertex of degree three in T and let u_1, u_2, u_3 be the vertices adjacent to u. Let T_i be the component of $T - uu_i$ containing u_i and let v_i be the vertex adjacent to u_i in T_i. Suppose T^3 contains such cycles and let C be the cycle which does not contain u. The cycle C must contain at least two vertices from at least two of the subtrees T_1, T_2, T_3 since each of the subtrees contains five vertices. Further, if C includes at least two vertices from T_i, it must include both u_i and v_i. If it includes only one of them, there cannot be two vertex-disjoint paths in C between the other vertex of C in T_i and a vertex of C not in T_i. If C includes both u_i and v_i, it must include all vertices in T_i, otherwise there cannot be two vertex-disjoint paths in the other cycle between a vertex in T_i and a vertex not in T_i. This implies that all vertices in at least two of the subtrees must be in C which must therefore have length at least ten, a contradiction. □

Lemma 5 *Let T be the tree $S_{11}(K_{1,5})$. Then T^4 does not contain four vertex-disjoint cycles of length* 14.

Proof: Let u be the vertex of degree five in T and let $T_i, 1 \leq i \leq 5$ be the components of $T - u$. We call the vertices in T_i which are at distance at most three from u in T as *connecting* vertices. There are 15 connecting vertices, three in each subtree. Suppose there exist four disjoint cycles of length 14 in T^4 and let C be a cycle not including u. If C includes at least two vertices from T_i, it must include at least two connecting vertices from T_i and if it includes only one vertex from T_i it must be a connecting vertex. Thus if C includes at least two vertices from two subtrees it must contain at least four connecting vertices. If C contains more than one vertex from only one subtree, it must contain at least three connecting vertices from other subtrees and hence totally at least six connecting vertices. Since there are three cycles not including u, and two cycles cannot contain two connecting vertices from the same subtree, at least one of the three cycles contains six connecting vertices and the others at least four each. Thus at least 14 connecting vertices are included in the cycles not including the u. The cycle containing u includes at least one vertex from two different subtrees and hence at least one connecting vertex from each of them. This implies there are at least 16 connecting vertices, a contradiction. □

4 Remarks

We have characterized the type of 2-factors that always exist in powers of general connected graphs based on the number of cycles. It may be possible to further characterize them in terms of the lengths of cycles. For example, the cube of any connected graph always contains a 2-factor with two cycles, one of which is a triangle. Further improvements may be possible by considering more restricted classes of graphs. Since the squares of 2-connected graphs and claw-free graphs [6] are Hamiltonian, the fourth powers of such graphs contain all 2-factors. It would be interesting to see what 2-factors are contained in the cubes of such graphs. More generally, it would be interesting to study the effect of forbidden subgraphs on the existence of 2-factors in powers of connected graphs.

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