IITB Tutorial : Interpolation 2015

Lecture 2: Interpolants

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Compile date: 2015-11-07

Where are we and where are we going?

We have seen

- proof generation

We will see

- computing interpolants using proofs

Topic 2.1

Interpolants

Interpolants

Definition 2.1

For mutually unsat formulas A and B, A formula I is interpolant between A and B if

1. $A \Rightarrow \mathrm{I}$,
2. $B \wedge \mathrm{I} \Rightarrow \bot$, *and*
3. $vars(\mathrm{I}) \subseteq (vars(A) \cap vars(B))$

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Consider:

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$vars(A) = \{x_1, x_2, x_3\}$ $vars(B) = \{x_1, x_3, x_4\}$ $vars(\mathrm{I}) \subseteq \{x_1, x_3\}$

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$vars(A) = \{x_1, x_2, x_3\}$ $\quad vars(B) = \{x_1, x_3, x_4\}$ $\quad vars(\mathrm{I}) \subseteq \{x_1, x_3\}$

$\mathrm{I} = x_1 + x_3 \leq 2$

A-local, B-local, global symbols

Definition 2.2

A symbol is A-local if it only occurs in A.

Definition 2.3

A symbol is B-local if it only occurs in B.

Definition 2.4

A symbol is global if it occurs both A and B.
The set of global symbols is written $Globals = vars(A) \cap vars(B)$

Interpolants, equivalent definition

Definition 2.5

Let $A \Rightarrow B$. A formula I is *interpolant* *between A and B if*

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Show both the definitions are equivalent.

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In this form, interpolant captures a general idea behind every human inquiry.

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Example 2.2

Human activity $\Rightarrow$ $\Rightarrow$ *Climate change*

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$$\textit{Human activity} \Rightarrow \underbrace{\textit{Increase in CO2 level}}_{\textit{interpolant}} \Rightarrow \textit{Climate change}$$

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A concise explanation of cause and effect

Interpolants from proofs

Back to the original definition of interpolants.

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As proofs have intermediate results, we should have equivalent concept of intermediate interpolants.

Partial interpolants

Definition 2.6

Let A,B and C be formulas such that $A \wedge B \Rightarrow C$.
A partial interpolant I_C *between A and B for C is a formula such that*

- $A \Rightarrow \mathrm{I}_C$
- $B \wedge \mathrm{I}_C \Rightarrow C$
- $vars(\mathrm{I}_C) \subseteq Globals \cup vars(C)$

Interpolation via proofs

We present here a proof based method for computing interpolants.

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2. annotate each intermediate derived formulas with partial interpolants inductively

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We present here a proof based method for computing interpolants.

Consider A and B are mutually unsat formulas in some given theory.

1. produce proof of unsatisfiablity of $A \wedge B$
2. annotate each intermediate derived formulas with partial interpolants inductively
3. Since $\bot$ is the final node of the proof, the annotation of $\bot$ is the interpolant between A and B.

We will write annotations in proof rules within square brackets after derived formulas.

$$\dots \frac{\dots}{C[\mathrm{I}_C]} \dots$$

Topic 2.2

Interpolation in $\mathcal{T}_{LRA}$

Annotation rules for conjunction of linear arithmetic formulas

Linear arithmetic proof system

$$hyp\frac{}{aX \leq c}aX \leq c \in A, B \quad comb\frac{a_1X \leq c_1 \quad a_2X \leq c_2}{(\lambda_1 a_1 + \lambda_2 a_2)X \leq (\lambda_1 c_1 + \lambda_2 c_2)}\lambda_1, \lambda_2 \geq 0$$

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Proof rules with partial interpolant annotations

$$\text{HYP-A}\frac{}{aX \leq c[aX \leq c]} aX \leq c \in A \qquad \text{HYP-B}\frac{}{aX \leq c[0 \leq 0]} aX \leq c \in B$$

$$\text{COMB}\frac{a_1X \leq c_1[a_1'X \leq c_1'] \quad a_2X \leq c_2[a_2'X \leq c_2']}{(\lambda_1 a_1 + \lambda_2 a_2)X \leq (\lambda_1 c_1 + \lambda_2 c_2)[(\lambda_1 a_1' + \lambda_2 a_2')X \leq (\lambda_1 c_1' + \lambda_2 c_2')]} \begin{matrix}\lambda_1 \geq 0\\ \lambda_2 \geq 0\end{matrix}$$

Example: LRA annotations

Example 2.3

Consider:

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$$\frac{x_1 + x_2 \leq 2 \qquad 6x_4 - 2x_1 \leq -8}{3x_4 + x_2 \leq -2}$$

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Theorem 2.1

The annotations in the proof rules are partial interpolants

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1. $A \Rightarrow a'X \leq c'$, //1st cond. of partial interpolants

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3. B-locals do not occur in $a'X \leq c'$, and

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1. $A \Rightarrow a'X \leq c'$, //1st cond. of partial interpolants
2. $B \Rightarrow (a - a')X \leq (c - c')$, //implies 2nd cond.(why?)
3. B-locals do not occur in $a'X \leq c'$, and
4. A-locals have same coefficient in $aX \leq c$ and $a'X \leq c'$. //3-4 imply 3rd cond.

