PERTURBED COMPRESSED SENSING BASED SINGLE SNAPSHOT DOA ESTIMATION

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ABSTRACT

Traditionally, direction of arrival (DOA) estimation techniques have been based on spectral estimation methods utilizing signal and noise subspaces [1]. Such techniques perform well when sensor measurements are available at multiple snapshots. Recently, compressed sensing (CS) based DOA estimation techniques have been introduced, which improve source localization in the single snapshot case by modeling the angle search as a sparse recovery problem. In this domain, various on-grid and off-grid methods have been proposed in the existing literature [2] [3] [4]. The on-grid methods rely on a fixed basis and solve traditional CS based sparse recovery problems while the latter has modifications based on first-order Taylor approximation of the array manifold matrix. In this paper, we present an off-grid CS based formulation, where we employ an alternating minimization strategy for fine-grid search of source directions based on coordinate descent. We show that our technique outperforms the first-order approximation techniques whose performance is limited by the signal-norm dependent Taylor error.

Index Terms— Array Signal Processing, Direction of Arrival Estimation, Perturbed Compressed Sensing

1. INTRODUCTION

The direction-of-arrival (DOA) estimation problem seeks to accurately locate the incoming directions of signal sources impinging on an array of sensors. This is a classical problem in array processing and finds applications in radar sensing, mobile communication, seismology, etc. Conventional non-parametric spectral estimation based methods have limited angular resolution during DOA estimation. Other DOA estimation techniques such as Capon beam-forming [5], MUSIC [6], ESPRIT [7], etc. have been introduced that work well when multiple time snapshots of signal measurements are available. MUSIC and its related algorithms are based on estimation of the eigenvalues of the signal or noise covariance matrix. Therefore, source localization under limited information like a single snapshot of the sensor measurements tends to be highly erroneous.

Recently, various compressed sensing (CS) based DOA estimation techniques have been proposed [8] [9][10]. These techniques are used to represent the DOA estimation problem as a sparse support recovery problem, where the domain is the discretized set of angles in the given range and the true sources are known to be located at a small subset of these locations. Unlike the gridless spectral estimation techniques, these techniques are mainly grid-based techniques, as the possible search space is discretized and the array matrix is fixed. One of the benefits of CS based methods is that, they allow the sensors to be located at non-uniform locations and the angular resolution is generally better than other methods typically with a single snapshot. Without further improvements, the resolution of this technique is limited by the fineness of the discretization. Various algorithms, including Bayesian techniques [11] have been proposed for solving this sparse recovery problem, both for the single and multi snapshot case. To improve the resolution of the angle search, apart from the fineness of discretization, certain off-grid techniques have been introduced [12][13], which model the search space with certain fixed points and the true source locations as small perturbations from these fixed locations. In [14],[15], an off-grid technique is introduced which considers non-grid locations as parametric perturbations in the array sensing matrix, modeled via a first-order Taylor approximation. This method is prone to large errors due to higher order terms in the Taylor expansion that are ignored, as the error is directly dependent on the signal norm and hence unbounded. In our approach, we model angle search as perturbations in the array manifold matrix which are solved via coordinate descent with brute-force search on each single parameter keeping the others fixed. For a fixed grid resolution, we show that our method performs better than the aforementioned Taylor approximation based methods, as we directly search for the true angular locations by coordinate descent and brute-force search. Note that in this work, we have specifically focused on the important case of single snapshot DOA estimation, which is required in situations where physical constraints prevent acquisition of multiple snapshots [16].

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In Section 2.1, we introduce the CS based formulation for DOA estimation and present our off-grid approach based on perturbed array manifold matrix. In Section 2.2, we present an alternating minimization algorithm for sparse recovery of the angles and signal magnitudes. In Section 3, we show angular recovery results compared to other single snapshot CS based methods. Finally, we conclude in Section 4.

