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Subdivisions of maximal 3-degenerate graphs of order $d+1$ in graphs of minimum degree d

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Abstract

We prove that every graph of minimum degree at least $d \geq 1$ contains a subdivision of *some* maximal 3-degenerate graph of order $d+1$. This generalizes the classic results of Dirac ($d = 3$) and Pelikán ($d = 4$). We conjecture that for any planar maximal 3-degenerate graph H of order $d+1$ and any graph G of minimum degree at least d, G contains a subdivision of H. We verify this in the case H is P_6^3 and P_7^3.

1 Introduction

A classic result of Dirac [2] states that every graph of minimum degree at least 3 contains a subdivision of K_4. Pelikán [6] proved that every graph of minimum degree at least 4 contains a subdivision of K_5^-, the graph obtained by deleting an edge from K_5. In general, Mader [4] first showed that there exists a function $f(k)$ such that every graph of minimum degree at least $f(k)$ contains a subdivision of K_k. Bollobás and Thomason [1] showed that $f(k)$ is $O(k^2)$ and this is best possible.

We consider the question in the other direction. For a given integer $d \geq 1$, for what graphs H is it true that every graph of minimum degree at least d contains a subdivision of H? Clearly, any such graph

H can have at most $d+1$ vertices, since K_{d+1} has minimum degree d. We consider graphs H of order exactly $d+1$. We call a graph H *good* if every graph of minimum degree at least $|H|-1$ contains a subdivision of H. For $1 \leq d \leq 3$, it follows that K_{d+1} is good. Since there are planar graphs of minimum degree 4, K_5 is not good, but Pelikán's theorem implies that K_5^- is good. We are interested in finding the maximal good graphs. Mader [5] showed that every graph of minimum degree at least $d \geq 2$ contains a pair of adjacent vertices with d internally disjoint paths between them. This implies that the graph $K_2 \vee \overline{K_{d-1}}$, consisting of $d-1$ triangles that share a common edge, is good. However, this graph has only $2d-1$ edges and is not a maximal good graph even for $d=3$. Turner [7] showed that the wheel $W_d = C_d \vee K_1$ is good, for all $d \geq 3$, but again this has size $2d$ and is not a maximal good graph for $d=4$.

Our main result is that every graph of minimum degree at least $d \geq 2$ contains a subdivision of *some* graph H of order $d+1$ and size $3d-3$. For $d=3,4$ this implies the theorems of Dirac and Pelikán, respectively, since K_4 and K_5^- are the only possible such graphs. Further, for $d=5$, this is the maximum possible number of edges in a good graph, since there exist planar graphs of minimum degree 5. We are unable to prove that any specific graph H of order $d+1$ and size $3d-3$ is good, for general d, but we can say something more about the structure of the graph H. We show that H can be chosen to be 3-degenerate, that is, every subgraph of H contains a vertex of degree at most 3. We conjecture that every planar 3-degenerate graph of order $d+1$ and size $3d-3$ is good. We prove this for two specific graphs P_6^3 and P_7^3. A weaker conjecture would be that P_n^3 is good for all $n \geq 2$.

2 Notation

All graphs considered are undirected, finite and simple. The vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$. The order of a graph G is $|V(G)|$ and $|E(G)|$ is its size. The subset of vertices adjacent to a vertex $v \in V(G)$ in a graph G is denoted by $N_G(v)$ and $d_G(v) = |N_G(v)|$ is the degree of the vertex v. If $S \subset V(G) \cup E(G)$, $G-S$ is the graph obtained from G by deleting all vertices and edges in S and also edges incident with vertices in S. A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $S \subset V(G)$,

$G - (V(G) \setminus S)$ is the subgraph of G induced by S. The union of two subgraphs H_1, H_2 of a graph G is the subgraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$.

A path P in a graph G is a sequence of distinct vertices $v_0, \ldots, v_l$ such that $v_i v_{i+1}$ is an edge in G for $0 \leq i < l$. We say P is a v_0–v_l path that joins v_0 to v_l. The vertices v_0, v_l are the endpoints of P and $\{v_1, \ldots, v_{l-1}\}$ are the internal vertices of P. The set of internal vertices of P is denoted $I(P)$. We will also consider P to be a subgraph of G with vertex set $\{v_0, \ldots, v_l\}$ and edge set $\{v_i v_{i+1} \mid 0 \leq i < l\}$. A path P is said to be an A–B path in G, for $A, B \subseteq V(G)$, if P joins a vertex in A to a vertex in B and $I(P) \cap (A \cup B) = \emptyset$. An A–B path is also said to join A to B. A set of paths $\mathcal{P}$ in G is said to be internally disjoint if for any two distinct paths $P, Q \in \mathcal{P}$, $I(P) \cap I(Q) = \emptyset$. If $\mathcal{P}$ is a set of internally disjoint paths, let $I(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} I(P)$ be the set of internal vertices of $\mathcal{P}$. If $A \subset V(G)$ and $u \in V(G) \setminus A$, a u–A fan is a set of internally disjoint u–A paths having distinct endpoints in A.

A graph G is said to contain a subdivision of a graph H if there exists a subset $B(H) \subseteq V(G)$ of vertices and a set $\mathcal{P}$ of internally disjoint $B(H)$–$B(H)$ paths in G such that:

1. There exist bijections $f : V(H) \to B(H)$ and $g : E(H) \to \mathcal{P}$.
2. If $uv \in E(H)$ then $g(uv)$ is an $f(u)$–$f(v)$ path in G.

We call the subgraph of G formed by the union of the paths in $\mathcal{P}$ a subdivision of H and denote it $\mathcal{T}(H)$. The vertex $f(v) \in V(\mathcal{T}(H))$ is said to correspond to the vertex $v \in V(H)$.

An ordered clique in a graph G is a complete subgraph of G together with a total ordering imposed on the vertices in the complete subgraph.

Let G be a graph and K an ordered clique in G. Let $u_1, u_2, \ldots, u_t$ be a sequence of vertices in $V(G) \setminus V(K)$ and $n_1, n_2, \ldots, n_t$ a sequence of positive integers. We say $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K in G by a set of paths $\mathcal{P}$ if the paths satisfy the following properties:

1. Every path in $\mathcal{P}$ is a $\{u_1, \ldots, u_t\}$–$V(K)$ path.
2. $\mathcal{P}$ is a set of internally disjoint paths.
3. No two paths in $\mathcal{P}$ have the same pair of endpoints.
4. Exactly n_i paths in $\mathcal{P}$ have u_i as an endpoint, for all $1 \leq i \leq t$.

