Interpolation Tutorial @IITB 2015

Lecture 1: Proof generation and Interpolation

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Where are we and where are we going?

I assume, you have seen

- CDCL (or DPLL)
- CDCL($\mathcal{T}$) (or DPLL($\mathcal{T}$))
- Theory of equality with uninterpreted functions($\mathcal{T}_{EUF}$)
- Theory of linear rational arithmetic ($\mathcal{T}_{LRA}$)

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We will see

- proof generation in CDCL($\mathcal{T}$) with $\mathcal{T}_{LRA}/\mathcal{T}_{EUF}$
- interpolation in boolean, LRA, EUF
- extension to Horn clauses (if time permits!)

Topic 1.1

Proof generation in CDCL

DPLL($\mathcal{T}$) - some notation

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Example 1.1

- $f(x) \approx g(h(x, y))$ *is a formula in QF_EUF.*
- $x > 0 \vee y + x \approx 3.5z$ *is a formula in QF_LRA.*

Boolean encoder

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Example 1.2

Let $F = x < 2 \vee (y > 0 \vee x \geq 2)$
and $e = \{x_1 \mapsto x < 2, x_2 \mapsto y > 0\}$
$e(F) = x_1 \vee (x_2 \vee \neg x_1)$

Recall: CDCL(T)

Algorithm 1.1: CDCL($\mathcal{T}$)

Input: CNF F, boolean encoder e

$\textsc{AddClauses}(e(F))$; m := $\textsc{UnitPropagation}()$; $dl := 0$; $dstack := \lambda x.0$;

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do

stands for decision level

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while $\exists x\ \{x \mapsto 0, x \mapsto 1\} \subseteq m$ **do**

  if $dl = 0$ **then return** *unsat*;

  $(C, dl) := \textsc{AnalyzeConflict}(m)$; // clause learning

  $m.resize(dstack(dl))$; $\textsc{AddClauses}(\{C\})$; m := $\textsc{UnitPropagation}()$;

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if $\forall x\ \{x \mapsto 0, x \mapsto 1\} \not\subseteq m$ **then**

  $(Cs, dl') := \textsc{TheoryDeduction}(\bigwedge e^{-1}(m))$;

  if $dl' < dl$ **then** $\{dl = dl'; m.resize(dstack(dl)); \}$;

  $\textsc{AddClauses}(e(Cs))$; m := $\textsc{UnitPropagation}()$;

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while *m is partial or* $\exists x\ \{x \mapsto 0, x \mapsto 1\} \subseteq m$;

return *sat*

Theory propagation

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Any implementation must comply with the following goals

- Correctness: boolean model is consistent with $\mathcal{T}$
- Termination: unsat partial models are never repeated

TheoryDeduction

TheoryDeduction solves conjunction of literals and returns a set of clauses and a decision level.

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Cs may contain the clauses of the form

$$(\bigwedge L) \Rightarrow \ell$$

where $\ell \in lits(F) \cup \{\bot\}$ and $L \subseteq e^{-1}(m)$.

Note: The RHS need not be a single literal

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- If $\bigwedge e^{-1}(m)$ is unsat in $\mathcal{T}$ then Cs must contain a clause with $\ell = \bot$.
- if $\bigwedge e^{-1}(m)$ is sat then $dl' = dl$.
  Otherwise, dl' is the decision level immediately after which the unsatisfiablity occurred (clearly stated shortly).

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Consider $F = (x = y \lor y = z) \land (y \neq z \lor z = u) \land (z = x)$
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After $\textsc{AddClauses}(e(Cs)); m := \textsc{UnitPropagation}()$
$m = \{x_4 \mapsto 1, x_2 \mapsto 0, x_1 \mapsto 1, x_1 \mapsto 0\}$ ← conflict

Topic 1.2

Resolution proofs for propositional logic

Resolution Proofs

A resolution proof rule is

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Implication graph for conflict graph

