Mathematical Logic

Lecture 13: Model existence theorem and its consequences

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Where are we and where are we going?

We have seen

▶ Syntax and semantics of FOL
▶ Herbrand model and Hinttika theorem

We will see

▶ Model existence theorem
▶ Compactness theorem
▶ Löwenheim-Skolem Theorem
Topic 13.1

Model existence theorem
Parameters

One often needs fresh symbols when instantiating an existential quantifiers.
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Example 13.1
Consider $S = (\{a/0, b/0\}, \{P/1\})$.
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Consider $S = \langle \{a/0, b/0\}, \{P/1\} \rangle$.
Is the following formula sat?

$$P(a) \land P(b) \land \exists x. \neg P(x)$$
Parameters

One often needs fresh symbols when instantiating an existential quantifiers.

Example 13.1

Consider $S = \{a/0, b/0\}, \{P/1\}$.

Is the following formula sat?

$$P(a) \land P(b) \land \exists x. \neg P(x)$$

We need to have a new constant symbol $c$ that denotes a value s.t. $\neg P(c)$ is true. Note that $a$ and $b$ can not do the job.
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Example 13.2
Consider $S = (\{\}, \{P/1\})$. Is the following formula sat?

$$\forall x. P(x) \land \exists x. \neg P(x)$$
Parameters

One often needs fresh symbols when instantiating an existential quantifiers.

Example 13.1

Consider $\mathbf{S} = (\{a/0, b/0\}, \{P/1\})$.

Is the following formula sat?

$$P(a) \land P(b) \land \exists x. \neg P(x)$$

We need to have a new constant symbol $c$ that denotes a value s.t. $\neg P(c)$ is true. Note that $a$ and $b$ can not do the job.

Example 13.2

Consider $\mathbf{S} = (\emptyset, \{P/1\})$. Is the following formula sat?

$$\forall x. P(x) \land \exists x. \neg P(x)$$

1. Instantiate existential quantifier with a fresh symbol $c$.

$$\neg P(c) \land \forall x. P(x) \land \exists x. \neg P(x)$$
Parameters

One often needs fresh symbols when instantiating an existential quantifiers.

Example 13.1

Consider $S = (\{a/0, b/0\}, \{P/1\})$.

Is the following formula sat?

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We need to have a new constant symbol $c$ that denotes a value s.t. $\neg P(c)$ is true. Note that $a$ and $b$ can not do the job.

Example 13.2

Consider $S = (\{\}, \{P/1\})$. Is the following formula sat?

$$\forall x. P(x) \land \exists x. \neg P(x)$$

1. Instantiate existential quantifier with a fresh symbol $c$.
   $$\neg P(c) \land \forall x. P(x) \land \exists x. \neg P(x)$$

2. Instantiate universal quantifier with a term $c$.
   $$P(c) \land \neg P(c) \land \forall x. P(x) \land \exists x. \neg P(x)$$
Parameters

We will need a supply of fresh symbols.
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We will need a supply of fresh symbols.

Let us define an extension of signature that ensures a supply of new constant symbols.
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We will need a supply of fresh symbols.

Let us define an extension of signature that ensures a supply of new constant symbols.

**Definition 13.1**

Let $S = (F, R)$ be a signature. Let $\text{par}$ be a infinite countable set of constant symbols disjoint from $S$. Let $S^\text{par} = (F, R \cup \text{par})$. 
Consistency property

Definition 13.2
Let $S = (F, R)$ be a signature. Let $C$ be a collection of sets of sentences in signature $S^{\text{par}}$. $C$ is a consistency property wrt to $S$ if for each $S \in C$ satisfies the following.
Consistency property

Definition 13.2
Let $S = (F, R)$ be a signature. Let $C$ be a collection of sets of sentences in signature $S_{\text{par}}$. $C$ is a consistency property wrt to $S$ if for each $S \in C$ satisfies the following.

1. for each $F \in A_{S_{\text{par}}}$, either $F \notin S$ or $\neg F \notin S$
2. if $\neg \neg F \in S$ then $\{F\} \cup S \in C$
3. if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \cup S \in C$
4. if $\beta \in S$ then $\{\beta_1\} \cup S \in C$ or $\{\beta_2\} \cup S \in C$
Consistency property

Definition 13.2
Let \( S = (F, R) \) be a signature. Let \( C \) be a collection of sets of sentences in signature \( S^{\text{par}} \). \( C \) is a consistency property wrt to \( S \) if for each \( S \in C \) satisfies the following.