2. PERTURBED COMPRESSED SENSING DOA ESTIMATION

2.1. Problem Definition

Consider a linear array $\mathcal{Y}$ of M identical sensors $[Y_1, \dots Y_M]$ which are assumed to be omnidirectional with same gain over the angular range of interest. We have k sources $u_i(t)$ $i \in 1 : k$ located at unknown angles $[\beta_1, \beta_2 \dots \beta_k]$. We invoke standard assumptions of far-field and narrow-band sources with a planar incoming wave front. Also the sources are assumed to be at rest w.r.t the sensor array. As shown in [1] [17], the time delay of the signal impinging on different sensors, corresponding to a source location at β_i, is modeled by steering vector $a(\beta_i) = [1 \; e^{-iw_c\tau_2} \dots e^{-iw_c\tau_M}]^T$ with output of M sensors as $\vec{y_i}(t) = a(\beta_i)u_i(t)$. Because of linearity of array, the total sensor output is $\boldsymbol{y}(t) = \sum_i \vec{y_i}(t)$. A uniform linear array (ULA) assumption leads us to steering vectors of form

$$a(\beta_i) = [1 \;\; e^{-\iota 2\pi \frac{d}{\lambda}\sin(\beta_i)} \dots e^{-\iota 2\pi (M-1)\frac{d}{\lambda}\sin(\beta_i)}]^T \quad (1)$$

where $\iota \triangleq \sqrt{-1}$, d is the distance between array elements and λ is wavelength of incoming source signals. The observation model is given as $\boldsymbol{y}(t) = A(\boldsymbol{\beta})\boldsymbol{u}(t) + \boldsymbol{\eta}(t)$ where $\boldsymbol{\beta} = [\beta_1 \dots \beta_k]$, $\boldsymbol{u}(t) = [u_1(t) \dots u_k(t)]^T$ and $\boldsymbol{\eta}(t) = [\eta_1 \dots \eta_k]$ is the noise vector. The matrix of steering vectors $A_{M\times k}(\beta) = [a(\beta_1) \dots a(\beta_k)]$ is called the array manifold matrix. The range of possible directions is $\hat{\beta}_j \in [\frac{-\pi}{2}, \frac{\pi}{2}]$. The mapping $\sin\hat{\beta} \leftarrow \hat{\theta}$ is one to one with $\boldsymbol{\theta} = [\sin\hat{\beta}_1, \dots, \sin\hat{\beta}_k] = [\theta_1, \dots \theta_k]$ where $\theta_i \in [-1, 1]$. Hence we can equivalently search for $\boldsymbol{\theta} \in [-1, 1]$. So we can write, $\boldsymbol{y}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{u}(t) + \boldsymbol{\eta}(t)$.

In single snapshot DOA estimation, we take the measurements only at a single time instant $t = t_o$. Therefore the observation model in single snapshot DOA estimation is

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{u} + \boldsymbol{\eta} \quad (2)$$

where $[A]_{M\times k}$ with $\boldsymbol{y_{M\times 1}} = \boldsymbol{y}(t_0)$, $\boldsymbol{u_{k\times 1}} = \boldsymbol{u}(t_0)$ and $\boldsymbol{\eta_{M\times 1}} = \boldsymbol{\eta}(t_0)$.

As shown in [8], we can utilize the compressed sensing framework to solve the above problem. The sources are considered to sparsely occupy few locations in the whole set of possible positions in $[-1, 1]$. We consider a discretized set $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_N]$ as a fixed grid covering the entire range from $[-1, 1]$ with corresponding coefficients $\boldsymbol{u} = [u_1, u_2, \dots, u_N]$. Only k coefficients of $\boldsymbol{u}$ have non-zero values corresponding to actual source locations and the rest of the coefficients are zero. Therefore, $\boldsymbol{u}$ is a k sparse vector and the sensing matrix $\boldsymbol{A_{M\times N}} = [a(\theta_1) \dots a(\theta_N)]$ in this model has N columns. This is essentially the on-grid model, where DOA estimation problem is basically a support recovery problem, where we want to identify the support of sparse vector $\boldsymbol{u}$ which would give non-zero source locations.

