Final exam: CS 663, Digital Image Processing, 21st November

Instructions: There are 180 minutes for this exam (5:30 pm to 8:30 pm). Answer all 8 questions. This exam is worth 25% of the final grade. Some formulae are listed at the end of the paper.

1. **Image Compression:** Consider an image whose intensity values are integers from 0 to 7, occurring with frequencies $0.1, 0.1, 0.2, 0.2, 0.2, 0.15, 0.025, 0.025$ respectively (note: there are 8 intensity values). Construct a Huffman tree for encoding these intensity values and find the corresponding average bit length (exact numerical values are not important, but show your steps clearly). [8 points]

2. **Color Imaging:** In color images, the hue θ at a pixel is calculated from the R,G,B values at that pixel, using the formula $\theta = \cos^{-1}(\dfrac{0.5(2R-G-B)}{\sqrt{(R-G)^2+(R-B)(G-B)}})$. What are the advantages and disadvantages of using hue in color image applications? [8 points]

3. **SVD:** Consider a matrix $\mathbf{A}$ of size $m \times n$. Explain how will you compute the SVD of $\mathbf{A}$, if you had access to a software routine that computed the eigenvectors and eigenvalues of a square matrix (and assuming you had no access to any software function that directly computed the SVD). State the time complexity of your procedure. [12 points]

4. **Color/Multichannel Imaging:** Consider a grayscale image $I(x,y)$. You know that the squared intensity change at point (x,y) along a direction $\mathbf{v} \in \mathbb{R}^2$ is given by $E(\mathbf{v}) = (\nabla I(x,y) \cdot \mathbf{v})^2$. Deduce along what direction $\mathbf{v}$, $E(\mathbf{v})$ is the maximum. Now consider a multichannel image $J(x,y,l)$ with $L > 1$ channels where l is an index for the channel. The squared intensity change at (spatial) point (x,y) along a direction $\mathbf{v} \in \mathbb{R}^2$ is given by $E(\mathbf{v}) = \sum_{l=1}^{L} (\nabla I(x,y,l) \cdot \mathbf{v})^2$. Deduce along which direction $\mathbf{v}$, $E(\mathbf{v})$ will be the maximum. Show how this expression reduces to your earlier answer when $L = 1$. Note that $\mathbf{v}$ in either case is a vector of unit magnitude. When $L > 1$, is $\mathbf{v}$ always guaranteed to be unique (upto a sign change)? Explain. [3+6+3+2=14 points]

5. **Fourier Transforms and More:** Consider a function $f(x,y)$ defined over a bounded rectangular domain. Consider the quantity $g(\rho,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$ where $x\cos\theta + y\sin\theta = \rho$ is the equation of a line in normal form and $\delta(z)$ is the Dirac delta function (i.e. $\delta(z) = \infty$ if $z = 0$, otherwise $\delta(z) = 0$). This quantity is called as the projection of $f(x,y)$ over the angle θ, and represents the measurements taken by modern day X Ray machines or CT scanners. Consider the quantity $G(\omega,\theta)$, defined as the 1D Fourier transform of $g(\rho,\theta)$ w.r.t. ρ where ω is a frequency variable. We have $G(\omega,\theta) = \int_{-\infty}^{+\infty} g(\rho,\theta)e^{-j2\pi\omega\rho}d\rho$. Starting from this, derive an expression for $G(\omega,\theta)$ in terms of $F(u,v)$, the Fourier transform of $f(x,y)$ where (u,v) stands for the frequency variables. Now, let us define the first order projection moment of $g(\rho,\theta)$ as $m_\theta = \int_{-\infty}^{\infty} g(\rho,\theta)\rho d\rho$, and let us define the (p,q)-order moment of the image f as $M_{pq} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^p y^q f(x,y)dxdy$. Then derive a relation between m_θ and $(M_{0,1}, M_{1,0})$. [7 + 7 = 14 points]

6. **PCA:** Consider a set of N vectors $\mathcal{X} = \{\boldsymbol{x_1}, \boldsymbol{x_2}, ..., \boldsymbol{x_N}\}$ each in $\mathbb{R}^d$, with average vector $\bar{\mathbf{x}}$. We have seen in class that the direction $\mathbf{e}$ such that $\sum_{i=1}^{N} \|\boldsymbol{x_i} - \bar{\mathbf{x}} - (\mathbf{e}\cdot(\boldsymbol{x_i} - \bar{\boldsymbol{x}}))\mathbf{e}\|^2$ is minimized, is obtained by maximizing $\mathbf{e}^t\mathbf{C}\mathbf{e}$, where $\mathbf{C}$ is the covariance matrix of the vectors in $\mathcal{X}$. This vector $\mathbf{e}$ is the eigenvector of matrix $\mathbf{C}$ with the highest eigenvalue. Prove that the direction $\mathbf{f}$ perpendicular to $\mathbf{e}$ for which $\mathbf{f}^t\mathbf{C}\mathbf{f}$ is maximized, is the eigenvector of $\mathbf{C}$ with the second highest eigenvalue. For simplicity, assume that all non-zero eigenvalues of $\mathbf{C}$ are distinct and that $\text{rank}(\mathbf{C}) > 2$. [12 points]