Example 1.4

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c_1 &= (\neg p_1 \vee p_2) \\
c_2 &= (\neg p_1 \vee p_3 \vee p_5) \\
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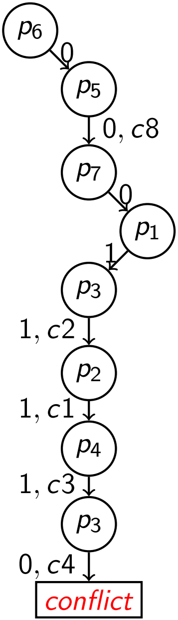

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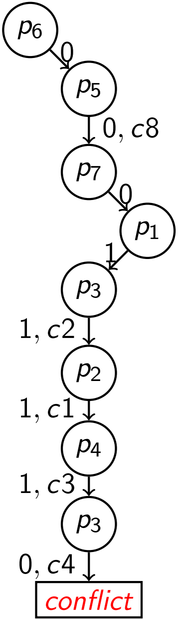

Implication graph

$\boxed{\neg p_6 @ 1}$

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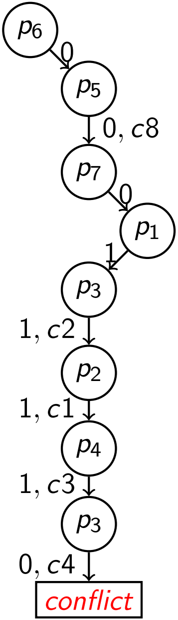

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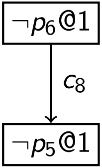

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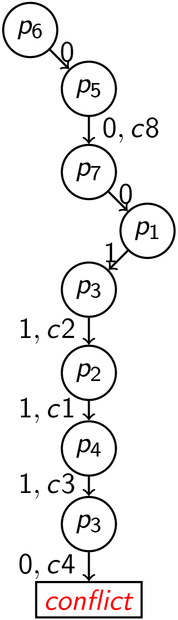

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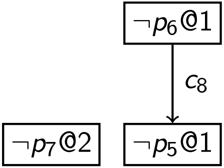

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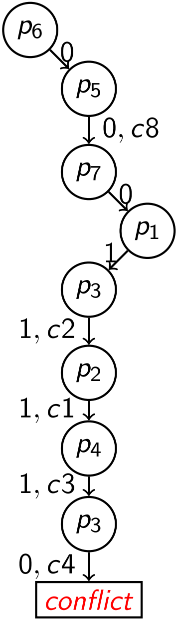

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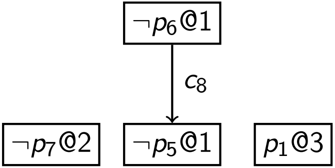

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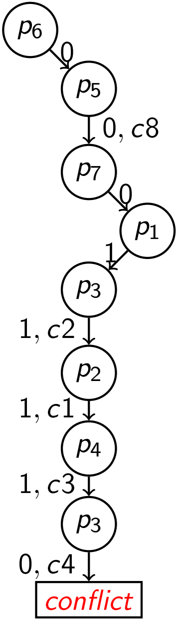

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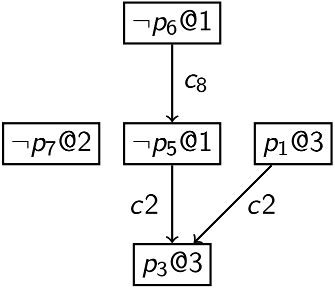

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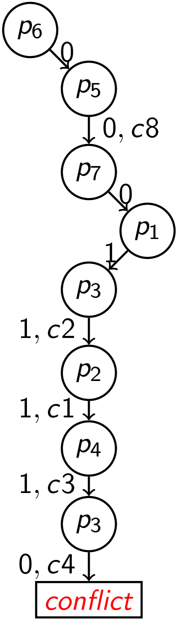
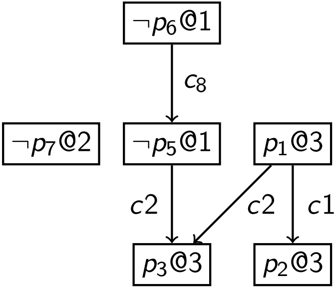