1. for each \( F \in A_{S^{\text{par}}} \), either \( F \notin S \) or \( \neg F \notin S \)
2. if \( \neg
\neg F \in S \) then \( \{F\} \cup S \in C \)
3. if \( \alpha \in S \) then \( \{\alpha_1, \alpha_2\} \cup S \in C \)
4. if \( \beta \in S \) then \( \{\beta_1\} \cup S \in C \) or \( \{\beta_2\} \cup S \in C \)
5. if \( \gamma \in S \) then \( \{\gamma(t)\} \cup S \in C \) for each \( t \in \hat{T}_{S^{\text{par}}} \)
6. if \( \delta \in S \) then \( \{\delta(c)\} \cup S \in C \) for some \( c \in \text{par} \)
Consistency property

Definition 13.2
Let $S = (F, R)$ be a signature. Let $C$ be a collection of sets of sentences in signature $S^{\text{par}}$. $C$ is a consistency property wrt to $S$ if for each $S \in C$ satisfies the following.

1. For each $F \in A_S^{\text{par}}$, either $F \notin S$ or $\neg F \notin S$
2. If $\neg \neg F \in S$ then $\{F\} \cup S \in C$
3. If $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \cup S \in C$
4. If $\beta \in S$ then $\{\beta_1\} \cup S \in C$ or $\{\beta_2\} \cup S \in C$
5. If $\gamma \in S$ then $\{\gamma(t)\} \cup S \in C$ for each $t \in \hat{T}_S^{\text{par}}$
6. If $\delta \in S$ then $\{\delta(c)\} \cup S \in C$ for some $c \in \text{par}$
7. $S \cup \{t \approx t\} \in C$ for each $t \in \hat{T}_S^{\text{par}}$
8. If $t_1 \approx u_1, \ldots, t_n \approx u_n \in S$ then $S \cup \{f(t_1, \ldots, t_n) \approx f(u_1, \ldots, u_n)\} \in C$ for each $f \in F$
9. If $t_1 \approx u_1, \ldots, t_n \approx u_n$, $P(t_1, \ldots, t_n) \in S$ then $S \cup \{P(u_1, \ldots, u_n)\} \in C$ for each $P \in R \cup \{\approx /2\}$
Model existence theorem

**Theorem 13.1**

*Let $C$ be a consistency property wrt to $S$, $S$ be a set of $S$-sentences. If $S \in C$, then $S$ is sat.*

Recall the proof in propositional case.

1. convert $C$ into finite character
2. show limit exists in finite character
3. construct a monotonic sequence of elements of $C$ starting from $S$
4. show its limit is a maximal element of $C$
5. show the limit is a Hinittika set

Naturally things are more complicated here.
Recall: subset closed consistency property

**Theorem 13.2**

*Every consistency property $C$ can be extended to a consistency property that is subset closed.*
Recall: subset closed consistency property

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*Every consistency property $C$ can be extended to a consistency property that is subset closed.*

**Proof.**

Let $C^+ := \{ S' | S' \subseteq S \text{ and } S \in C \}$. We show $C^+$ is consistency property.
Theorem 13.2

Every consistency property $C$ can be extended to a consistency property that is subset closed.

Proof.

Let $C^+ := \{S' | S' \subseteq S \text{ and } S \in C\}$. We show $C^+$ is consistency property. Consider $S' \in C^+$. By definition, there is $S \in C$ s.t. $S' \subseteq S$.\[\]
Recall: subset closed consistency property

**Theorem 13.2**

*Every consistency property $C$ can be extended to a consistency property that is subset closed.*

**Proof.**

Let $C^+ := \{ S' | S' \subseteq S \text{ and } S \in C \}$. We show $C^+$ is a consistency property. Consider $S' \in C^+$. By definition, there is $S \in C$ s.t. $S' \subseteq S$.

1. Therefore, $S'$ does not contain contradictory literals.
2. If $\neg \neg F \in S'$. Therefore, $\neg \neg F \in S$. Therefore, $\{F\} \cup S \in C$. Therefore, $\{F\} \cup S' \in C^+$.
3. .... (trivially extends to all 9 cases)
Recall: finite character

Definition 13.3
A consistency property \( C \) has **finite character** if \( S \in C \) iff every finite subset of \( S \) is in \( C \).

Theorem 13.3
if \( C \) is of finite character then \( C \) is subset closed.

Theorem 13.4
Let consistency property \( C \) is of finite character. If \( S_1, S_2, \ldots \) is sequence of members of \( C \) such that \( S_1 \subseteq S_2 \subseteq \ldots \). Then, \( \bigcup_i S_i \in C \).

Proofs of the above theorems were given in lecture 6.
Extendable to finite character

**Theorem 13.5**

A subset closed consistency property $C$ is extendable to one of finite character.
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A subset closed consistency property \( C \) is extendable to one of finite character.
Extendable to finite character

Theorem 13.5

A subset closed consistency property $\mathcal{C}$ is extendable to one of finite character.

(counter case).