3 Unavoidable configurations

The main technique used in this paper is essentially the same as used by Mader in [5]. We consider ordered pairs of the form (G, K), where K is an ordered clique in the graph G. We define a reduction operation on such pairs.

Let G be a graph and K an ordered clique in G, such that $V(K) \subset V(G)$. Let $v_1 < v_2 < \cdots < v_k$ be the ordering of the vertices in K.

1. Suppose there exists a vertex $w \in V(G) \setminus V(K)$ that is adjacent to all vertices in $V(K)$. Let K' be the ordered clique in G with $V(K') = V(K) \cup \{w\}$ and the ordering $w < v_1 < \cdots < v_k$ of $V(K')$. We say the pair (G, K') is obtained from the pair (G, K) by adding the vertex w.

2. Suppose every vertex in $V(G) \setminus V(K)$ is not adjacent to at least one vertex in $V(K)$. For every vertex $u \in N_G(v_1) \setminus V(K)$ let $f(u)$ be the smallest index such that $v_{f(u)} \notin N_G(u)$. Let G' be the graph obtained from G by deleting the vertex v_1 and adding the edge $uv_{f(u)}$, for all vertices $u \in N_G(v_1) \setminus V(K)$. Let K' be the ordered clique in G' with $V(K') = V(K) \setminus \{v_1\}$ and the ordering $v_2 < \cdots < v_k$ of $V(K')$. We say the pair (G', K') is obtained from (G, K) by deleting the vertex v_1.

Note that for any pair (G, K) with $V(K) \subset V(G)$, exactly one of the two operations can be applied. We say a pair (G', K') can be derived from the pair (G, K), denoted $(G, K) \to (G', K')$, if (G', K') can be obtained from (G, K) by a sequence of vertex deletion or addition operations.

Lemma 1 *If $(G, K) \to (G', K')$ then the following properties hold.*

1. *$V(G') \setminus V(K') \subseteq V(G) \setminus V(K)$.*
2. *For every vertex $u \in V(G') \setminus V(K')$, $d_{G'}(u) = d_G(u)$.*
3. *For any subset $S \subseteq V(G' - V(K')) \cup E(G' - V(K'))$, $(G - S, K) \to (G' - S, K')$.*
4. *$|V(G')| + |V(G') \setminus V(K')| < |V(G)| + |V(G) \setminus V(K)|$.*

Proof: The proof follows by induction on the number of reduction operations used to derive (G', K') from (G, K). It is easy to check that each operation satisfies the required properties. □

Lemma 2 *Suppose (G', K') is obtained from (G, K) by deleting a vertex. If $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K' in G', then it is also $(n_1, \ldots, n_t)$-joined to K in G.*

Proof: Let K be the ordered clique $v_1 < v_2 < \cdots < v_k$ and suppose K' is obtained from K by deleting v_1, keeping the order of the remaining vertices the same. Let $\mathcal{P}$ be the set of paths that $(n_1, \ldots, n_t)$-joins $(u_1, \ldots, u_t)$ to K' in G'. By the definition of the reduction operation, the only edges in G' that are not in G are edges of the form $uv_{f(u)}$ for every vertex $u \in N_G(v_1) \setminus V(K)$. We call such edges *bad* edges. Note that there is at most one bad edge incident with any vertex $u \in V(G') \setminus V(K')$, and it must have one endpoint in $V(K')$. Also, if $uv_{f(u)}$ is a bad edge, by the definition of $f(u)$, uv_i is an edge in G, for all $1 \leq i < f(u)$.

Let $\mathcal{P}_i$ be the set of n_i paths in $\mathcal{P}$ that form a u_i–$V(K')$ fan. If none of these paths contain a bad edge, these form a u_i–$V(K)$ fan in G. If any of these paths contains a bad edge, it must be the last edge in the path. Let $w_m v_{j_m}$, for $1 \leq m \leq l$, be the bad edges contained in the paths in $\mathcal{P}_i$, where $1 = j_0 < j_1 < j_2 < \cdots < j_l$, and $1 \leq l \leq n_i$. Then, replacing the bad edge $w_m v_{j_m}$ by the edge $w_m v_{j_{m-1}}$, for $1 \leq m \leq l$, gives a set of n_i paths that form a u_i–$V(K)$ fan in G. These paths have the same set of internal vertices as the paths in $\mathcal{P}_i$. Since the paths in $\mathcal{P}_i$ and $\mathcal{P}_j$ can only have vertices in $V(K')$ in common for $i \neq j$, this step can be done independently for each u_i. Thus we get a set of paths that $(n_1, \ldots, n_t)$-joins $(u_1, \ldots, u_t)$ to K in G. □

Lemma 3 *Suppose (G, K') is obtained from (G, K) by adding a vertex w. Suppose $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K' in G by a set of paths $\mathcal{P}$. If the sequence $n_1, \ldots, n_t$ does not have a unique maximum, and at most one path in $\mathcal{P}$ has w as an endpoint, then $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K in G.*

Proof: If none of the paths in $\mathcal{P}$ has w as an endpoint, then $\mathcal{P}$ is a set of paths that $(n_1, \ldots, n_t)$-joins $(u_1, \ldots, u_t)$ to $K = K' - w$ in G. Suppose $\mathcal{P}$ contains exactly one path terminating in w. Without loss of generality, we may assume u_1 is its other endpoint. Since the sequence $n_1, \ldots, n_t$ does not have a unique maximum, there exists an $i > 1$ such that $n_i \geq n_1$. Without loss of generality, assume $i = 2$. Since $\mathcal{P}$ contains n_2 u_2–$V(K')$ paths having distinct endpoints in $V(K') \setminus \{w\}$, we must have $|V(K')| > n_2 \geq n_1$. This implies $|V(K)| \geq n_1$. Since $\mathcal{P}$ contains n_1 u_1–$V(K')$ paths, one of which terminates in w, there exists a vertex $v \in V(K') \setminus \{w\}$, such that there is no u_1–v path in $\mathcal{P}$. Since w is adjacent to all vertices in K, adding the edge wv to the u_1–w path in $\mathcal{P}$, together with all other paths in $\mathcal{P}$, gives a set of paths that $(n_1, \ldots, n_t)$-joins $(u_1, \ldots, u_t)$ to K in G. □

In view of Lemma 2, we will henceforth only need to consider cases where (G, K') is obtained from (G, K) by adding a vertex w. Suppose $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K' in G by a set of paths $\mathcal{P}$. In any such case, we will denote by $\mathcal{P}' \subseteq \mathcal{P}$ the subset of paths that terminate in w, and by $U' \subseteq \{u_1, \ldots, u_t\}$ the endpoints of paths in $\mathcal{P}'$ other than w. We will only consider cases where the sequence $n_1, \ldots, n_t$ does not have a unique maximum, and hence by Lemma 3, we only need to consider cases where $|\mathcal{P}'| = |U'| \geq 2$.