7. **Image Restoration:** Given a blurred and noisy image g, an inquisitive student wants to know how to determine the blur kernel k besides the underlying image f. (Recall that in class, we assumed that k was known). For this, (s)he tries to minimize the objective function $E_1(k, f) = \|g - k * f\|^2 + \sum_{i=1}^{N} f_x^2(i) + f_y^2(i)$, where N is the number of image pixels, i is an index for a pixel location, and $f_x(i)$ and $f_y(i)$ represent the gradients of image f at location i, in X and Y directions respectively. What answer will the student obtain? Do not worry about the exact procedure/algorithm for minimization, just assume that there was a magic routine that did the job for you. [12 points]

8. **Compression:** Consider a set of N vectors $\mathcal{X} = \{\boldsymbol{x_1}, \boldsymbol{x_2}, ..., \boldsymbol{x_N}\}$ each in $\mathbb{R}^d$ $(N > d)$. Assume their mean vector is $\mathbf{0}$. Let $\mathbf{V} \in \mathbb{R}^{d \times d}$ be the orthonormal matrix containing the principal components of this dataset arranged in descending order of the eigenvalues (assume all eigenvalues are distinct). Let us denote the order k $(k < d)$ linear approximation of vector $\boldsymbol{x_i}$ using $\mathbf{V}$ as $L(\boldsymbol{x_i}^{(k)}; \mathbf{V}) = \boldsymbol{V_k} \boldsymbol{\alpha_i^{(k)}}$ where $\boldsymbol{V_k}$ is a $d \times k$ matrix containing the first k columns of $\mathbf{V}$, and $\boldsymbol{\alpha_i^{(k)}} = \boldsymbol{V_k}^t \boldsymbol{x_i}$. Let us denote the order k $(k < d)$ non-linear approximation of vector $\boldsymbol{x_i}$ using $\mathbf{V}$ as $N(\boldsymbol{x_i}^{(k)}; \mathbf{V}) = \mathbf{V} \boldsymbol{\alpha_i}$ where $\boldsymbol{\alpha_i} = \arg\min_{\boldsymbol{c_i}} \|\boldsymbol{x_i} - \mathbf{V}\boldsymbol{c_i}\|^2$ subject to the constraint that vector $\boldsymbol{c_i}$ has at the most k non-zero elements. The total reconstruction errors for the linear and non-linear approximations are respectively $E_L(\mathbf{V}) = \sum_{i=1}^{N} \|\boldsymbol{x_i} - L(\boldsymbol{x_i}^{(k)}; \mathbf{V})\|^2$ and $E_N(\mathbf{V}) = \sum_{i=1}^{N} \|\boldsymbol{x_i} - N(\boldsymbol{x_i}^{(k)}; \mathbf{V})\|^2$. Which of the following statements is true and why:

   (a) $E_L(\mathbf{V}) \leq E_N(\mathbf{V})$

   (b) $E_L(\mathbf{V}) \geq E_N(\mathbf{V})$

   (c) $E_L(\mathbf{V}) = E_N(\mathbf{V})$

   (d) One cannot make a conclusion about which error is greater.

   Also devise an efficient algorithm to obtain the order k non-linear approximation of $\boldsymbol{x_i}$ given $\mathbf{V}$, and state its time complexity. Argue why your algorithm is correct.

   Based on what you have studied about PCA in class, can you conclude the following: There cannot exist an orthonormal basis $\mathbf{W}$ such that $E_N(\mathbf{W}) < E_N(\mathbf{V})$ for some fixed k. Justify your answer. [8 + 8 + 4 = 20 points]

LIST OF FORMULAE:

1. Gaussian pdf in 1D centered at μ and having standard deviation σ: $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$.

2. 1D Fourier transform and inverse Fourier transform:
   $F(u) = \int_{-\infty}^{+\infty} f(x) e^{-j2\pi ux} dx, f(x) = \int_{-\infty}^{+\infty} F(u) e^{j2\pi ux} du$

3. 2D Fourier transform and inverse Fourier transform:
   $F(u,v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy, f(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u,v) e^{j2\pi(ux+vy)} du dv$

4. Convolution theorem: $\mathcal{F}(f(x) * g(x))(u) = F(u)G(u); \mathcal{F}(f(x)g(x))(u) = F(u) * G(u)$

5. Fourier transform of $g(x-a)$ is $e^{-j2\pi ua}G(u)$. Fourier transform of $\frac{df^n(x)}{dx^n} = (j2\pi u)^n F(u)$ ($n > 0$ is an integer).

6. 1D DFT: $F(u) = \dfrac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N}, f(x) = \dfrac{1}{\sqrt{N}} \sum_{u=0}^{N-1} F(u) e^{j2\pi ux/N}$

7. 2D DFT: $F(u,v) = \dfrac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux+vy)/N}, f(x,y) = \dfrac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux+vy)/N}$