$\mathcal{C}^+ := \{S' | \text{all finite subsets of } S' \text{ are in } \mathcal{C}\}$ is consistency property. Let $S' \in \mathcal{C}^+$.

6. case $\delta \in S'$: Consider finite set $T \subseteq S' \cup \{\delta(c)\}$ for some $c \in \text{par}$. Therefore, $(T - \{\delta(c)\}) \subseteq S'$.
Extendable to finite character

Theorem 13.5

A subset closed consistency property $\mathcal{C}$ is extendable to one of finite character.

(counter case).

$\mathcal{C}^+ := \{ S' | \text{all finite subsets of } S' \text{ are in } \mathcal{C} \}$ is consistency property. Let $S' \in \mathcal{C}^+$.

6. case $\delta \in S'$: Consider finite set $T \subseteq S' \cup \{ \delta(c) \}$ for some $c \in \text{par}$. Therefore, $(T - \{ \delta(c) \}) \subseteq S'$.

Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \subseteq S'$. Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \in \mathcal{C}$. 
Theorem 13.5
A subset closed consistency property $\mathcal{C}$ is extendable to one of finite character.

(counter case).

$\mathcal{C}^+ := \{S' \mid \text{all finite subsets of } S' \text{are in } \mathcal{C}\}$ is consistency property. Let $S' \in \mathcal{C}^+$.

6. case $\delta \in S'$: Consider finite set $T \subseteq S' \cup \{\delta(c)\}$ for some $c \in \text{par}$.

Therefore, $(T - \{\delta(c)\}) \subseteq S'$.

Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \subseteq S'$. Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \in \mathcal{C}$. Since $\mathcal{C}$ is consistency property, $\{\delta\} \cup (T - \{\delta(c)\}) \cup \{\delta(c')\} \in \mathcal{C}$.
Extendable to finite character

Theorem 13.5
A subset closed consistency property $C$ is extendable to one of finite character.

(counter case).

$C^+ := \{ S' | \text{all finite subsets of } S' \text{are in } C \}$ is consistency property. Let $S' \in C^+$.

6. case $\delta \in S'$: Consider finite set $T \subseteq S' \cup \{ \delta(c) \}$ for some $c \in \text{par}$. Therefore, $(T - \{\delta(c)\}) \subseteq S'$.

Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \subseteq S'$. Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \in C$.

Since $C$ is consistency property, $\{\delta\} \cup (T - \{\delta(c)\}) \cup \{\delta(c')\} \in C$.

Therefore, $\{\delta\} \cup T \cup \{\delta(c)\} \in C$. 


Extendable to finite character

**Theorem 13.5**

A subset closed consistency property $C$ is extendable to one of finite character.

(counter case).

$C^+ := \{ S' | \text{all finite subsets of } S' \text{ are in } C \}$ is consistency property. Let $S' \in C^+$.

6. case $\delta \in S'$: Consider finite set $T \subseteq S' \cup \{ \delta(c) \}$ for some $c \in \text{par}$. Therefore, $(T - \{ \delta(c) \}) \subseteq S'$.

Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \subseteq S'$. Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \in C$. Since $C$ is consistency property, $\{ \delta \} \cup (T - \{ \delta(c) \}) \cup \{ \delta(c') \} \in C$.

Therefore, $\{ \delta \} \cup T \cup \{ \delta(c) \} \in C$.

Since $C$ is subset closed, $T \in C$. Therefore, $S' \cup \{ \delta(c) \} \in C^+$. 

□
Expanded consistency property

Definition 13.4

Let $S = (F, R)$ be a signature. Let $C$ be a collection of sets of sentences in signature $S_{\text{par}}$. $C$ is a expanded consistency property wrt to $S$ if for each $S \in C$ satisfies the following.

1. for each $F \in A_{S_{\text{par}}}$, either $F \not\in S$ or $\neg F \not\in S$

2. if $\neg\neg F \in S$ then $\{F\} \cup S \in C$

3. if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \cup S \in C$

4. if $\beta \in S$ then $\{\beta_1\} \cup S \in C$ or $\{\beta_2\} \cup S \in C$

5. if $\gamma \in S$ then $\{\gamma(t)\} \cup S \in C$ for each $t \in \hat{T}_{S_{\text{par}}}$

6. if $\delta \in S$ then $\{\delta(c)\} \cup S \in C$ for each $c \in \text{par}$ and not occurring in $S$

7. $S \cup \{t \approx t\} \in C$ for each $t \in \hat{T}_{S_{\text{par}}}$

8. if $t_1 \approx u_1, \ldots, t_n \approx u_n \in S$ then $S \cup \{f(t_1, \ldots, t_n) \approx f(u_1, \ldots, u_n)\} \in C$ for each $f/n \in F$

