Dealing with Frequency Perturbations in Compressive Reconstructions with Fourier Sensing Matrices

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Here, we sketch a partial proof of the convergence of the solution iterates of Algorithm A1.

1. Convergence of Algorithm A1

While the empirical results show the algorithm working well across a large number of simulated scenarios, we also characterize the formulation by providing partial theoretical analysis for convergence of the algorithm. We analyze the convergence of Algorithm A1 (or its modified version) from the main paper under a specific condition mentioned further. Let $\boldsymbol{F}(\boldsymbol{\delta})$ denote the Fourier transform computed at the frequencies values $\boldsymbol{u}+\boldsymbol{\delta}$ where $\boldsymbol{\delta} = h(\boldsymbol{\beta}, \boldsymbol{u})$. Assign $\boldsymbol{z} = \{\boldsymbol{x}, \boldsymbol{\beta}\}$. Recall that our objective is to determine the solution $\boldsymbol{z}^*$ that minimizes the objective function $J(\boldsymbol{z}) \triangleq \|\boldsymbol{x}\|_1 + \lambda\|\boldsymbol{y} - \boldsymbol{F}(\boldsymbol{\delta})\boldsymbol{x}\|_2$, namely $\boldsymbol{z}^* = \mathrm{argmin}_{\boldsymbol{z}} J(\boldsymbol{z})$.

Let $\boldsymbol{z_t} = \{\boldsymbol{x_t}, \boldsymbol{\beta_t}\}$ be the present solution of our alternating search algorithm at iteration t. Our alternating search algorithm ensures that the sequence of function values $\{J(\boldsymbol{z_t})\}_{t\in\mathbb{N}}$ is monotonically decreasing. As J is bounded below by 0, the sequence $\{J(\boldsymbol{z_t})\}_{t\in\mathbb{N}}$ converges to a limit value $E \in \mathbb{R}^+$ by the monotone convergence theorem.

However, this does not yet establish the convergence of the solution sequence $\{\boldsymbol{z_t}\}$. To this end, let $\boldsymbol{x}(\boldsymbol{\beta})$ denote the minimizer for the convex objective function on $\boldsymbol{x}$ with $\boldsymbol{\beta}$ held fixed, namely $\boldsymbol{x}(\boldsymbol{\beta}) = \mathrm{argmin}_{\boldsymbol{x}} J_{\boldsymbol{\beta}}(\boldsymbol{x})$, where $J_{\boldsymbol{\beta}}(\boldsymbol{x}) = J(\boldsymbol{z})$ with $\boldsymbol{\beta}$ held constant. In the context of our alternating search algorithm, we have $\boldsymbol{x}_{t+1} = \boldsymbol{x}(\boldsymbol{\beta}_t)$. Letting $\boldsymbol{z}_{t+\frac{1}{2}} = \{\boldsymbol{x}_{t+1}, \boldsymbol{\beta}_t\}$ we find

$$
\begin{aligned}
\|\boldsymbol{x}_{t+1}\|_2 \leq \|\boldsymbol{x}_{t+1}\|_1 \leq J\left(\boldsymbol{z}_{t+\frac{1}{2}}\right) \\
= J_{\boldsymbol{\beta}_t}\left(\boldsymbol{x}_{t+1}\right) \leq J_{\boldsymbol{\beta}_t}(\boldsymbol{0}) = \lambda\|\boldsymbol{y}\|_2
\end{aligned}
$$

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giving an upper bound on the norm of $\boldsymbol{x}_t$. The last but one inequality follows from that fact that $\boldsymbol{x}_{t+1}$ minimizes $J_{\boldsymbol{\beta}_t}(\boldsymbol{x})$. Further, as $-r \leq \beta_i \leq r$ for each i, we see that the sequence $\{\boldsymbol{z}_t\}_{t\in\mathbb{N}}$ lie within a compact space. Hence as per Theorem 4.9 in [1], this sequence has atleast one accumulation point. Another statement in the same theorem states that if a certain condition is satisfied, then $\lim_{t\to\infty}\|\boldsymbol{z}_{t+1} - \boldsymbol{z}_t\| = 0$, which establishes convergence of the solution. The condition is that for each such accumulation point, the minimization of $J(\boldsymbol{z})$ gives (i) a unique solution for $\boldsymbol{x}$ if $\boldsymbol{\beta}$ is fixed, and (ii) a unique solution for $\boldsymbol{\beta}$ if $\boldsymbol{x}$ is fixed.