Lemma 4 *Suppose (G, K') is obtained from (G, K) by adding a vertex w. Suppose $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K' in G by a set of paths $\mathcal{P}$ and suppose $|\mathcal{P}'| \geq 2$. Then $(u_1, u_2, \ldots, u_t, w)$ is $(n'_1, \ldots, n'_t, m)$-joined to K in $G - I(\mathcal{P}')$, where $n'_i = n_i - 1$ if $u_i \in U'$ else $n'_i = n_i$, and $m = \max_{1 \leq i \leq t} \; n'_i$.*

Proof: Since the paths in $\mathcal{P}$ are internally disjoint, $\mathcal{P} \setminus \mathcal{P}'$ is a set of paths that $(n'_1, \ldots, n'_t)$-joins $(u_1, \ldots, u_t)$ to K in $G - I(\mathcal{P}')$. Since any two paths in $\mathcal{P}$ can have at most one endpoint in common, $|V(K)| \geq m = \max_{1 \leq i \leq t} \; n'_i$. Since w is adjacent to every vertex in $V(K)$, adding m edges joining w to $V(K)$ to the set of paths $\mathcal{P} \setminus \mathcal{P}'$ gives the required set of paths that $(n'_1, \ldots, n'_t, m)$-joins $(u_1, \ldots, u_t, w)$ to K in $G - I(\mathcal{P}')$. □

Let $\mathcal{C}$ be a set of graphs such that $A = \{a_1, \ldots, a_t\} \subseteq V(H)$, for all graphs $H \in \mathcal{C}$. Suppose each vertex $a_i \in A$ is assigned a positive integer weight n_i, for $1 \leq i \leq t$. We call such a set of graphs $\mathcal{C}$ a *configuration* with *terminal* vertices $(a_1, \ldots, a_t)$ having weights $(n_1, \ldots, n_t)$.

Let $\mathcal{C}$ be a configuration with terminal vertices $(a_1, \ldots, a_t)$ having weights $(n_1, \ldots, n_t)$. We say that $\mathcal{C}$ is *unavoidable* if for every graph G and (G', K') such that $(G, \emptyset) \to (G', K')$, the following property holds.

- If $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K' in G', then G contains a subdivision of some graph $H \in \mathcal{C}$ such that the vertex u_i in G corresponds to the vertex a_i in H, for $1 \leq i \leq t$.

The basic idea to prove that a configuration $\mathcal{C}$ is unavoidable is to use induction on the length of the sequence of reductions $(G, \emptyset) =$

(G_0, K_0), $(G_1, K_1), \ldots, (G_l, K_l)$ such that (G_{i+1}, K_{i+1}) is obtained from (G_i, K_i) by addition or deletion of vertices. If $\mathcal{C}$ has t terminals $(a_1, \ldots, a_t)$ of weights $(n_1, \ldots, n_t)$, we assume $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K_l in G_l. In all configurations that we consider, the sequence of weights does not have a unique maximum. Lemmas 2 and 3 then imply that if (G_l, K_l) is obtained from (G_{l-1}, K_{l-1}) by deleting a vertex or if $|\mathcal{P}'| = 1$, we can apply induction. If $|\mathcal{P}'| \geq 2$, we use Lemma 4 and an appropriate configuration $\mathcal{C}'$ that is either known or assumed to be unavoidable as part of the induction hypothesis, and apply induction. This may require that several configurations are proved unavoidable simultaneously.

The following lemma gives a starting point for applying this argument to a graph of minimum degree at least d.

Lemma 5 *Let G be a graph of minimum degree at least $d \geq 2$. Then there exists a pair (G', K') such that $(G, \emptyset) \to (G', K')$ and $G' - V(K')$ contains an edge u_1u_2 such that (u_1, u_2) is $(d-1, d-1)$-joined to K' in G'.*

Proof: Let $(G, \emptyset) = (G_0, K_0)$, $(G_1, K_1), \ldots, (G_l, K_l)$ be a maximal sequence of pairs such that (G_i, K_i) is obtained from (G_{i-1}, K_{i-1}) by either deleting or adding a vertex, for $1 \leq i \leq l$. Such a sequence exists since $|V(G_{i+1})| + |V(G_{i+1}) \setminus V(K_{i+1})| < |V(G_i)| + |V(G_i) \setminus V(K_i)|$. Then we must have $V(G_l) = V(K_l)$, otherwise we can add one more pair to the sequence. Let $i < l$ be the largest index such that $G_i - V(K_i)$ contains an edge u_1u_2. The choice of i implies that (G_{i+1}, K_{i+1}) is obtained from (G_i, K_i) be adding either the vertex u_1 or u_2 to K_i, otherwise u_1u_2 is an edge in $G_{i+1} - V(K_{i+1})$. Without loss of generality, $V(K_{i+1}) = V(K_i) \cup \{u_1\}$. Then u_2 cannot be adjacent to any vertex other than u_1 in $G_i - V(K_i)$. Since G has minimum degree at least d, u_2 has at least $d-1$ neighbors in $V(K_i)$ and thus $|V(K_i)| \geq d-1$. Since u_1 is adjacent to every vertex in $V(K_i)$, it has at least $d-1$ neighbors in $V(K_i)$. Thus (u_1, u_2) is $(d-1, d-1)$-joined to K_i in G_i, and (G_i, K_i) is the required pair. $\square$

Let H be any graph of order $d+1$ and $\mathcal{C}(H)$ the configuration containing all possible graphs $H - a_1a_2$, for every edge $a_1a_2 \in E(H)$, with terminal vertices (a_1, a_2) having weights $(d-1, d-1)$. If this configuration is unavoidable, Lemma 5 implies that H is good.