9. if $t_1 \approx u_1, \ldots, t_n \approx u_n$, $P(t_1, \ldots, t_n) \in H$ then $S \cup \{P(u_1, \ldots, u_n)\} \in C$ for each $P/n \in R \cup \{\approx /2\}$
Converting to extended consistency property

Definition 13.5
A parameter substitution $\pi$ is $\text{par} \rightarrow \text{par}$. Let $F_\pi$ be a formula obtained by replacing parameter $c$ by $\pi(c)$ in $F$ for every $c \in \text{par}$. The substitution naturally extends to a set of formulas.
Converting to extended consistency property

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A parameter substitution $\pi$ is $\text{par} \rightarrow \text{par}$. Let $F_\pi$ be a formula obtained by replacing parameter $c$ by $\pi(c)$ in $F$ for every $c \in \text{par}$. The substitution naturally extends to a set of formulas.

Theorem 13.6
For subset-closed consistency property $C$, let $C^+ := \{S \mid \text{there is } \pi \text{ s.t. } S_\pi \in C\}$.

1. $C^+$ extends $C$ and subset closed
2. $C^+$ is expanded consistency property
Converting to extended consistency property

Definition 13.5
A parameter substitution $\pi$ is $\text{par} \rightarrow \text{par}$. Let $F_\pi$ be a formula obtained by replacing parameter $c$ by $\pi(c)$ in $F$ for every $c \in \text{par}$. The substitution naturally extends to a set of formulas.

Theorem 13.6
For subset-closed consistency property $C$, let $C^+ := \{ S \mid \text{there is } \pi \text{ s.t. } S_\pi \in C \}$.

1. $C^+$ extends $C$ and subset closed
2. $C^+$ is expanded consistency property

Proof.
Part 1 can be easily proved.
Proof for part 2.
For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Converting to extended consistency property (contd.)

Proof for part 2.
For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Consider $S \in C^+$. 

1. Choose closed atom $F$.
   Assume $\{F, \neg F\} \in S$.
   There is a $\pi$ s.t. $S_\pi \in C$.
   Since $\{F_\pi, (\neg F)_\pi\} \subseteq S_\pi$,
   $\{F_\pi, \neg (F_\pi)\} \subseteq S_\pi$.
   Contradiction.

6. case $\delta \in S$:
   Choose $c \in \text{par}$ s.t. $c$ does not occur in $S$.
   Since there is a $\pi$ s.t. $S_\pi \in C$,
   there is a $c' \in \text{par}$ s.t. $S_\pi \cup \{\delta(\pi[c \mapsto c'])\} \in C$.
   Therefore, $S_\pi \cup \{\delta(\pi[c \mapsto c'])\} \in C^+$.
Converting to extended consistency property (contd.)

Proof for part 2.
For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Consider $S \in C^+$.

1. Choose closed atom $F$.
   Assume $\{F, \neg F\} \in S$. 

2. Consider case $\delta \in S$:
   Choose $c \in \text{par}$ s.t. $c$ does not occur in $S$.
   Since there is a $\pi$ s.t. $S\pi \in C$,
   there is a $c' \in \text{par}$ s.t. $S\pi \cup \{\delta\pi(c')\} \in C$.
   Therefore, $(S \cup \{\delta(c)\})(\pi[c \mapsto c']) \in C^+$. 


Proof for part 2.
For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Consider $S \in C^+$. 

1. Choose closed atom $F$.
Assume $\{F, \neg F\} \in S$.
There is a $\pi$ s.t. $S\pi \in C$. 

Proof for part 2.

For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Consider $S \in \mathcal{C}^+$. 

1. Choose closed atom $F$.
   Assume $\{F, \neg F\} \in S$.
   There is a $\pi$ s.t. $S\pi \in \mathcal{C}$.
   Since $\{F\pi, (\neg F)\pi\} \subseteq S\pi$, $\{F\pi, \neg (F\pi)\} \subseteq S\pi$. Contradiction.
Converting to extended consistency property (contd.)

Proof for part 2.

For part 2 we can easily check that conditions 2-5 and 7-8 holds.

Consider $S \in C^+$. 

1. Choose closed atom $F$.
   Assume $\{F, \neg F\} \in S$.
   There is a $\pi$ s.t. $S\pi \in C$.
   Since $\{F\pi, (\neg F)\pi\} \subseteq S\pi$, $\{F\pi, \neg (F\pi)\} \subseteq S\pi$. Contradiction.

6. case $\delta \in S$:
   Choose $c \in \text{par}$ s.t. $c$ does not occur in $S$. 
Converting to extended consistency property (contd.)

Proof for part 2.
For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Consider $S \in \mathcal{C}^+$.

1. Choose closed atom $F$.
   Assume $\{F, \neg F\} \in S$.
   There is a $\pi$ s.t. $S\pi \in \mathcal{C}$.
   Since $\{F\pi, (\neg F)\pi\} \subseteq S\pi$, $\{F\pi, \neg(F\pi)\} \subseteq S\pi$. Contradiction.