When sources are located at off-grid locations, the above model can cause significant errors. The accuracy can only be increased by making the grid finer but that can increase the coherence of sensing matrix $\boldsymbol{A}$ violating sparse recovery conditions [18]. Hence, we consider an off-grid model with perturbations $\boldsymbol{\delta} = [\delta_1, \delta_2, ..., \delta_N]$ on the grid points $\boldsymbol{\theta}$. Then we have a sensing matrix $\boldsymbol{A_\theta(\delta)}$, as a function of unknown perturbations $\boldsymbol{\delta}$ and fixed grid $\boldsymbol{\theta}$ which is uniform in $[-1, 1]$. In this way, we can achieve higher accuracy for a coarser sampling grid. Note that our model and algorithm is different from other off-grid DOA estimation approches like [14]. We propose to jointly estimate $\boldsymbol{\delta}$ and $\boldsymbol{u}$, by solving the following problem P1:

$$\mathsf{P1}: \min_{\boldsymbol{u}, \boldsymbol{\delta}\in[-r,r]^M} \|\boldsymbol{u}\|_1 \text{ s. t. } \|\boldsymbol{y} - \boldsymbol{A_\theta(\delta)u}\|_2^2 \le \varepsilon, \quad (3)$$

where r is an upper bound (that we assume is known) on each δ_i value, and ε is an upper bound on the noise. Since the above optimization problem is non-convex, we will use an alternating minimization algorithm to solve for $\boldsymbol{u}, \boldsymbol{\delta}$.

2.2. Recovery Algorithm

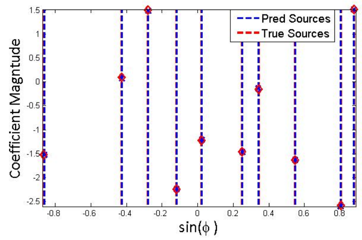
To solve P1, the following two sub-problems are considered.

- **Step I**: Given the current estimate of $\boldsymbol{\delta}$, solve for the best $\boldsymbol{u}$ as: $\text{argmin}_{\hat{\boldsymbol{u}}} \|\hat{\boldsymbol{u}}\|_1$ s.t. $\|\boldsymbol{y} - (\boldsymbol{A_\theta(\delta)}\hat{\boldsymbol{u}})\|_2^2 \le \varepsilon$.
- **Step II**: Given the current estimate of $\boldsymbol{u}$, solve for $\boldsymbol{\delta}$ as: $\boldsymbol{\delta} = \text{argmin}_{\hat{\boldsymbol{\delta}}} \|\boldsymbol{y} - (\boldsymbol{A_\theta(\delta)u})\|_2^2$ s.t. $|\delta_i| \le r$.

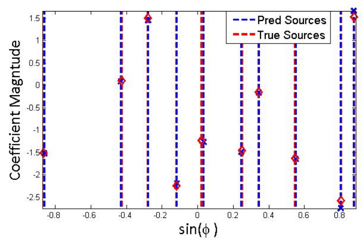
where δ_i is the i^{th} element in $\boldsymbol{\delta}$ and it is bounded by $r \triangleq \frac{1}{2N}$ to prevent overlap between the different $\theta_i + \delta_i$ values. The problem in Step I is solved via a standard basis pursuit denoising (BPDN) algorithm and the parameter ε is dependent on the noise variance. The problem in Step II is a highly nonconvex problem which is solved by coordinate descent with a constrained search for each δ_i in the range $[-r, +r]$. The alternating minimization steps are performed iteratively till convergence. Convergence of the function value is guaranteed by the monotone convergence theorem [19] as objective value decreases in each step and is bounded below by zero.

3. EXPERIMENTAL RESULTS

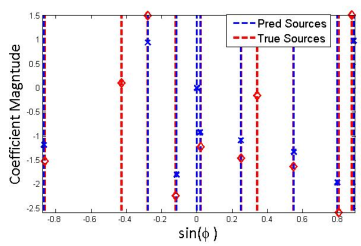
We perform various simulations to compare our algorithm (henceforth referred to as 'BF search') for DOA recovery with the Taylor approximation based methods given in [12, 14, 15] and spectral estimation based methods namely, Root-MUSIC



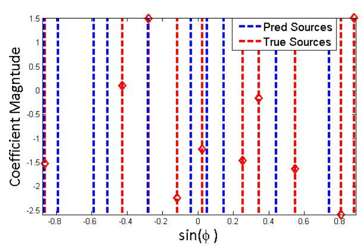
(a)