We illustrate the method by restating the proof of Mader's theorem in terms of unavoidable configurations. Let $\mathcal{C}(d)$ be the configuration containing the single graph $K_{2,d}$, with the two vertices in the part of size 2 being the terminal vertices having weight d. We claim that for all $d \geq 1$, the configuration $\mathcal{C}(d)$ is unavoidable.

Applying the general strategy, we may assume (G_l, K_l) is obtained from (G_{l-1}, K_{l-1}) by adding a vertex w and $U' = \{u_1, u_2\}$. Then the union of the two paths in $\mathcal{P}'$ is a u_1–u_2 path P in $G_{l-1} - V(K_{l-1})$ that contains w. If $d = 1$, this gives a subdivision of $K_{2,1}$ in G, otherwise by Lemma 4, (u_1, u_2) is $(d-1, d-1)$-joined to K_{l-1} in $G_{l-1} - I(P)$. By induction, $G - I(P)$ contains a subdivision of $K_{2,d-1}$ with vertices u_1, u_2 corresponding to the two terminals in $K_{2,d-1}$. The union of this with the path P gives the required subdivision of $K_{2,d}$. The unavoidability of $\mathcal{C}(d)$ and Lemma 5 proves Mader's theorem for $d \geq 2$.

Turner's theorem for wheels can be proved in a similar way. In this case, we consider the configuration $W_d - a_1a_2$, where a_1a_2 is a spoke and a_1 the center of the wheel. Both a_1, a_2 have weight $d-1$. We also need another configuration $W_d - \{a_1a_2, a_1a_3, a_2a_3\}$, where $\{a_1, a_2, a_3\}$ induce a triangle in W_d with a_1 being the center of the wheel. If $d = 3$, a_1, a_2, a_3 all have weight 1, while for $d \geq 4$, a_1, a_2 have weight $d-2$ and a_3 has weight $d-3$. It can be argued in a similar way that both these configurations are unavoidable for all $d \geq 3$.

4 Maximal 3-degenerate graphs

A maximal 3-degenerate graph of order $n \geq 3$ is a graph whose vertices can be ordered $v_1, \ldots, v_n$ such that $\{v_1, v_2, v_3\}$ induce a K_3 and v_i is adjacent to exactly 3 vertices in $\{v_1, \ldots, v_{i-1}\}$, for $4 \leq i \leq n$.

Theorem 1 *Every graph of minimum degree at least $d \geq 2$ contains a subdivision of some maximal 3-degenerate graph of order $d+1$.*

Proof: The proof follows the same general strategy. We define a set of configurations and show that they are unavoidable. The theorem then follows by applying Lemma 5.

Consider the following configurations.

1. $\mathcal{C}_1(d)$ for $d \geq 1$, contains all graphs of order $d+2$ with two terminal vertices a_1, a_2, and d other vertices $b_1, \ldots, b_d$, ordered so that b_1 is adjacent to a_1 and a_2, and b_i is adjacent to exactly 3 vertices in $\{a_1, a_2, b_1, \ldots, b_{i-1}\}$, for $2 \leq i \leq d$. The two terminal vertices a_1, a_2 have weight d.

2. $\mathcal{C}_2(d)$ for $d \geq 1$, contains all graphs of order $d+3$ with 3 terminal vertices a_1, a_2, a_3, and d other vertices $b_1, b_2, \ldots, b_d$, ordered so that b_i is adjacent to exactly 3 vertices in $\{a_1, a_2, a_3, b_1, \ldots, b_{i-1}\}$, for $1 \leq i \leq d$. The 3 terminal vertices have weight d each.

3. $\mathcal{C}_3(d)$ for $d \geq 1$, contains all graphs of order $d+3$ with 3 terminal vertices a_1, a_2, a_3, such that a_1 adjacent to a_2, and d other vertices $b_1, \ldots, b_d$, ordered so that b_i is adjacent to exactly 3 vertices in $\{a_1, a_2, a_3, b_1, \ldots, b_{i-1}\}$ for $1 \leq i \leq d$. The vertices a_1, a_2 have weight $d+1$, while a_3 has weight d.

4. $\mathcal{C}_4(d, t)$ for $d \geq 0$, $t \geq 3$, contains all graphs with t terminal vertices $a_1, a_2, \ldots, a_t$ such that a_1 is adjacent to a_i for $2 \leq i \leq t$, and d other vertices $b_1, \ldots, b_d$ such that b_i is adjacent to exactly 3 vertices in $\{a_1, \ldots, a_t, b_1, \ldots, b_{i-1}\}$ for $1 \leq i \leq d$. The weight of a_1 is $d+t-1$, and the weight of a_i is $d+i-1$, for $2 \leq i \leq t$.

We show that the configurations $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ are unavoidable. We consider each of the 4 configurations.

Case 1. Consider the configuration $\mathcal{C}_1$. If $d = 1$, this just contains the graph $K_{2,1}$ with 2 terminal vertices of weight 1. This is unavoidable, as argued in the proof of Mader's theorem. Suppose $d \geq 2$. We may assume $\mathcal{P}'$ contains exactly 2 paths. Lemma 4 implies (u_1, u_2, w) is $(d-1, d-1, d-1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of some graph in $\mathcal{C}_2(d-1)$, with vertices u_1, u_2, w corresponding to a_1, a_2, a_3 , respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of a graph in $\mathcal{C}_1(d)$.

Case 2. Consider the configuration $\mathcal{C}_2$. If $d = 1$, this contains the graph $K_{3,1}$ with 3 terminal vertices of weight 1. If $\mathcal{P}'$ contains 3 paths, this gives a subdivision of $K_{3,1}$ in $G_{l-1} - V(K_{l-1})$, with u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. Suppose $|\mathcal{P}'| = 2$, and assume without loss of generality $U' = \{u_1, u_2\}$. Lemma 4 implies (u_3, w) is $(1, 1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. This implies $G - I(\mathcal{P}')$ contains a u_3–w path. Adding this to the paths in $\mathcal{P}'$ gives a subdivision of $K_{3,1}$ in which u_1, u_2, u_3 correspond to a_1, a_2, a_3, respectively.