6. case $\delta \in S$:
   Choose $c \in \text{par}$ s.t. $c$ does not occur in $S$.
   Since there is a $\pi$ s.t. $S\pi \in \mathcal{C}$, there is a $c' \in \text{par}$ s.t. $S\pi \cup \{\delta\pi(c')\} \in \mathcal{C}$. 
Converting to extended consistency property (contd.)

Proof for part 2.

For part 2 we can easily check that conditions 2-5 and 7-8 holds. Consider $S \in \mathcal{C}^+$. 

1. Choose closed atom $F$.
   
   Assume $\{F, \neg F\} \in S$.
   
   There is a $\pi$ s.t. $S\pi \in \mathcal{C}$. 
   
   Since $\{F\pi, (\neg F)\pi\} \subseteq S\pi$, $\{F\pi, \neg (F\pi)\} \subseteq S\pi$. Contradiction.

6. case $\delta \in S$: 
   
   Choose $c \in \text{par}$ s.t. $c$ does not occur in $S$.
   
   Since there is a $\pi$ s.t. $S\pi \in \mathcal{C}$, there is a $c' \in \text{par}$ s.t. $S\pi \cup \{\delta\pi(c')\} \in \mathcal{C}$. 
   
   Therefore, $S\pi \cup \{\delta(c)(\pi[c \mapsto c'])\} \in \mathcal{C}$. 
Converting to extended consistency property (contd.)

Proof for part 2.

For part 2 we can easily check that conditions 2-5 and 7-8 holds.

Consider $S \in \mathcal{C}^+$. 

1. Choose closed atom $F$.
   Assume $\{F, \neg F\} \in S$.
   There is a $\pi$ s.t. $S\pi \in \mathcal{C}$.
   Since $\{F\pi, (\neg F)\pi\} \subseteq S\pi$, $\{F\pi, \neg(F\pi)\} \subseteq S\pi$. Contradiction.

6. case $\delta \in S$:
   Choose $c \in \text{par}$ s.t. $c$ does not occur in $S$.
   Since there is a $\pi$ s.t. $S\pi \in \mathcal{C}$, there is a $c' \in \text{par}$ s.t. $S\pi \cup \{\delta\pi(c')\} \in \mathcal{C}$. 
   Therefore, $S\pi \cup \{\delta(c)(\pi[c \mapsto c'])\} \in \mathcal{C}$.
   Therefore, $(S \cup \{\delta(c)\})(\pi[c \mapsto c']) \in \mathcal{C}$.
Converting to extended consistency property (contd.)

Proof for part 2.

For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Consider $S \in C^+$. 

1. Choose closed atom $F$.
   Assume $\{F, \neg F\} \in S$.
   There is a $\pi$ s.t. $S\pi \in C$.
   Since $\{F\pi, (\neg F)\pi\} \subseteq S\pi$, $\{F\pi, (\neg (F\pi))\} \subseteq S\pi$. Contradiction.

6. case $\delta \in S$:
   Choose $c \in \text{par}$ s.t. $c$ does not occur in $S$.
   Since there is a $\pi$ s.t. $S\pi \in C$, there is a $c' \in \text{par}$ s.t. $S\pi \cup \{\delta\pi(c')\} \in C$.
   Therefore, $S\pi \cup \{\delta(c)(\pi[c \mapsto c'])\} \in C$.
   Therefore, $(S \cup \{\delta(c)\})(\pi[c \mapsto c']) \in C$.
   Therefore, $(S \cup \{\delta(c)\}) \in C^+$. 

$\square$
Extension to finite character

Theorem 13.7
A subset-closed expanded consistency property \( C \) is extendable to one of finite character.

Proof.
\( C^+ := \{ S' | \text{all finite subsets of } S' \text{ are in } C \} \) is consistency property. Let \( S' \in C^+ \)

6. case \( \delta \in S' \):
   Consider finite set \( T \subseteq S' \cup \{ \delta(c) \} \) for some fresh \( c \in \text{par wrt } S' \).
   Therefore, \( (T - \{\delta(c)\}) \subseteq S' \).
Extension to finite character

Theorem 13.7
A subset-closed expanded consistency property $\mathcal{C}$ is extendable to one of finite character.

Proof.
$\mathcal{C}^+ := \{S' | \text{all finite subsets of } S' \text{ are in } \mathcal{C} \}$ is consistency property. Let $S' \in \mathcal{C}^+$

6. case $\delta \in S'$:
   Consider finite set $T \subseteq S' \cup \{\delta(c)\}$ for some fresh $c \in \text{par}$ wrt $S'$.
   Therefore, $(T - \{\delta(c)\}) \subseteq S'$.
   Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \subseteq S'$. Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \in \mathcal{C}$. 