(b)



(c)



(d)

Fig. 1: Comparison of DOA estimation for (a) BF, (b) Baseline, (c) Taylor approx., (d) Root-MUSIC methods for $k = 10$ sources, $M = 30$ sensors, 1% measurement noise, $N = 90$ grid size. The sine of the angular locations of sources are plotted as vertical lines. On y-axis we have the signal strengths from each source.

and ESPRIT, for the single snapshot case. We also compare our algorithm to a baseline approach where we directly solve the standard BPDN problem with a fixed array matrix without considering any perturbations (henceforth referred to as 'Baseline', as implemented for example in [2, 16]). We show that the performance of such a method is limited by the fineness of the grid.

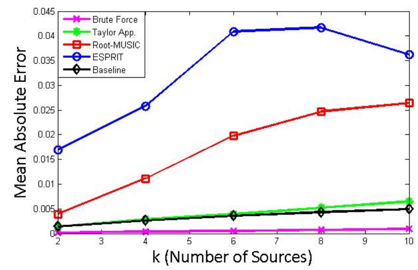


Fig. 2: Comparison of angle estimation error for (a) BF, (b) Baseline, (c) Taylor approx., (d) Root-MUSIC methods for $k \in \{2, 4, 6, 8, 10\}$ sources, $M = 30$ sensor measurements, 1% measurement noise, $N = 90$ grid size, averaged over 100 realizations

Sources	Signal Reconstruction MAE		
k	Baseline	Taylor App.	BF search
2	0.0238	0.3267	0.0219
4	0.044	0.441	0.0228
6	0.0254	0.4857	0.0268
8	0.0477	0.499	0.0316
10	0.0557	0.5638	0.0437

Table 1: Comparison of mean absolute error (MAE) in signal reconstruction for the CS based techniques, Same setup as in Fig.2

Sources	Incorrect Source Predictions				
k	BF	Taylor	Root-MUSIC	ESPRIT	Baseline
2	0	0.15	0.26	0.480	0
4	0.010	0.085	0.175	0.330	0
6	0.010	0.125	0.483	0.490	0.001
8	0.013	0.155	0.427	0.421	0.005
10	0.032	0.213	0.404	0.369	0.012

Table 2: Ratio of incorrectly predicted sources to total no of sources. Baseline has least number of incorrect predictions but this should be seen with the angle estimation error results of Fig 2.

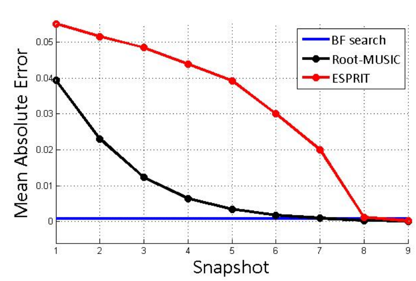


Fig. 3: Comparison of error in angle estimation for single snapshot BF search and multi-snapshot Root-MUSIC and ESPRIT for $k = 8$ sources, $M = 30$ measurements.

We note that it is possible for CS based algorithms to predict more than one source close to the original source location such that the sum of powers of these sources is equal to the original source. We use an approach where these predictions are clubbed together as a single prediction. Let $\delta_{i_1}, \delta_{i_2}, ..., \delta_{i_p}$ be the angles predicted by the algorithm that are closest (within a threshold, say $t \triangleq 0.1$ in our experiments, s.t. $|\delta_{i_k} - \delta_{itrue}| < t$) to a particular true source location say δ_{itrue}. These predictions are clustered together as a single source estimate δ_{ipred} . Let the coefficients of these p predictions be $u_{i_1}, u_{i_2}, ..., u_{i_p}$, then we combine them together as,

$$\delta_{ipred} = \frac{\sum_{j=1}^{p} |u_{i_j}| \delta_{i_j}}{\sum_{j=1}^{p} |u_{i_j}|}, u_{ipred} = \sum_{j=1}^{p} u_{i_j}.$$

Similar techniques for performance evaluation have also been used in [15]. In all our experiments we choose $\frac{d}{\lambda} = 0.5$ which satisfies $d < \lambda$. In Fig 1, we plot the prediction results for a sample experiment for the different methods with $k = 10, M = 30, N = 90$, where k is number of sources, M is number of sensor measurements and N is the grid size. We can observe from the plots that the prediction of source locations for our method is more accurate when compared to the other methods.