A similar argument holds if $d \geq 2$. If $\mathcal{P}'$ contains 3 paths, by Lemma 4, (u_1, u_2, u_3) is $(d-1, d-1, d-1)$-joined to K_{l-1} in $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$. By induction, $G - (I(\mathcal{P}') \cup \{w\})$ contains a subdivision of some graph in $\mathcal{C}_2(d-1)$, with vertices u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. Adding the vertex w and the paths in $\mathcal{P}'$ to this, gives a subdivision of a graph in $\mathcal{C}_2(d)$. Suppose $|\mathcal{P}'| = 2$ and assume without loss of generality, $U' = \{u_1, u_2\}$. Lemma 4 implies

(u_3, w, u_1) is $(d, d, d-1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. Therefore $G - I(\mathcal{P}')$ contains a subdivision of some graph in $\mathcal{C}_3(d-1)$, with vertices u_3, w, u_1 corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of a graph in $\mathcal{C}_2(d)$.

Case 3. Consider the configuration $\mathcal{C}_3$. Suppose $|\mathcal{P}'| = 3$. If $d = 1$, then (u_1, u_2) is $(1,1)$-joined to K_{l-1} in $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$. This implies $G - (I(\mathcal{P}') \cup \{w\})$ contains a u_1–u_2 path. Adding w and the paths in $\mathcal{P}'$ to this, gives the required subdivision of the graph in $\mathcal{C}_3(1)$. If $d \geq 2$, then (u_1, u_2, u_3) is $(d, d, d-1)$-joined to K_{l-1} in $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$. By induction, $G - (I(\mathcal{P}') \cup \{w\})$ contains a subdivision of some graph in $\mathcal{C}_3(d-1)$, with vertices u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. Adding the vertex w to this along with the paths in $\mathcal{P}'$, gives the required subdivision of a graph in $\mathcal{C}_3(d)$.

Suppose $|\mathcal{P}'| = 2$ and $U' = \{u_1, u_2\}$. The union of the two paths in $\mathcal{P}'$ is a u_1–u_2 path P in $G_{l-1} - V(K_{l-1})$ that contains w. Lemma 4 implies (u_1, u_2, u_3) is (d, d, d)-joined to K_{l-1} in $G_{l-1} - I(P)$. By induction, $G - I(P)$ contains a subdivision of some graph in $\mathcal{C}_2(d)$, with vertices u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. Adding the path P to this gives a subdivision of some graph in $\mathcal{C}_3(d)$.

Suppose $|\mathcal{P}'| = 2$ and $U' = \{u_2, u_3\}$. The case when $U' = \{u_1, u_3\}$ is symmetric. Then (u_1, u_2, w) is $(d+1, d, d+1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of some graph in $\mathcal{C}_4(d-1, 3)$, with vertices u_1, u_2, w corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ to it, gives a subdivision of a graph in $\mathcal{C}_3(d)$.

Case 4. Consider the configuration $\mathcal{C}_4$.

Case 4.1 Suppose $d = 0$. The only graph in $\mathcal{C}_4(0, t)$ has $t \geq 3$ terminals $a_1, \ldots, a_t$ with edges $a_1 a_i$, for $2 \leq i \leq t$. The weight of a_1 is $t-1$ and that of a_i is $i-1$ for $2 \leq i \leq t$. In this case, we need to show that there exist $t-1$ paths in G that form a u_1–$\{u_2, \ldots, u_t\}$ fan.

Suppose $u_1 \in U'$. Let i be the smallest index greater than 1 such that $u_i \in U'$. The union of the u_1–w and u_i–w paths in $\mathcal{P}'$ is a u_1–u_i path P in $G_{l-1} - V(K_{l-1})$. If $t > 3$, then $(u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t)$ is $(t-2, 1, \ldots, i-2, i-1, \ldots, t-2)$-joined to K_{l-1} in $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$. By induction, $G - (I(\mathcal{P}') \cup \{w\})$ contains $t-2$ internally disjoint paths that form a u_1–$\{u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t\}$ fan. Adding the path P to this gives the required set of $t-1$ paths. If $t = 3$, then (u_1, u_{5-i}) is $(1,1)$-joined to K_{l-1} in $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$. Thus $G - (I(\mathcal{P}') \cup \{w\})$ contains a u_1–u_{5-i} path. Adding the path P to this gives the required paths that form a u_1–$\{u_2, u_3\}$ fan.

Suppose $u_1 \notin U'$. Again, let i be the smallest index such that $u_i \in U'$. Then $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t, w)$ is $(t-1, 1, 2, \ldots, t-2, t-1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains $t-1$ internally disjoint paths that form a u_1–$\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t, w\}$ fan. The union of the u_1–w path in this set with the u_i–w path in $\mathcal{P}'$ is a u_1–u_i path in G that is internally disjoint from the other paths in the set. Replacing the u_1–w path in the fan by this gives $t-1$ paths that form a u_1–$\{u_2, \ldots, u_t\}$ fan.

Case 4.2 Suppose $d > 0$. If $|\mathcal{P}'| \geq 3$, then $(u_1, \ldots, u_t)$ is $(d+t-2, d, \ldots, d+t-2)$-joined to K_{l-1} in $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$. By induction, $G - (I(\mathcal{P}') \cup \{w\})$ contains a subdivision of some graph in $\mathcal{C}_4(d-1, t)$ with vertex u_i corresponding to a_i, for $1 \leq i \leq t$. Adding the vertex w and any 3 paths in $\mathcal{P}'$, we get a subdivision of graph in $\mathcal{C}_4(d, t)$ that is contained in G.

Suppose $|\mathcal{P}'| = 2$ and $u_1 \in U'$. Let $u_i, i > 1$ be the other vertex in U'. Then the union of the two paths in $\mathcal{P}'$ is a u_1–u_i path P in $G_{l-1} - V(K_{l-1})$. If $t = 3$, then (u_1, u_{5-i}, u_i) is $(d+1, d+1, d)$-joined to K_{l-1} in $G_{l-1} - I(P)$. By induction, $G - I(P)$ contains a subdivision of some graph in $\mathcal{C}_3(d)$ with vertices u_1, u_{5-i}, u_i corresponding to a_1, a_2, a_3 , respectively. Adding the u_1–u_i path P to this gives a subdivision of a graph in $\mathcal{C}_4(d, 3)$. If $t > 3$, then $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t)$ is $(d+t-2, d+1, \ldots, d+i-2, d+i-1, \ldots, d+t-2)$-joined to K_{l-1} in $G_{l-1} - I(P)$. By induction, $G - I(P)$ contains a subdivision of some graph in $\mathcal{C}_4(d, t-1)$ with vertices $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t$ corresponding to $a_1, \ldots, a_{t-1}$, respectively. Adding the path P to this gives a subdivision of a graph in $\mathcal{C}_4(d, t)$.