Exercise 13.1
Prove case 8.
Extension to finite character

Theorem 13.7

A subset-closed expanded consistency property $\mathcal{C}$ is extendable to one of finite character.

Proof.

$\mathcal{C}^+ := \{ S' | \text{all finite subsets of } S' \text{ are in } \mathcal{C} \}$ is consistency property. Let $S' \in \mathcal{C}^+$.

6. case $\delta \in S'$: Consider finite set $T \subseteq S' \cup \{ \delta(c) \}$ for some fresh $c \in \text{par}$ wrt $S'$.
   Therefore, $(T - \{ \delta(c) \}) \subseteq S'$.
   Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \subseteq S'$. Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \in \mathcal{C}$.
   Since $c$ does not occur in $\{ \delta \} \cup (T - \{ \delta(c) \})$ and $\mathcal{C}$ is expanded consistency property, $\{ \delta \} \cup (T - \{ \delta(c) \}) \cup \{ \delta(c) \} \in \mathcal{C}$.

Exercise 13.1

Prove case 8.
Extension to finite character

Theorem 13.7
A subset-closed expanded consistency property $\mathcal{C}$ is extendable to one of finite character.

Proof.
$\mathcal{C}^+ := \{S' | \text{all finite subsets of } S' \text{ are in } \mathcal{C}\}$ is consistency property. Let $S' \in \mathcal{C}^+$

6. case $\delta \in S'$:
Consider finite set $T \subseteq S' \cup \{\delta(c)\}$ for some fresh $c \in \text{par}$ wrt $S'$.
Therefore, $(T - \{\delta(c)\}) \subseteq S'$.
Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \subseteq S'$. Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \in \mathcal{C}$.
Since $c$ does not occur in $\{\delta\} \cup (T - \{\delta(c)\})$ and $\mathcal{C}$ is expanded consistency property, $\{\delta\} \cup (T - \{\delta(c)\}) \cup \{\delta(c)\} \in \mathcal{C}$.
Therefore, $\{\delta\} \cup T \cup \{\delta(c)\} \in \mathcal{C}$. 

Exercise 13.1
Prove case 8.
Extension to finite character

Theorem 13.7

A subset-closed expanded consistency property $C$ is extendable to one of finite character.

Proof.

$C^+ := \{S' \mid \text{all finite subsets of } S' \text{ are in } C\}$ is consistency property. Let $S' \in C^+$

6. case $\delta \in S'$:

Consider finite set $T \subseteq S' \cup \{\delta(c)\}$ for some fresh $c \in \text{par wrt } S'$. Therefore, $(T - \{\delta(c)\}) \subseteq S'$.

Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \subseteq S'$. Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \in C$.

Since $c$ does not occur in $\{\delta\} \cup (T - \{\delta(c)\})$ and $C$ is expanded consistency property, $\{\delta\} \cup (T - \{\delta(c)\}) \cup \{\delta(c)\} \in C$.

Therefore, $\{\delta\} \cup T \cup \{\delta(c)\} \in C$.

Since $C$ is subset closed, $T \in C$. Therefore, $S' \cup \{\delta(c)\} \in C^+$. 

Exercise 13.1

Prove case 8.
Theorem 13.7

A subset-closed expanded consistency property \( \mathcal{C} \) is extendable to one of finite character.

Proof.

\( \mathcal{C}^+ := \{ S' \mid \text{all finite subsets of } S' \text{ are in } \mathcal{C} \} \) is consistency property. Let \( S' \in \mathcal{C}^+ \).

6. case \( \delta \in S' \):

Consider finite set \( T \subseteq S' \cup \{ \delta(c) \} \) for some fresh \( c \in \text{par wrt } S' \).

Therefore, \( (T - \{ \delta(c) \}) \subseteq S' \).

Therefore, \( \{ \delta \} \cup (T - \{ \delta(c) \}) \subseteq S' \). Therefore, \( \{ \delta \} \cup (T - \{ \delta(c) \}) \in \mathcal{C} \).

Since \( c \) does not occur in \( \{ \delta \} \cup (T - \{ \delta(c) \}) \) and \( \mathcal{C} \) is expanded consistency property, \( \{ \delta \} \cup (T - \{ \delta(c) \}) \cup \{ \delta(c) \} \in \mathcal{C} \).

Therefore, \( \{ \delta \} \cup T \cup \{ \delta(c) \} \in \mathcal{C} \).

Since \( \mathcal{C} \) is subset closed, \( T \in \mathcal{C} \). Therefore, \( S' \cup \{ \delta(c) \} \in \mathcal{C}^+ \).

Other cases are similarly proven.
Extension to finite character

Theorem 13.7
A subset-closed expanded consistency property $C$ is extendable to one of finite character.

