# SUPPLEMENT MATERIAL

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## 1. PERFORMANCE BOUNDS FOR Q1 AND Q2

**Theorem 1:** - Bounds for Q1
Abiding by the notations in [1], let $B(\cdot)$ be an $(\delta, k)$ RIP obeying linear operator [1], i.e. $\forall X \neq 0$ and $\|X\|_0 \leq k$

$$\left| \frac{\|B(X)\|_2^2}{\|X\|_2^2} - 1 \right| < \delta \tag{1}$$

Let $X^*$ be the true rank 1 matrix that satisfies the constraints in Q1, and $\tilde{X}$ be the solution to Q1 for an appropriate choice of parameters. We define $\Delta = \tilde{X} - X^\star$. For a matrix $X \in \mathbb{R}^{d\times d}$, $X_T$ denotes a matrix with all values zero except the indices in $T$, which are set to the corresponding values of $X$. $\forall \lambda \geq \frac{2d}{1-\rho} + \frac{d}{k^{\frac{1}{2}}}$ and $\delta \leq \sqrt{2} - 1$ we get

$$\|\Delta\|_2 \leq \frac{2\alpha\varepsilon(1-\rho)^{-1} + 2(1+\rho)(1-\rho)^{-1}k^{-\frac{1}{2}}\|\tilde{X} - X_k^\star\|_1}{1 - (\frac{2d}{1-\rho} + \frac{d}{k^{\frac{1}{2}}})\frac{1}{\lambda}} \tag{2}$$

where $\rho = \frac{\sqrt{2}\delta}{1-\delta} \leq 1$, $X_k^\star$ is the matrix with the $k$ largest elements in $X^\star$ at the corresponding indices and the others elements to be 0, $\varepsilon \sim \mathcal{O}(\sqrt{n})$ and $\alpha = \frac{4\sqrt{(I+c)(1+\delta)}}{1-\delta}$. I is the true intensity of the underlying measurement, i.e. $\|B(X^\star)\|_1$, which is assumed to be known for naturally acquired measurements.

**Theorem 2:** - Bounds for Q2
Given that the assumptions in section 2.2 hold, and $\mathbf{\Psi}$ has sufficiently small RIP constant $\delta$, then, there exist positive absolute constants $C_1$, $C_2$, and $C_3$ such that if $m \geq c_1 k \log \frac{d}{k}$, and $n \geq C_1 m$ then any estimate $\widehat{X}$ of the Algorithm obeys

$$\left\| \widehat{X} - X^\star \right\|_F \leq C_2,$$

for all rank-one and $k \times k$-sparse matrices $X^\star \succcurlyeq 0$ with probability exceeding $1 - e^{-C_3 n}$. The constant $C_2$ depends on $\sqrt{I}$ and since $\varepsilon \sim \mathcal{O}(\sqrt{n})$, we don't get a dependence on $\frac{1}{\sqrt{n}}$ like that in [2]. The proofs of these theorems are inspired by the performance bounds for compressive phase retrieval presented in [3] by Ohlsson et. al for the standard CPRL problem (P1) , and in [2] by Bahamani et. al for the 2-stage sparse recovery method (P2). We adapted these proofs for Poisson corrupted signals using the techniques in [4] and [5].

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## 2. PROVING PERFORMANCE BOUNDS

We briefly present here the proof outline for performance bounds of Q1 and Q2.

### 2.1. Proof for Q1

As defined in theorem 1, let $B(\cdot)$ be an $(\delta, k)$ RIP obeying linear operator [1], i.e. $\forall X \neq 0$ and $\|X\|_0 \leq k$

$$\left| \frac{\|B(X)\|_2^2}{\|X\|_2^2} - 1 \right| < \delta \tag{3}$$

Let $X^*$ be the true rank 1 matrix that satisfies the constraints in Q1, and $\tilde{X}$ be the solution to Q1 for an appropriate choice of parameters. We define $\Delta = \tilde{X} - X^\star$. For a matrix $X \in \mathbb{R}^{d\times d}$, $X_T$ denotes a matrix with all values zero except the indices in $T$, which are set to the corresponding values of $X$. $T_0$ is the index set of the $k$ largest elements of $X^\star$ in absolute value, and $T_0^c$ is the corresponding complement set. Let $T_1$ be the index set associated with the $k$ largest elements in absolute value of $\Delta_{T_0^c}$ and $T_{0,1} \triangleq T_0 \cup T_1$ be the union. Let $T_2$ be the index set associated with the $k$ largest elements in absolute value of $\Delta_{T_{0,1}^c}$, and so on. Using the fact that -:

$$\|\Delta\|_2 = \|\Delta_{T_{0,1}} + \Delta_{T_{0,1}^c}\|_2 \leq \|\Delta_{T_{0,1}}\|_2 + \|\Delta_{T_{0,1}^c}\|_2. \tag{4}$$

We independently derive bounds for both the terms on the RHS, and then combine the result to get a bound on $\|\Delta\|_2$.