Implication graph



Implication graph for conflict graph

Example 1.4

$$c_1 = (\neg p_1 \vee p_2)$$
$$c_2 = (\neg p_1 \vee p_3 \vee p_5)$$
$$c_3 = (\neg p_2 \vee p_4)$$
$$c_4 = (\neg p_3 \vee \neg p_4)$$
$$c_5 = (p_1 \vee p_5 \vee \neg p_2)$$
$$c_6 = (p_2 \vee p_3)$$
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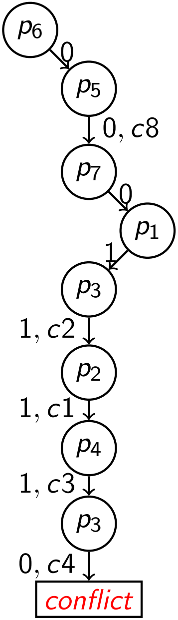

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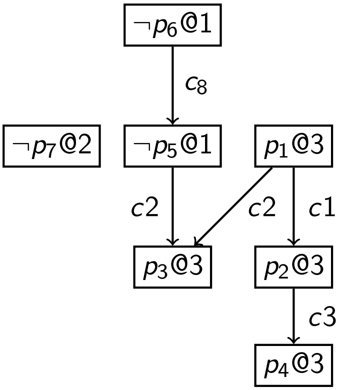

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c_8 &= (p_6 \lor \neg p_5)
\end{aligned}
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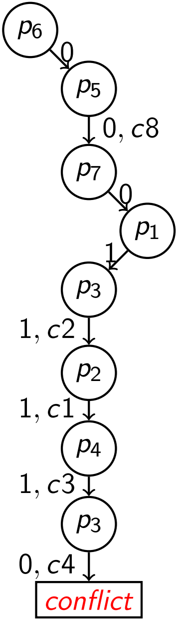

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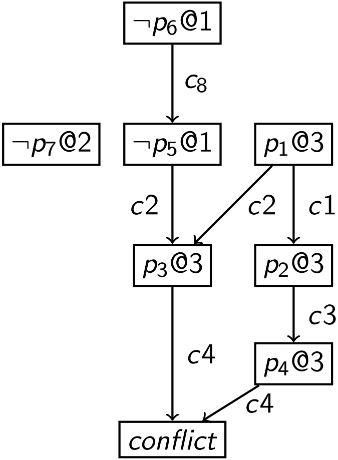
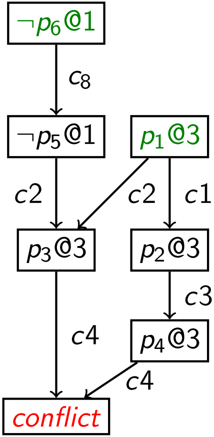


Reading proofs from implication graphs

- For each learned clause we assign a resolution proof that proves that the learned clause is implied by the clauses in the solver so far.

We demonstrate the process using an example.

Example 1.5



Input clauses:

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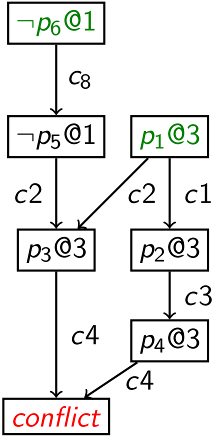

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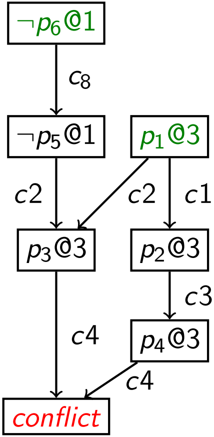

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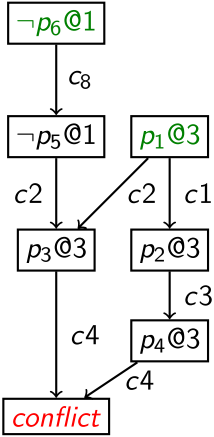