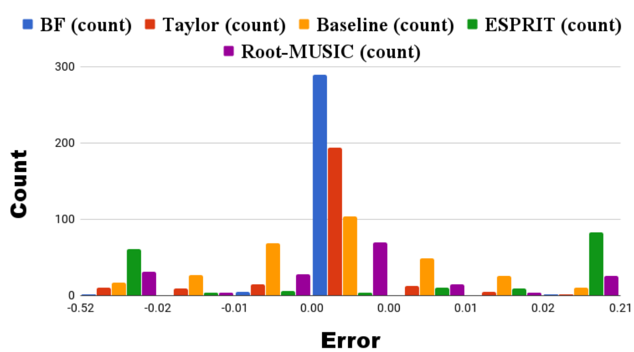


Fig. 4: Histogram plot of angular error values for all methods with k = 6 sources, M = 45 measurements, N = 90 grid size

We perform simulations for $P = 100$ different signals with varying number of sources k and compare the mean absolute error (MAE), $\frac{\sum_P \sum_i |\delta_i - \hat{\delta}_i|}{Pk}$ for our method with the other methods. For Root-MUSIC and ESPRIT, a single time instant is used for comparison with our single snapshot method. We can observe from Fig 2 that the BF search method gives the least error. We also note that the algorithms may miss certain sources in the prediction results or give an incorrect prediction which is far from any true source location. From Table 2, we observe that the ratio of incorrect predictions is small for our method as compared to Taylor method and spectral methods. Though the Baseline has least number of incorrect predictions, these should be seen in combination with the angle estimation error results of Fig 2 where our method performs better.

Grid Size N	Angle estimation MAE		
	BF search	Taylor Approx.	Baseline
90	0.0007	0.0036	0.0075
180	0.0004	0.0006	0.0013

Table 3: Comparison of angular estimation error for coarse grid $N = 90$ and fine grid $N = 180$ for CS based methods averaged over 100 simulations.

From Fig 3, we can observe that it requires around 8 snapshots for Root-MUSIC method to give similar results to our single snapshot technique. We also compare the signal reconstruction MAE $\frac{\sum_P \sum_i |s_i - \hat{s_i}|}{Pk}$ (see Table 1) for the CS based methods as spectral methods do not give the signal estimates.

In Table 3, we can see that for a coarse grid of $N = 90$, our algorithm works much better than baseline algorithm as well as the Taylor approximation based method. Only when we increase the grid fineness to $N = 180$, do these algorithms begin to perform reasonably well. But this also increases the size of the $\boldsymbol{A}$ matrix (especially in extensions to 3D-DOA estimation) as well as its coherence, which is against the well-known sufficient conditions for sparse recovery [18].

In Fig 4, we plot histogram of error in estimation of each angle $\delta_{ipred} - \delta_{itrue}$ across 100 experiments for different methods with $M = 45, k = 6$ and grid size $N = 90$. We can clearly see that the error values in our method are clustered around zero while in other methods they are spread out.

We find that the proposed Brute Force (BF) search based formulation (3) performs considerably well for DOA estimation as compared to Taylor approximation based method of [14, 15] without requiring higher resolution of the grid. We also observe that our algorithm is better than the single snapshot performance of Root-MUSIC and ESPRIT, which is useful in many applications where we have limited availability of sensor measurements.

4. CONCLUSION

In this work, we introduced a perturbed compressed sensing based DOA estimation method. In the single-snapshot case, we showed superior performance of our method for varying number of sensors and sources across many simulations, as compared to (1) standard CS-based on-grid DOA estimation technique, (2) Taylor approximation based off-grid techniques (since the Taylor error is quite large and signal-dependent), and (3) spectral estimation based Root-MUSIC and ESPRIT under the single snapshot case. As future work, we intend to extend our approach to the multi-snapshot case by solving a joint sparsity based Multiple Measurement Vector (MMV) problem [20] under the alternating minimization framework.

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