Bounds for $\|\Delta_{T_{0,1}^c}\|_2$ can be derived exactly as in the proof of theorem 8 in [1]. For bounding $\|\Delta_{T_{0,1}}\|_2$, we first use a technique from [5] to upper bound $\|B(\Delta)\|_2$, which handles Poisson corruption for signals in Q1 and Q2. Some algebraic manipulation gives

$$\begin{aligned}\|B(\Delta)\|_2^2 &= \|B(\tilde{X} - X^\star)\|_2^2 \\ &= \Sigma_{i=1}^N((\sqrt{(B(\tilde{X})_i + c} - \sqrt{(B(X^\star))_i + c})^2 \\ &\quad (\sqrt{(B(\tilde{X})_i + c} + \sqrt{(B(X^\star))_i + c})^2).\end{aligned}$$

1. Since $\|\sqrt{y+c} - \sqrt{B(X)+c}\|_2$ is upper bounded by $\varepsilon$, using triangular inequality we get $\|\sqrt{B(\tilde{X})+c} - \sqrt{B(X^\star)+c}\|_2 \leq \|\sqrt{y+c} - \sqrt{B(\tilde{X})+c}\|_2 + \|\sqrt{y+c} - \sqrt{B(X^\star)+c}\|_2 \leq 2\varepsilon$.

2. For scalars $v_1 \geq 0, v_2 \geq 0$, we have $(\sqrt{v_1} + \sqrt{v_2})^2 \leq 4\max(v_1, v_2)$. Likewise we also have $(B\tilde{X}))_i \leq I$, where I is the true intensity of the underlying measurement, i.e. $\|B(X^\star)\|_1$, which is assumed to be known for naturally acquired measurements. Hence we get $(\sqrt{(B(\tilde{X}))_i + c} + \sqrt{(B(X^\star))_i + c})^2 \leq 4(I + c)$.
3. Using the results of all the steps above, we have $\|B(\Delta)\|_2 \leq 4\varepsilon\sqrt{I + c}$.

By a derivation similar to that in [4], one can show that

$$\|\Delta_{T_{0,1}}\|_2 \leq \frac{\alpha\varepsilon}{1-\rho} + \frac{2\rho}{1-\rho} k^{-1/2} \|\tilde{X} - X^\star{}_k\|_1 - \frac{1}{\lambda}\frac{1}{1-\rho} \operatorname{Tr}\Delta. \tag{5}$$

where $\rho = \frac{\sqrt{2}\delta}{1-\delta} \leq 1$, $X_k^\star$ is the matrix with the $k$ largest elements in $X^\star$ at the corresponding indices and the others elements to be 0, and $\alpha = \frac{4\sqrt{(I+c)(1+\delta)}}{1-\delta}$. The $\operatorname{Tr}\Delta$ term makes the bound a bit unpractical, and can be eliminated using the inequality - $\operatorname{Tr}\Delta \leq d\|\Delta\|_2$. This inequality can obtained by upper bounding trace by $l_1$ norm and then using the Cauchy Schwartz inequality subsequently as done in [1]. We observe that this inequality differs slightly from that in the proof of the corollary 9 in [1] since $\Delta$ is not k-sparse. Using this inequality, and the bounds obtained for $\|\Delta_{T_{0,1}}\|_2$ and $\|\Delta_{T_{0,1}^c}\|_2$, $\forall\lambda \geq \frac{2d}{1-\rho} + \frac{d}{k^{\frac{1}{2}}}$ and $\delta \leq \sqrt{2} - 1$we get

$$\|\Delta\|_2 \leq \frac{2\alpha\varepsilon(1-\rho)^{-1} + 2(1+\rho)(1-\rho)^{-1}k^{-\frac{1}{2}}\|\tilde{X} - X_k^\star\|_1}{1 - (\frac{2d}{1-\rho} + \frac{d}{k^{\frac{1}{2}}})\frac{1}{\lambda}} \tag{6}$$

### 2.2. Proof for Q2

First we find an upper bound on $\|\widehat{A} - A^\star\|_2$.
$\widehat{A}$ is the solution to Q2 and $A^\star$ is the true low rank matrix. The performance bounds for estimation of $A$ can be derived exactly as done in [2], with a modification to the proof of theorem 1.2 presented [6] to account for the VST based constraint to handle Poisson noise. In section 6 of [6], on taking into account our constraint, and following the steps to bound $\|B(\Delta)\|_2$ as done for Q1 above, equation 6.2 in section 6 of [6] changes to $\|\mathcal{A}(H)\|_2 \leq 4\varepsilon\sqrt{(I + c)}$, where $H = \Delta$ and $\mathcal{A}$ corresponds to the linear operator $B(\cdot)$ in Q1. Since the upper bound for $\|\mathcal{A}(H)\|_2$ is changed only by a multiplicative constant, the rest of the proof proceeds in a similar manner to [6], giving us a bound of the form similar to that in [2] with a different constant on the R.H.S, i.e.

$$\|\widehat{A} - A^\star\|_F \leq \frac{C_2''\varepsilon}{\sqrt{n}}, \tag{7}$$

for all valid $A^\star$ , thereby for all valid $X^\star$, with probability exceeding $1 - e^{-C_3 n}$ for some constant $C_3$. $C_2''$ is a constant that depends on $\sqrt{I + c}$ since the $\varepsilon$ gets scaled by that factor due to the VST constraint. We see that the bound we obtained has a dependence on $\sqrt{n}$ as opposed to the theorem 1.2 in [6]. This difference arises due to the fact that the noise vector in [6] is assumed to be a sphere of radius $\sqrt{n}$, and hence we see a dependence on $\sqrt{n}$ in the 2-stage sparse recovery algorithm as no such assumption is taken for the noise vector in this case. The $\widehat{A}$ estimated from above step now can be used to get the sparse estimate $\widehat{X}$. The upper bound on $\|\widehat{X} - X^\star\|_2$ can now be derived exactly as done in [2]. This completes the proof sketch for Q1 and Q2.

## 3. REFERENCES


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