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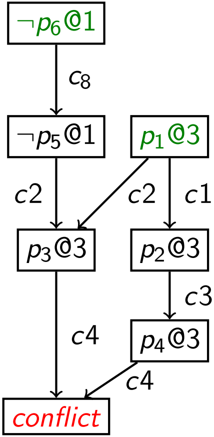

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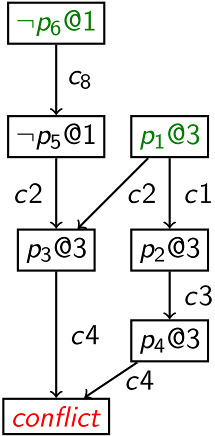

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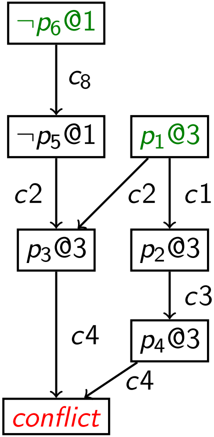

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Resolution proofs for conflict clauses

Example 1.6 (contd.)

$$\dfrac{\dfrac{\dfrac{\neg p_1 \vee p_3 \vee p_5 \qquad \dfrac{p_6 \vee \neg p_5 \quad \neg p_6}{\neg p_5}}{\neg p_1 \vee p_3} \qquad p_1}{p_3} \qquad \dfrac{\neg p_3 \vee \neg p_4 \qquad \dfrac{\neg p_2 \vee p_4 \qquad \dfrac{\neg p_1 \vee p_2 \quad p_1}{p_2}}{p_4}}{\neg p_3}}{\bot}$$

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The above is a resolution proof of the conflict clause.

Resolution proofs for conflict clauses

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One more issue:

There may be a leaf of the above proof that is a conflict clause in itself.

- In the case, there must be a resolution proof for the conflict clause.
- We “stitch” that proof with the above proof .

CDCL($\mathcal{T}$) with proof generation

Algorithm 1.2: CDCL($\mathcal{T}$)

Input: CNF F, boolean encoder e

$\textsc{AddClauses}(e(F))$; m := $\textsc{UnitPropagation}()$; $dl := 0$; $dstack := \lambda x.0$; $proofs = \lambda C.C$;

do

 // backtracking

 while $\exists x\ \{x \mapsto 0, x \mapsto 1\} \subseteq m$ **do**

  $(C, dl, proof) := \textsc{AnalyzeConflict}(m, proofs)$; $proofs(C) := proof$;

  if $C = \emptyset$ **then return** *unsat(proof)*;

  $m.resize(dstack(dl))$; $\textsc{AddClauses}(\{C\})$; m := $\textsc{UnitPropagation}()$;

 // Boolean decision

 if *m is partial* **then**

  $dstack(dl) := m.size()$;

  $dl := dl + 1$; m := $\textsc{Decide}()$; m := $\textsc{UnitPropagation}()$;

 // Theory propagation

 if $\forall x\ \{x \mapsto 0, x \mapsto 1\} \not\subseteq m$ **then**

  $(Cs, dl', proofs') := \textsc{TheoryDeduction}(\bigwedge e^{-1}(m))$;

  if $dl' < dl$ **then** $\{dl = dl'; m.resize(dstack(dl)); \}$;

  update proofs; $\textsc{AddClauses}(e(Cs))$; m := $\textsc{UnitPropagation}()$;

while *m is partial or* $\exists x\ \{x \mapsto 0, x \mapsto 1\} \subseteq m$;

return *sat*

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We also need proofs of Cs from the theory solvers

Topic 1.3

Proofs from theory solvers

Theory solvers

Each theory needs to have its own proof rules and instrumentation of the employed decision procedure to obtain proofs.