Annotation correctness(contd.)

Proof(contd.)

□

base case:

$$\textsc{hyp-a}\frac{}{aX \leq c[aX \leq c]} aX \leq c \in A$$

Annotation correctness(contd.)

Proof(contd.)

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base case:

$$\text{HYP-A}\frac{}{aX \leq c[aX \leq c]} aX \leq c \in A$$

1. $A \Rightarrow aX \leq c$

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Proof(contd.)

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$$\textsc{Hyp-B}\frac{}{aX \leq c[0 \leq 0]} aX \leq c \in B$$

Annotation correctness(contd.)

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1. $A \Rightarrow 0 \le 0$

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1. $A \Rightarrow 0 \leq 0$
2. $B \Rightarrow (a-0)X \leq (c-0)$
3. B-locals do not occur in $0 \leq 0$, and
4. A-locals have same coefficient in $0 \leq 0$ and $aX \leq c$

Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\text{COMB}\frac{a_1 X \le c_1[a'_1 X \le c'_1] \quad a_2 X \le c_2[a'_2 X \le c'_2]}{(\lambda_1 a_1 + \lambda_2 a_2)X \le (\lambda_1 c_1 + \lambda_2 c_2)[(\lambda_1 a'_1 + \lambda_2 a'_2)X \le (\lambda_1 c'_1 + \lambda_2 c'_2)]}\begin{matrix}\lambda_1 \ge 0\\ \lambda_2 \ge 0\end{matrix}$$

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Due to ind. hyp., the conditions of the partial interpolants of the antecedents holds.

1. $A \Rightarrow a'_1 X \leq c'_1$
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Annotation correctness(contd.)

Proof(contd.)

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1. $A \Rightarrow a'_2 X \le c'_2$
2. $B \Rightarrow (a_2 - a'_2)X \le (c_2 - c'_2)$
3. No B-locals in $a'_2 X \le c'_2$
4. A-locals have same coefficient in $a_2 X \le c_2$ and $a'_2 X \le c'_2$.

Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\text{COMB}\frac{a_1X \le c_1[a'_1X \le c'_1] \quad a_2X \le c_2[a'_2X \le c'_2]}{(\lambda_1 a_1 + \lambda_2 a_2)X \le (\lambda_1 c_1 + \lambda_2 c_2)[(\lambda_1 a'_1 + \lambda_2 a'_2)X \le (\lambda_1 c'_1 + \lambda_2 c'_2)]}\begin{matrix}\lambda_1 \ge 0\\ \lambda_2 \ge 0\end{matrix}$$

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Rest is exercises □

Exercise 2.2

Show the above four properties hold for the annotation of the consequent (Only proving 4th condition is non-trivial.)

Topic 2.3

Interpolation in Propositional logic

Annotation rules for propositional formulas

Resolution proof system

$$hyp \frac{}{C} C \in A, B \quad comb \frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

Annotation rules for propositional formulas

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$$hyp\frac{}{C}\, C \in A, B \quad comb\frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

Let $C/_B = \{l \in C | vars(l) \notin vars(B)\}$ and $C|_B = \{l \in C | vars(l) \in vars(B)\}$.
Proof rules with partial interpolant annotations

$$\textsc{hyp-A}\frac{}{C[C|_B \vee C/_B]}\, C \in A \quad \textsc{hyp-B}\frac{}{C[\top \vee C/_B]}\, C \in B$$

Annotation rules for propositional formulas

Resolution proof system

$$hyp \frac{}{C} C \in A, B \quad comb \frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

Let $C/_B = \{l \in C | vars(l) \notin vars(B)\}$ and $C|_B = \{l \in C | vars(l) \in vars(B)\}$.
Proof rules with partial interpolant annotations

$$\textsc{hyp-A} \frac{}{C[C|_B \vee C/_B]} C \in A \quad \textsc{hyp-B} \frac{}{C[\top \vee C/_B]} C \in B$$

$$\textsc{Res-B} \frac{C \vee x[\mathrm{I}_1 \vee C/_B] \quad D \vee \neg x[\mathrm{I}_2 \vee D/_B]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B]} x \in vars(B)$$

Annotation rules for propositional formulas

Resolution proof system

$$hyp \frac{}{C} C \in A, B \quad comb \frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

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$$\textsc{Res-A} \frac{C \vee x[\mathrm{I}_1 \vee (C/_B \vee x)] \quad D \vee \neg x[\mathrm{I}_2 \vee (D/_B \vee \neg x)]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B]} x \notin vars(B)$$

Resolution Annotation correctness

Theorem 2.2

The annotations in the above proof rules are partial interpolants

Proof.