If the number of measurements $M \geq N$ (i.e. number of measurements M is greater than or equal to signal dimension N), then Condition (i) is easy to satisfy as the problem will be strongly convex in $\boldsymbol{x}$ if $\boldsymbol{\beta}$ is fixed, and if the columns of the matrix in $\boldsymbol{F}(\boldsymbol{\delta})$ are linearly independent. The latter condition on $\boldsymbol{F}(\boldsymbol{\delta})$ will hold with probability 1 if the frequencies are chosen uniformly at random. In the case that $M < N$, we refer to Lemma 4 of [2] which establishes uniqueness of the LASSO with probability 1 if the columns of the sensing matrix are drawn from a continuous distribution. In our case, this requires the frequencies of the columns of $\boldsymbol{F}(\boldsymbol{\delta})$ to be drawn uniformly at random. This condition is easy to satisfy because the values in $\boldsymbol{\delta}$ (which are arbitrary real numbers) will leave the columns of $\boldsymbol{F}(\boldsymbol{\delta})$ linearly independent with probability 1. Furthermore, Lemma 5 of [2] extends the result of uniqueness with probability 1, to cost functions with a differentiable data fidelity term that is strictly convex in its argument. This is satisfied by $J(\boldsymbol{z})$ which is based on the square-root LASSO, since $\|\boldsymbol{y} - \boldsymbol{F}(\boldsymbol{\delta})\boldsymbol{x}\|_2$ is strictly convex in the argument $\boldsymbol{F}(\boldsymbol{\delta})\boldsymbol{x}$, even if it is not strictly convex in $\boldsymbol{x}$.

We do not have a proof for Condition (ii), but we have *always* observed uniqueness in practice, especially since the values in $\boldsymbol{\beta}$ are bounded between $-r$ to $+r$. As an example, in Fig. 1, we show a plot of the function $\|\boldsymbol{y} - \boldsymbol{F}(\boldsymbol{\delta})\boldsymbol{x}\|_2$ keeping $\boldsymbol{x}$ and all but one value in $\boldsymbol{\delta}$ fixed. Note that here $\boldsymbol{x}$ denotes the estimated signal value upon (empirically observed) convergence of Algorithm A1. We would like to emphasize that Theorem 4.9 in [1] only requires continuity of the function J and no other conditions like biconvexity. Given the non-convexity of J, global guarantees are very difficult to establish.

References

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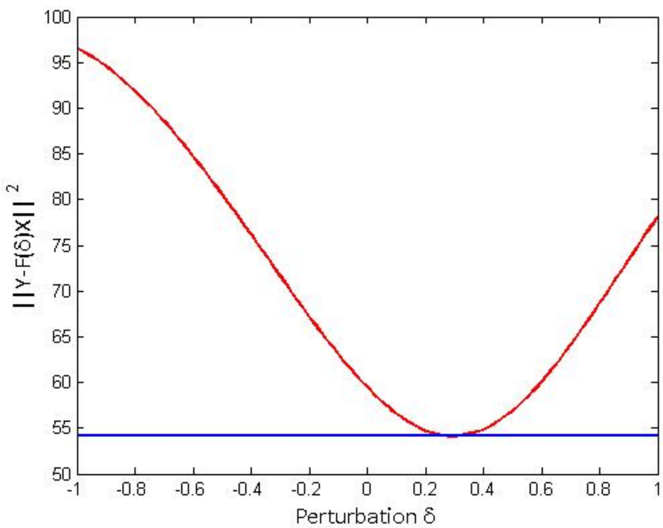

Figure 1: Uniqueness of the solution for $\boldsymbol{\delta}$ keeping $\boldsymbol{x}$ fixed, where $\boldsymbol{x}$ is the estimated signal at empirically observed convergence of Algorithm A1.