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Here, we will look at two examples

- Theory of linear rational arithmetic ($\mathcal{T}_{LRA}$)
- Theory of equality with uninterpreted functions($\mathcal{T}_{EUF}$)

Proof generation in $\mathcal{T}_{LRA}$

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The following is the only proof rule for the theory.

$$\frac{a_1 x \leq b_1 \quad a_1 x \leq b_1}{(\lambda_1 a_1 + \lambda_2 a_2)x \leq (\lambda_1 b_1 + \lambda_2 b_2)} \lambda_1, \lambda_2 \geq 0$$

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Example 1.7

Consider: $3x_1 \leq -6 \wedge x_1 - 3x_2 \leq 1 \wedge x_1 + x_2 \leq 2$

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There are many decision procedures for solving LRA.

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The algorithm is not the most efficient one, but it will help us illustrate the proof extraction.

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equisatisfiable but without x

Exercise 1.1

a. Add support for equality, dis-equality, and strict inequalities.

b. What is the complexity?

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Consider: $-x_1 + x_2 + 2x_3 \leq 0 \wedge x_1 - x_2 \leq 0 \wedge x_1 - x_3 \leq 0 \wedge 1 \leq x_3$

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After simplification:

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Since x_2 is bounded only from one side, we can trivially eliminate x_2

$x_3 \leq 0 \wedge 1 \leq x_3$

Example: Fourier-Motzkin

Consider: $-x_1 + x_2 + 2x_3 \leq 0 \wedge x_1 - x_2 \leq 0 \wedge x_1 - x_3 \leq 0 \wedge 1 \leq x_3$

Suppose we eliminate x_1 first. We transform the constraints in our format.

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After simplification:

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$1 \leq 0 \leftarrow$ false formula therefore unsat

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Proof rules of $\mathcal{T}_{EUF}$

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a. Run $DP_{EUF}.push$ *on* $x \approx f(x) \wedge f(f(f(x))) \not\approx f(f(x))$.

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Exercise 1.2

a. Run $DP_{EUF}.push$ *on* $x \approx f(x) \wedge f(f(f(x))) \not\approx f(f(x))$.

b. Give a fix in push such that it works on the above example.

DP_{EUF} implementation - union-find

Equivalence classes are usually implemented using union-find data structure

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Prove the above complexity

Example: union-find

Consider:

$$\underbrace{t_1 = t_8}_{1} \wedge \underbrace{t_7 = t_2}_{2} \wedge \underbrace{t_7 = t_1}_{3} \wedge \underbrace{t_6 = t_7}_{4} \wedge \underbrace{t_9 = t_3}_{5} \wedge \underbrace{t_5 = t_4}_{6} \wedge \underbrace{t_4 = t_3}_{7} \wedge \underbrace{t_5 = t_7}_{8} \wedge \underbrace{t_1 \neq t_4}_{9}$$

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$$t_1 \xrightarrow{1} t_8$$

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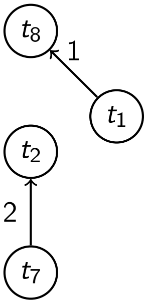

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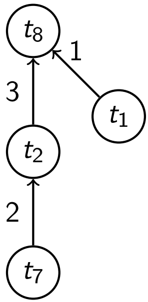

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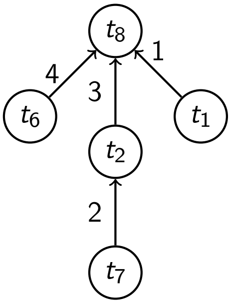

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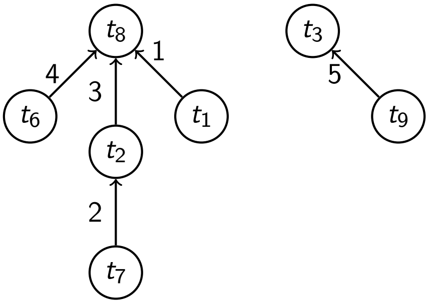

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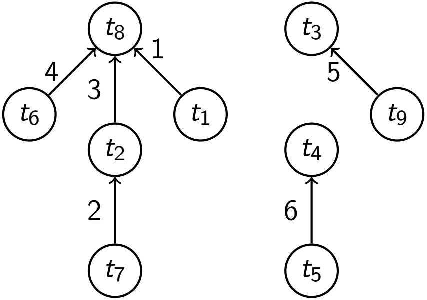