Proof.
$C^+ := \{ S' \mid \text{all finite subsets of } S' \text{ are in } C \}$ is consistency property. Let $S' \in C^+$

6. case $\delta \in S'$:
Consider finite set $T \subseteq S' \cup \{ \delta(c) \}$ for some fresh $c \in \text{par wrt } S'$.
Therefore, $(T - \{ \delta(c) \}) \subseteq S'$.
Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \subseteq S'$. Therefore, $\{ \delta \} \cup (T - \{ \delta(c) \}) \in C$.
Since $c$ does not occur in $\{ \delta \} \cup (T - \{ \delta(c) \})$ and $C$ is expanded consistency property, $\{ \delta \} \cup (T - \{ \delta(c) \}) \cup \{ \delta(c) \} \in C$.
Therefore, $\{ \delta \} \cup T \cup \{ \delta(c) \} \in C$.
Since $C$ is subset closed, $T \in C$. Therefore, $S' \cup \{ \delta(c) \} \in C^+$.

Other cases are similarly proven.

Exercise 13.1
Prove case 8.
Model existence theorem

Theorem 13.8
Let \( C \) be a consistency property wrt to \( S \). If \( S \in C \) then \( S \) is sat.
Model existence theorem

Theorem 13.8

Let $C$ be a consistency property wrt to $S$. If $S \in C$ then $S$ is sat.

Proof.

Wlog, we assume $C$ is of finite character and expanded (why?).
Model existence theorem

Theorem 13.8

Let $C$ be a consistency property wrt to $S$. If $S \in C$ then $S$ is sat.

Proof.

Wlog, we assume $C$ is of finite character and expanded (why?). Let $F_1, F_2, ..$ be an enumeration of all the sentences of $S^{par}$ in an order (why?).
Model existence theorem

Theorem 13.8
Let $\mathcal{C}$ be a consistency property wrt to $\mathcal{S}$. If $S \in \mathcal{C}$ then $S$ is sat.

Proof.
Wlog, we assume $\mathcal{C}$ is of finite character and expanded (why?). Let $F_1, F_2, \ldots$ be an enumeration of all the sentences of $\mathcal{S}^{\text{par}}$ in an order (why?).

Let us define a sequence $S_1, S_2, \ldots$ as follows.

$s_1 = S$

$s_{n+1} = \begin{cases} 
S_n \cup \{F_n, \delta(c)\} & S_n \cup \{F_n\} \in \mathcal{C} \text{ and } F_n = \delta \\
S_n \cup \{F_n\} & S_n \cup \{F_n\} \in \mathcal{C} \text{ and } F_n \neq \delta \\
S_n & \text{otherwise}
\end{cases}
$

where $c$ is a fresh parameter wrt $S_n \cup \{F_n\}$. 
Model existence theorem

Theorem 13.8
Let $C$ be a consistency property wrt to $S$. If $S \in C$ then $S$ is sat.

Proof.
Wlog, we assume $C$ is of finite character and expanded (why?). Let $F_1, F_2, \ldots$ be an enumeration of all the sentences of $\text{S}^{\text{par}}$ in an order (why?).

Let us define a sequence $S_1, S_2, \ldots$ as follows.

$$S_1 = S \quad S_{n+1} = \begin{cases} S_n \cup \{F_n, \delta(c)\} & S_n \cup \{F_n\} \in C \text{ and } F_n = \delta \\ S_n \cup \{F_n\} & S_n \cup \{F_n\} \in C \text{ and } F_n \neq \delta \\ S_n & \text{otherwise} \end{cases}$$

where $c$ is a fresh parameter wrt $S_n \cup \{F_n\}$.

Since $S_n$ are in $C$ and $C$ is of finite character, $\bigcup_n S_n \in C$. Let $M := \bigcup_n S_n$. 
Model existence theorem (contd.)

Proof.

Claim: $M$ is maximal in $C$. (same argument as in propositional logic)
Model existence theorem (contd.)

Proof.

Claim: $M$ is maximal in $C$. (same argument as in propositional logic)
Assume $M' \in C$ s.t. $M \subset M'$. There is $F_n$ such that $F_n \in M'$ and $F_n \notin M$.

Exercise 13.2 Prove $\delta$ case to show that $M$ is a Hinttika set.
Model existence theorem (contd.)

Proof.

Claim: \( M \) is maximal in \( \mathcal{C} \). (same argument as in propositional logic)

Assume \( M' \in \mathcal{C} \) s.t. \( M \subset M' \). There is \( F_n \) such that \( F_n \in M' \) and \( F_n \notin M \).

By def. of \( M \), \( S_n \cup \{F_n\} \notin \mathcal{C} \).
Model existence theorem (contd.)

Proof.