Finally, suppose $|\mathcal{P}'| = 2$ and $u_1 \notin U'$. Then $(u_1, u_2, \ldots, u_t, w)$ is $(d+t-1, d, \ldots, d+t-2, d+t-1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of some graph in $\mathcal{C}_4(d-1, t+1)$, with vertices $u_1, \ldots, u_t, w$ corresponding to $a_1, \ldots, a_{t+1}$, respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of a graph in $\mathcal{C}_4(d, t)$.

This completes all cases and we conclude that all the 4 configurations are unavoidable. The theorem then follows from Lemma 5 and the fact that the configuration $\mathcal{C}_1(d-1)$ is unavoidable. Note that for any graph in $\mathcal{C}_1(d-1)$, adding an edge between the two terminal vertices a_1, a_2 gives a maximal 3-degenerate graph of order $d+1$. $\square$

5 Planar Maximal 3-degenerate Graphs

Since there exist non-planar 3-degenerate graphs of order 6, not every maximal 3-degenerate graph is good. However, we do not know of any planar graph that is not good. This suggests the following problem.

Conjecture 1 *Every planar maximal 3-degenerate graph is good.*

A specific family of planar maximal 3-degenerate graphs is P_n^3 with vertices $v_1, \ldots, v_n$ and v_i adjacent to v_j iff $1 \leq |j-i| \leq 3$. We verify Conjecture 1 for two graphs P_6^3 and P_7^3. Note that P_4^3 is K_4, P_5^3 is K_5^- and P_6^3 is the only planar maximal 3-degenerate graph of order 6.

Theorem 2 *Every graph of minimum degree at least 5 contains a subdivision of P_6^3.*

Proof: The proof is again based on the same technique, using more restricted configurations than those used in Theorem 1. Consider the following set of configurations.

1. $\mathcal{C}_5$ contains a subset of the graphs in the configuration $\mathcal{C}_1(4)$. The graphs have 6 vertices $\{a_1, a_2, b_1, b_2, b_3, b_4\}$, where a_1, a_2 are terminal vertices of weight 4. The edge sets of the 3 graphs are
   (a) $\{a_1b_1, a_2b_1, a_1b_2, a_2b_2, b_1b_2, a_1b_3, a_2b_3, b_2b_3, a_1b_4, b_2b_4, b_3b_4\}$.
   (b) $\{a_1b_1, a_2b_1, a_1b_2, a_2b_2, b_1b_2, a_1b_3, b_1b_3, b_2b_3, b_1b_4, b_2b_4, b_3b_4\}$.
   (c) $\{a_1b_1, a_2b_1, a_1b_2, a_2b_2, b_1b_2, a_2b_3, b_1b_3, b_2b_3, a_2b_4, b_2b_4, b_3b_4\}$.
2. $\mathcal{C}_6$ contains a subset of the graphs in the configuration $\mathcal{C}_2(3)$. The graphs have 6 vertices $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, where a_1, a_2, a_3 are terminal vertices of weight 3. The edge sets of the 3 graphs are
   (a) $\{a_1b_1, a_2b_1, a_3b_1, a_1b_2, a_2b_2, b_1b_2, a_1b_3, b_1b_3, b_2b_3\}$.
   (b) $\{a_1b_1, a_2b_1, a_3b_1, a_1b_2, a_3b_2, b_1b_2, a_3b_3, b_1b_3, b_2b_3\}$.
   (c) $\{a_1b_1, a_2b_1, a_3b_1, a_2b_2, a_3b_2, b_1b_2, a_2b_3, b_1b_3, b_2b_3\}$.
3. $\mathcal{C}_7$ contains only one graph from the configuration $\mathcal{C}_2(2)$. This graph has 5 vertices $\{a_1, a_2, a_3, b_1, b_2\}$, where a_1, a_2, a_3 are terminal vertices of weight 2. The edges in the graph are $\{a_1b_1$, a_2b_1, a_3b_1, a_1b_2, a_2b_2, $b_1b_2\}$.
4. $\mathcal{C}_8$ contains only one graph from the configuration $\mathcal{C}_3(2)$. This graph has 5 vertices $\{a_1, a_2, a_3, b_1, b_2\}$, where a_1, a_2, a_3 are terminal vertices, a_1, a_2 have weight 3 and a_3 has weight 2. The edge set of the graph is $\{a_1a_2, a_1b_1, a_2b_1, a_3b_1, a_1b_2, a_2b_2, b_1b_2\}$.

5. $\mathcal{C}_9$ contains two graphs with 5 vertices $\{a_1, a_2, a_3, a_4, b_1\}$, where a_1, a_2, a_3, a_4 are terminals, a_1, a_2 have weight 1 and a_3, a_4 have weight 2. The edge sets of the two graphs are
   (a) $\{a_1b_1, a_2b_1, a_3b_1, a_4b_1\}$.
   (b) $\{a_1b_1, a_2b_1, a_3a_4, a_3b_1\}$.

We show that $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$, $\mathcal{C}_8$ and $\mathcal{C}_9$ are unavoidable.

Case 1. Consider the configuration $\mathcal{C}_5$. The only case to be considered here is if $|\mathcal{P}'| = 2$. Lemma 4 implies that (u_1, u_2, w) is $(3,3,3)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of one of the graphs (a), (b) or (c) in $\mathcal{C}_6$, with vertices u_1, u_2, w corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of the corresponding graph (a), (b) or (c) in $\mathcal{C}_5$ with u_1, u_2 corresponding to a_1, a_2 and w corresponding to b_1.

Case 2. Consider the configuration $\mathcal{C}_6$. If $|\mathcal{P}'| = 3$, then (u_1, w, u_2) is $(2,2,2)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of the graph in $\mathcal{C}_7$, with vertices u_1, w, u_2 corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of the graph (a) in $\mathcal{C}_6$.