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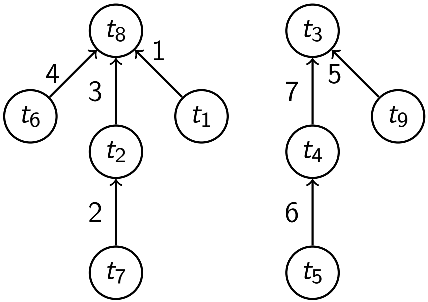

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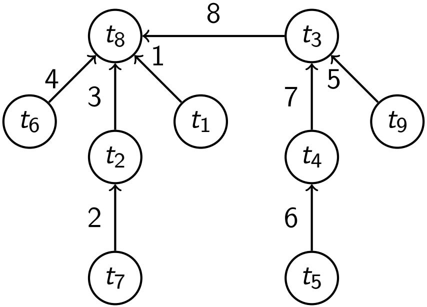

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The proof is constructed bottom up.

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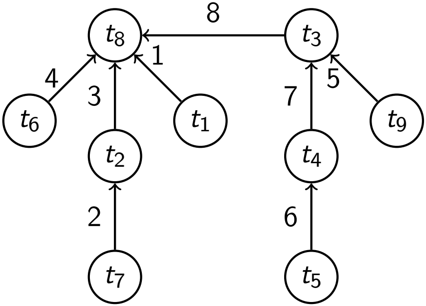
   Stitch the proofs appropriately.

For improved algorithm: R. Nieuwenhuis and A. Oliveras. Proof-producing congruence closure. RTA'05, LNCS 3467

Example: union-find proof generation

Consider:

$$\underbrace{t_1 = t_8}_{1} \wedge \underbrace{t_7 = t_2}_{2} \wedge \underbrace{t_7 = t_1}_{3} \wedge \underbrace{t_6 = t_7}_{4} \wedge \underbrace{t_9 = t_3}_{5} \wedge \underbrace{t_5 = t_4}_{6} \wedge \underbrace{t_4 = t_3}_{7} \wedge \underbrace{t_5 = t_7}_{8} \wedge \underbrace{t_1 \neq t_4}_{9}$$



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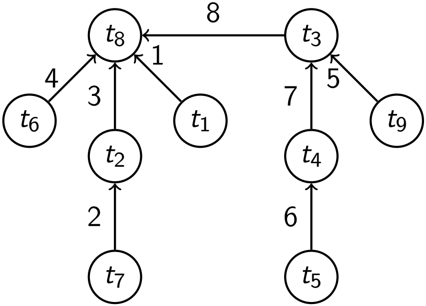

1. $t_1 \neq t_4$ is violated.

$$\frac{t_1 \neq t_4 \qquad\qquad\qquad\qquad\qquad\qquad}{\bot}$$

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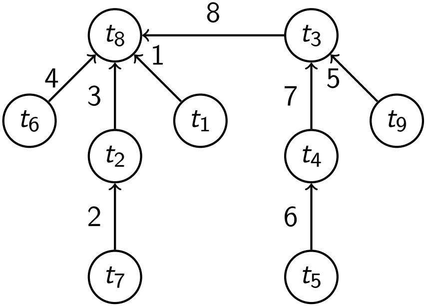


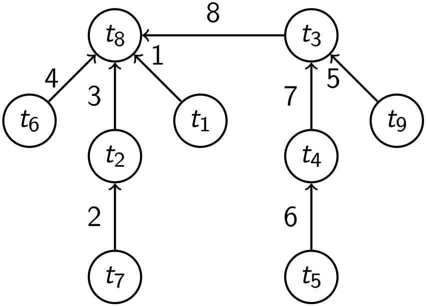
1. $t_1 \neq t_4$ is violated.
2. 8 is the latest edge in the path between t_1 and t_4

$$\dfrac{t_1 \neq t_4 \qquad t_1 = t_4}{\bot}$$

Example: union-find proof generation

Consider:

$$\underbrace{t_1 = t_8}_{1} \wedge \underbrace{t_7 = t_2}_{2} \wedge \underbrace{t_7 = t_1}_{3} \wedge \underbrace{t_6 = t_7}_{4} \wedge \underbrace{t_9 = t_3}_{5} \wedge \underbrace{t_5 = t_4}_{6} \wedge \underbrace{t_4 = t_3}_{7} \wedge \underbrace{t_5 = t_7}_{8} \wedge \underbrace{t_1 \neq t_4}_{9}$$