We will prove two stronger conditions than partial interpolation conditions. We have disjunction $(\mathrm{I} \vee C/_B)$ in each annotation.

1. $A \Rightarrow (\mathrm{I} \vee C/_B)$
2. $B \wedge \mathrm{I} \Rightarrow C|_B$, which implies the second condition $B \wedge (\mathrm{I} \vee C/_B) \Rightarrow C$ (why?)

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base case:

$$\textsc{hyp-A}\ \frac{}{C[C|_B \vee C/_B]}\ C \in A$$

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$$\text{HYP-A}\,\frac{}{C[C|_B \vee C/_B]}\, C \in A$$

1. $A \Rightarrow C|_B \vee C/_B$
2. $B \wedge (C|_B) \Rightarrow C|_B$

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3. $\mathrm{I} \subseteq Globals$, which implies the 3rd condition

base case:

$$\text{HYP-A}\ \frac{}{C[C|_B \vee C/_B]}\ C \in A$$

1. $A \Rightarrow C|_B \vee C/_B$
2. $B \wedge (C|_B) \Rightarrow C|_B$
3. $vars(C|_B) \subseteq Globals$ (why?)

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base case:

$$\textsc{hyp}\text{-A}\frac{}{C[C|_B \vee C/_B]}\, C \in A \qquad \textsc{hyp}\text{-B}\frac{}{C[\top \vee C/_B]}\, C \in B$$

1. $A \Rightarrow C|_B \vee C/_B$
2. $B \wedge (C|_B) \Rightarrow C|_B$
3. $\mathit{vars}(C|_B) \subseteq \mathit{Globals}$ (why?)

Resolution Annotation correctness

Theorem 2.2

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We have disjunction $(\mathrm{I} \vee C/_B)$ in each annotation.

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3. $\mathrm{I} \subseteq \mathit{Globals}$, which implies the 3rd condition

base case:

$$\textsc{hyp-A}\,\frac{}{C[C|_B \vee C/_B]}\, C \in A$$

1. $A \Rightarrow C|_B \vee C/_B$
2. $B \wedge (C|_B) \Rightarrow C|_B$
3. $\mathit{vars}(C|_B) \subseteq \mathit{Globals}$ (why?)

$$\textsc{hyp-B}\,\frac{}{C[\top \vee C/_B]}\, C \in B$$

1. $A \Rightarrow \top \vee C/_B$

Resolution Annotation correctness

Theorem 2.2

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Proof.

We will prove two stronger conditions than partial interpolation conditions. We have disjunction $(\mathrm{I} \vee C/_B)$ in each annotation.

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2. $B \wedge \mathrm{I} \Rightarrow C|_B$, which implies the second condition $B \wedge (\mathrm{I} \vee C/_B) \Rightarrow C$ (why?)
3. $\mathrm{I} \subseteq \mathit{Globals}$, which implies the 3rd condition

base case:

$$\textsc{hyp-A}\ \frac{}{C[C|_B \vee C/_B]}\ C \in A$$

1. $A \Rightarrow C|_B \vee C/_B$
2. $B \wedge (C|_B) \Rightarrow C|_B$
3. $\mathit{vars}(C|_B) \subseteq \mathit{Globals}$ (why?)

$$\textsc{hyp-B}\ \frac{}{C[\top \vee C/_B]}\ C \in B$$

1. $A \Rightarrow \top \vee C/_B$
2. $B \wedge \top \Rightarrow C|_B$ (why?)

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The annotations in the above proof rules are partial interpolants

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We will prove two stronger conditions than partial interpolation conditions. We have disjunction $(\mathrm{I} \vee C/_B)$ in each annotation.

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2. $B \wedge \mathrm{I} \Rightarrow C|_B$, which implies the second condition $B \wedge (\mathrm{I} \vee C/_B) \Rightarrow C$ (why?)
3. $\mathrm{I} \subseteq Globals$, which implies the 3rd condition

base case:

$$\textsc{hyp-A}\ \frac{}{C[C|_B \vee C/_B]}\ C \in A$$

1. $A \Rightarrow C|_B \vee C/_B$
2. $B \wedge (C|_B) \Rightarrow C|_B$
3. $vars(C|_B) \subseteq Globals$ (why?)

$$\textsc{hyp-B}\ \frac{}{C[\top \vee C/_B]}\ C \in B$$

1. $A \Rightarrow \top \vee C/_B$
2. $B \wedge \top \Rightarrow C|_B$ (why?)
3. $vars(\top) \subseteq Globals$

Resolution Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\textsc{Res-B}\frac{C \vee x[\mathrm{I}_1 \vee C/_B] \quad D \vee \neg x[\mathrm{I}_2 \vee D/_B]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B]} x \in vars(B)$$

Resolution Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\textsc{Res-B}\,\frac{C \vee x[\mathrm{I}_1 \vee C/_B] \quad D \vee \neg x[\mathrm{I}_2 \vee D/_B]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B]}\, x \in \mathit{vars}(B)$$

1. Since $A \Rightarrow \mathrm{I}_1 \vee C/_B$ and $A \Rightarrow \mathrm{I}_2 \vee D/_B$, $A \Rightarrow (\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B$.(why?)