Suppose $U' = \{u_1, u_2\}$. The other cases can be argued symmetrically. Then (u_3, w, u_1) is $(3,3,2)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of the graph in $\mathcal{C}_8$, with vertices u_3, w, u_1 corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of the graph (b) in $\mathcal{C}_6$.

Case 3. Consider the configuration $\mathcal{C}_7$. If $|\mathcal{P}'| = 3$, then (u_1, u_2, w) is $(1,1,1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. Since the configuration $\mathcal{C}_2(1)$ is unavoidable, $G - I(\mathcal{P}')$ contains a subdivision of $K_{3,1}$ in which the vertices u_1, u_2, w correspond to the vertices in the part of size 3. Adding the paths in $\mathcal{P}'$ to this, gives a subdivision of the graph in $\mathcal{C}_7$.

Suppose $U' = \{u_1, u_2\}$. Then (u_1, u_2, w, u_3) is $(1,1,2,2)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$ and $G - I(\mathcal{P}')$ contains a subdivision of one of the two graphs in $\mathcal{C}_9$, with vertices u_1, u_2, w, u_3 corresponding to a_1, a_2, a_3, a_4, respectively. In either case, adding the paths in $\mathcal{P}'$ to this gives a subdivision of the graph in $\mathcal{C}_7$.

Suppose $U' = \{u_2, u_3\}$. Then (u_1, w, u_2) is $(2,2,1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$ and $G - I(\mathcal{P}')$ contains a subdivision of the graph in $\mathcal{C}_3(1)$, with vertices u_1, w, u_2 corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of the graph in $\mathcal{C}_7$. The case when $U' = \{u_1, u_3\}$ can be argued symmetrically.

Case 4. Consider the configuration $\mathcal{C}_8$. If $|\mathcal{P}'| = 3$ then (u_1, u_2, w)

is $(2,2,1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of the graph in $\mathcal{C}_3(1)$, with vertices u_1, u_2, w corresponding to a_1, a_2, a_3, respectively. Together with the paths in $\mathcal{P}'$, this gives a subdivision of the graph in $\mathcal{C}_8$.

Suppose $U' = \{u_1, u_2\}$. The union of the two paths in $\mathcal{P}'$ is a u_1–u_2 path P in $G_{l-1} - V(K_{l-1})$ that contains w. Since (u_1, u_2, u_3) is $(2,2,2)$-joined to K_{l-1} in $G_{l-1} - I(P)$, $G - I(P)$ contains a subdivision of the graph in $\mathcal{C}_7$, with vertices u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. Adding the path P to this gives a subdivision of the graph in $\mathcal{C}_8$.

Suppose $U' = \{u_2, u_3\}$. Then (u_1, u_2, w) is $(3,2,3)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of the graph in $\mathcal{C}_4(1,3)$, with vertices u_1, u_2, w corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ to this gives a subdivision of the graph in $\mathcal{C}_8$. The case when $U' = \{u_1, u_3\}$ is similar.

Case 5. Consider the configuration C_9. If $|\mathcal{P}'| = 4$, then $G_{l-1} - V(K_{l-1})$ contains a subdivision of the graph (a) in $\mathcal{C}_9$, with vertices u_1, u_2, u_3, u_4, w corresponding to a_1, a_2, a_3, a_4, b_1, respectively.

If $|\mathcal{P}'| = 3$, then (u_i, w) is $(1,1)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$, where $u_i \notin U'$. Then $G - I(\mathcal{P}')$ contains a u_i–w path, which together with the paths in $\mathcal{P}'$ gives a subdivision of the graph (a) in $\mathcal{C}_9$.

Suppose $|\mathcal{P}'| = 2$ and $U' \neq \{u_3, u_4\}$. Then (w, u_i, u_j) is $(2,1,2)$-joined to K_{l-1} in $G_{l-1} - I(\mathcal{P}')$, where $u_i, u_j \notin U'$ and $1 \leq i < j \leq 4$. By induction, $G - I(\mathcal{P}')$ contains a subdivision of the graph in $\mathcal{C}_4(0,3)$ with vertices w, u_i, u_j corresponding to a_1, a_2, a_3, respectively. Adding the paths in $\mathcal{P}'$ gives a subdivision of the graph (a) in $\mathcal{C}_9$.

The only other possibility is that $U' = \{u_3, u_4\}$. The union of the 2 paths in $\mathcal{P}'$ is a u_3–u_4 path P in $G_{l-1} - V(K_{l-1})$. Since (u_1, u_2, u_3) is $(1,1,1)$-joined to K_{l-1} in $G_{l-1} - I(P)$, $G - I(P)$ contains a subdivision of the graph in $\mathcal{C}_2(1)$, with vertices u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. Adding the path P to this gives a subdivision of the graph (b) in $\mathcal{C}_9$.

This completes all cases and shows that the configurations $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$, $\mathcal{C}_8$ and $\mathcal{C}_9$ are unavoidable. Theorem 2 then follows from Lemma 5, since adding the edge a_1a_2 to any graph in $\mathcal{C}_5$ gives the graph P_6^3. $\square$

We next consider planar maximal 3-degenerate graphs of order 7. There are 3 different such graphs, but we consider only the graph P_7^3. While it is possible to use the same technique, the number of configurations required appears to be large. We can reduce the number of configurations required by starting with an initial graph other than an edge.

Let $\mathcal{C}$ be a configuration with terminal vertices $a_1, \ldots, a_t$ of weights $n_1, \ldots, n_t$, respectively. We say a pair (G, K) contains the configuration $\mathcal{C}$ if $G - V(K)$ contains a subdivision $\mathcal{T}(H)$ of some graph $H \in \mathcal{C}$, such that vertices $u_1, \ldots, u_t$ correspond to $a_1, \ldots, a_t$, respectively, and $(u_1, \ldots, u_t)$ is $(n_1, \ldots, n_t)$-joined to K in $G - V(\mathcal{T}(H)) \setminus \{u_1, \ldots, u_t\}$.

Consider the following set of configurations.