1. $t_1 \neq t_4$ is violated.
2. 8 is the latest edge in the path between t_1 and t_4
3. 8 corresponds to $t_5 = t_7$

$$\dfrac{t_1 \neq t_4 \qquad \dfrac{\dfrac{\dfrac{t_5 = t_7}{t_7 = t_5}}{\quad}}{t_1 = t_4}}{\bot}$$

Example: union-find proof generation

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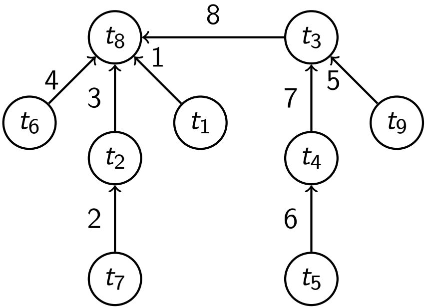

1. $t_1 \neq t_4$ is violated.
2. 8 is the latest edge in the path between t_1 and t_4
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4. Now look for proof of $t_1 = t_7$ and $t_4 = t_5$

$$\cfrac{t_1 \neq t_4 \qquad \cfrac{\cfrac{\cfrac{t_5 = t_7}{t_7 = t_5}}{}}{t_1 = t_4}}{\bot}$$

Example: union-find proof generation

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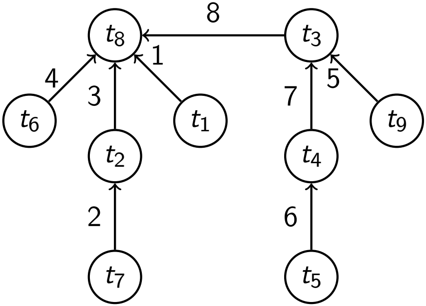

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4. Now look for proof of $t_1 = t_7$ and $t_4 = t_5$
5. 3 is the latest edge between t_1 and t_7, which corresponds to $t_1 = t_7$.

Example: union-find proof generation

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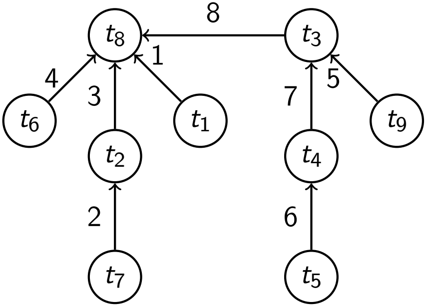

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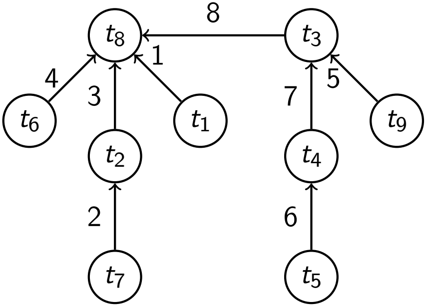
5. 3 is the latest edge between t_1 and t_7, which corresponds to $t_1 = t_7$.
6. Similarly, $t_4 = t_5$ is edge 6

Example: union-find proof generation

Consider:

$$\underbrace{t_1 = t_8}_{1} \wedge \underbrace{t_7 = t_2}_{2} \wedge \underbrace{t_7 = t_1}_{3} \wedge \underbrace{t_6 = t_7}_{4} \wedge \underbrace{t_9 = t_3}_{5} \wedge \underbrace{t_5 = t_4}_{6} \wedge \underbrace{t_4 = t_3}_{7} \wedge \underbrace{t_5 = t_7}_{8} \wedge \underbrace{t_1 \neq t_4}_{9}$$



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End of Lecture 1