Resolution Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\textsc{Res-B}\frac{C \vee x[\mathrm{I}_1 \vee C/_B] \quad D \vee \neg x[\mathrm{I}_2 \vee D/_B]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B]} x \in vars(B)$$

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2. Since $B \wedge \mathrm{I}_1 \Rightarrow (C \vee x)|_B$ and $B \wedge \mathrm{I}_2 \Rightarrow (D \vee \neg x)|_B$,

$$B \wedge \mathrm{I}_1 \wedge \mathrm{I}_2 \Rightarrow (C|_B \vee x) \wedge (D|_B \vee \neg x) \Rightarrow (C \vee D)|_B.$$

Resolution Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\textsc{Res-B}\frac{C \vee x[\mathrm{I}_1 \vee C/_B] \quad D \vee \neg x[\mathrm{I}_2 \vee D/_B]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B]} x \in \mathit{vars}(B)$$

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2. Since $B \wedge \mathrm{I}_1 \Rightarrow (C \vee x)|_B$ and $B \wedge \mathrm{I}_2 \Rightarrow (D \vee \neg x)|_B$,

$$B \wedge \mathrm{I}_1 \wedge \mathrm{I}_2 \Rightarrow (C|_B \vee x) \wedge (D|_B \vee \neg x) \Rightarrow (C \vee D)|_B.$$

3. $\mathit{vars}(\mathrm{I}_1 \wedge \mathrm{I}_2) \subseteq \mathit{Globals}$

Resolution Annotation correctness(contd.)

Proof(contd.)

$$\textsc{Res-A}\ \frac{C \vee x[\mathrm{I}_1 \vee (C/_B \vee x)] \quad D \vee \neg x[\mathrm{I}_2 \vee (D/_B \vee \neg x)]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B]} x \notin vars(B)$$

Resolution Annotation correctness(contd.)

Proof(contd.)

$$\textsc{Res-A}\,\frac{C \vee x[\mathrm{I}_1 \vee (C/_B \vee x)] \quad D \vee \neg x[\mathrm{I}_2 \vee (D/_B \vee \neg x)]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B]}\, x \notin vars(B)$$

1. Since $A \Rightarrow \mathrm{I}_1 \vee C/_B \vee x$ and $A \Rightarrow \mathrm{I}_2 \vee D/_B \vee \neg x$,

$$A \Rightarrow (\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B.(why?)$$

Resolution Annotation correctness(contd.)

Proof(contd.)

$$\textsc{Res-A}\,\frac{C \vee x[\mathrm{I}_1 \vee (C/_B \vee x)] \quad D \vee \neg x[\mathrm{I}_2 \vee (D/_B \vee \neg x)]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B]}\, x \notin vars(B)$$

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$$A \Rightarrow (\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B.(why?)$$

2. Since $B \wedge \mathrm{I}_1 \Rightarrow C|_B$ and $B \wedge \mathrm{I}_2 \Rightarrow D|_{B(\text{why?})}$,

$$B \wedge (\mathrm{I}_1 \vee \mathrm{I}_2) \Rightarrow C|_B \vee D|_B.$$

Resolution Annotation correctness(contd.)

Proof(contd.)

$$\textsc{Res-A}\ \frac{C \vee x[\mathrm{I}_1 \vee (C/_B \vee x)] \quad D \vee \neg x[\mathrm{I}_2 \vee (D/_B \vee \neg x)]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B]}\, x \notin vars(B)$$

1. Since $A \Rightarrow \mathrm{I}_1 \vee C/_B \vee x$ and $A \Rightarrow \mathrm{I}_2 \vee D/_B \vee \neg x$,

   $$A \Rightarrow (\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B.(why?)$$

2. Since $B \wedge \mathrm{I}_1 \Rightarrow C|_B$ and $B \wedge \mathrm{I}_2 \Rightarrow D|_{B(\text{why?})}$,

   $$B \wedge (\mathrm{I}_1 \vee \mathrm{I}_2) \Rightarrow C|_B \vee D|_B.$$

3. $vars(\mathrm{I}_1 \vee \mathrm{I}_2) \subseteq Globals$.

End of Lecture 2