1. $\mathcal{C}_{10}(d)$ for $d \geq 1$ is the configuration containing only the graph K_2 with 2 terminal vertices of weight d.
2. $\mathcal{C}_{11}(d)$ for $d \geq 1$ is the configuration containing only the graph K_3 with 3 terminal vertices of weight d.
3. $\mathcal{C}_{12}(d)$ for $d \geq 1$ is the configuration containing only the graph K_4^-, obtained by deleting an edge from K_4. There are 3 terminal vertices a_1, a_2, a_3 with a_1, a_3 of weight $d+1$ and a_2 of weight d. The missing edge is $a_1 a_3$.
4. $\mathcal{C}_{13}(d)$ for $d \geq 1$ is the configuration containing only the graph K_4 with 3 terminal vertices a_1, a_2, a_3 of weight d.

Lemma 6 *Let G be a graph of minimum degree at least $d \geq 4$. Then there exists a pair (G', K') such that $(G, \emptyset) \to (G', K')$ and (G', K') contains the configuration $\mathcal{C}_{13}(d-3)$.*

Proof: Let $(G, \emptyset) = (G_0, K_0), (G_1, K_1), \ldots, (G_l, K_l)$ be a maximal sequence of pairs such that (G_{i+1}, K_{i+1}) is obtained from (G_i, K_i) by adding or deleting a vertex, for $0 \leq i < l$. Let i be the smallest index such that (G_i, K_i) contains the configuration $\mathcal{C}_{10}(d-1)$. Lemma 5 implies there exists such an index i. Since $d \geq 4$, we have $i > 0$. Then (G_i, K_i) must be obtained from (G_{i-1}, K_{i-1}) by adding a vertex w, and $U' = \{u_1, u_2\}$. This implies (G_{i-1}, K_{i-1}) contains the configuration $\mathcal{C}_{11}(d-2)$ with vertices u_1, u_2, w corresponding to a_1, a_2, a_3, respectively.

Let j be the smallest index such that (G_j, K_j) contains $\mathcal{C}_{11}(d-2)$. Since $d \geq 4$, we have $j > 0$. Again, (G_j, K_j) must be obtained from (G_{j-1}, K_{j-1}) by adding a vertex w. If $|\mathcal{P}'| = 3$, then (G_{j-1}, K_{j-1}) contains $\mathcal{C}_{13}(d-3)$, with vertices u_1, u_2, u_3 corresponding to the vertices a_1, a_2, a_3, respectively. Then (G_{j-1}, K_{j-1}) is the required pair.

Suppose $|\mathcal{P}'| = 2$, and without loss of generality, $U' = \{u_2, u_3\}$. Then (G_{j-1}, K_{j-1}) contains $\mathcal{C}_{12}(d-3)$ with vertices u_1, u_2, w corresponding to a_1, a_2, a_3, respectively.

Let m be the smallest index such that (G_m, K_m) contains $\mathcal{C}_{12}(d-3)$. Since $d \geq 4$, we have $m > 0$. Again, (G_m, K_m) must be obtained from (G_{m-1}, K_{m-1}) by adding a vertex w. If $|\mathcal{P}'| = 3$, then (G_{m-1}, K_{m-1}) contains the configuration $\mathcal{C}_{13}(d-3)$ with vertices u_1, u_3, w corresponding to a_1, a_2, a_3, respectively. If $U' = \{u_1, u_3\}$ then (G_{m-1}, K_{m-1}) contains $\mathcal{C}_{13}(d-3)$ with vertices u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. If $U' = \{u_2, u_3\}$ then (G_{m-1}, K_{m-1}) contains $\mathcal{C}_{12}(d-3)$, with vertices u_1, u_3, w corresponding to a_1, a_2, a_3, respectively. This contradicts the choice of m. Similarly, if $U' = \{u_1, u_2\}$, then (G_{m-1}, K_{m-1}) contains $\mathcal{C}_{12}(d-3)$, with vertices u_3, u_1, w corresponding to a_1, a_2, a_3, respectively. Again, this contradicts the choice of m. Therefore (G_{m-1}, K_{m-1}) must contain $\mathcal{C}_{13}(d-3)$. □

Theorem 3 *Every graph of minimum degree at least 6 contains a subdivision of* P_7^3.

Proof: Let G be a graph of minimum degree at least 6. Lemma 6 implies there exists a pair (G', K') such that $(G, \emptyset) \to (G', K')$ and (G', K') contains the configuration $\mathcal{C}_{13}(3)$. Thus $G' - V(K')$ contains a subdivision H of K_4, with vertices u_1, u_2, u_3 corresponding to the vertices a_1, a_2, a_3, respectively, such that (u_1, u_2, u_3) is $(3,3,3)$-joined to K' in $G' - (V(H) \setminus \{u_1, u_2, u_3\})$. Since $\mathcal{C}_6$ is unavoidable, $G - (V(H) \setminus \{u_1, u_2, u_3\})$ contains a subdivision of one of the graphs (a), (b), or (c) in $\mathcal{C}_6$, with vertices u_1, u_2, u_3 corresponding to a_1, a_2, a_3, respectively. In all cases, the union of this graph with H gives a subdivision of P_7^3 in G. □

6 Remarks

We have verified Conjecture 1 for the other two planar maximal 3-degenerate graphs of order 7. Although the method is the same, the number of configurations required is larger, and we omit the details. A planar maximal 3-degenerate graph is also a maximal planar graph. An interesting question is whether all maximal planar graphs are good? The smallest case to consider is the octahedron, obtained by deleting a perfect matching from K_6. While we do not know a graph of minimum degree 5 that does not contain a subdivision of this, the technique used in this paper cannot be applied since the required configuration is avoidable. It would be interesting to see if there is any characterization of unavoidable configurations. Perhaps the first question to answer would be to find the maximum number of edges in a good graph of order $d+1$. For $2 \leq d \leq 5$, this is exactly $3d-3$. Does this hold in general? An even simpler question would be to find the largest number m such that every graph of minimum degree d contains a subdivision of some graph of order $d+1$ and size m. Theorem 1 shows that $m \geq 3d-3$ and the bound is tight for $2 \leq d \leq 5$. Does this hold for all d? Finally, it would be interesting to consider non-separating versions of these results. Kriesell [3] generalized Dirac's theorem to show that every connected graph G with minimum degree at least 4 contains a subdivision H of K_4 such that $G - V(H)$ is connected. Can the results in this paper be extended in a similar way, by increasing the minimum degree bound by one?

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