Claim: $M$ is maximal in $\mathcal{C}$. (same argument as in propositional logic)

Assume $M' \in \mathcal{C}$ s.t. $M \subseteq M'$. There is $F_n$ such that $F_n \in M'$ and $F_n \notin M$.

By def. of $M$, $S_n \cup \{F_n\} \notin \mathcal{C}$.

Since $S_n \cup \{F_n\} \subseteq M'$ and $\mathcal{C}$ is subset closed, $S_n \cup \{F_n\} \in \mathcal{C}$. Contradiction.
Proof.

**Claim:** \( M \) is maximal in \( \mathcal{C} \). (same argument as in propositional logic)

Assume \( M' \in \mathcal{C} \) s.t. \( M \subset M' \). There is \( F_n \) such that \( F_n \in M' \) and \( F_n \notin M \).

By def. of \( M \), \( S_n \cup \{F_n\} \notin \mathcal{C} \).

Since \( S_n \cup \{F_n\} \subseteq M' \) and \( \mathcal{C} \) is subset closed, \( S_n \cup \{F_n\} \in \mathcal{C} \). **Contradiction.**

**Claim:** \( M \) is a Hinttika set.
Model existence theorem (contd.)

Proof.

Claim: $M$ is maximal in $\mathcal{C}$. (same argument as in propositional logic)
Assume $M' \in \mathcal{C}$ s.t. $M \subset M'$. There is $F_n$ such that $F_n \in M'$ and $F_n \notin M$.
By def. of $M$, $S_n \cup \{F_n\} \notin \mathcal{C}$.
Since $S_n \cup \{F_n\} \subset M'$ and $\mathcal{C}$ is subset closed, $S_n \cup \{F_n\} \in \mathcal{C}$. Contradiction.

Claim: $M$ is a Hinttika set.
If $\alpha \in M$ then $\{\alpha_1, \alpha_2\} \cup M \in \mathcal{C}$. Since $M$ is maximal, $\{\alpha_1, \alpha_2\} \subset M$. 
Proof.

**Claim:** $M$ is maximal in $\mathcal{C}$. (same argument as in propositional logic)

Assume $M' \in \mathcal{C}$ s.t. $M \subset M'$. There is $F_n$ such that $F_n \in M'$ and $F_n \notin M$.

By def. of $M$, $S_n \cup \{F_n\} \notin \mathcal{C}$.

Since $S_n \cup \{F_n\} \subseteq M'$ and $\mathcal{C}$ is subset closed, $S_n \cup \{F_n\} \in \mathcal{C}$. **Contradiction.**

**Claim:** $M$ is a Hinttika set.

If $\alpha \in M$ then $\{\alpha_1, \alpha_2\} \cup M \in \mathcal{C}$. Since $M$ is maximal, $\{\alpha_1, \alpha_2\} \subseteq M$.

Other conditions hold similarly, except $\delta$ case.
Model existence theorem (contd.)

Proof.

Claim: $M$ is maximal in $C$. (same argument as in propositional logic)
Assume $M' \in C$ s.t. $M \subset M'$. There is $F_n$ such that $F_n \in M'$ and $F_n \notin M$.
By def. of $M$, $S_n \cup \{F_n\} \notin C$.
Since $S_n \cup \{F_n\} \subseteq M'$ and $C$ is subset closed, $S_n \cup \{F_n\} \in C$. Contradiction.

Claim: $M$ is a Hinttika set.
If $\alpha \in M$ then $\{\alpha_1, \alpha_2\} \cup M \in C$. Since $M$ is maximal, $\{\alpha_1, \alpha_2\} \subseteq M$.
Other conditions hold similarly, except $\delta$ case.

Since $M$ is a Hinttika set, $M$ is sat. Since $S \subseteq M$, $S$ is sat.

Exercise 13.2
Prove $\delta$ case to show that $M$ is a Hinttika set.
Topic 13.2

Consequences of model existence theorem
Compactness

Theorem 13.9

Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If each finite subset of $S$ is sat then $S$ is sat.

Proof.

Let $C := \{S' \subseteq \text{Sp}ar$-sentences | all finite subsets of $S'$ are sat and there are infinitely many parameters in $\text{par}$ that do not occur in $S'$\}.
Compactness

Theorem 13.9
Let \( S = (F, R) \) be a signature and \( S \) be a set of \( S \)-sentences. If each finite subset of \( S \) is sat then \( S \) is sat.

Proof.
Let \( C := \{ S' \subseteq \text{Spar}-\text{sentences} \mid \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \} \).

Claim: \( C \) is a consistency property.
Compactness

Theorem 13.9
Let \( S = (F, R) \) be a signature and \( S \) be a set of \( S \)-sentences. If each finite subset of \( S \) is sat then \( S \) is sat.

Proof.
Let \( C := \{ S' \subset S^{\text{par}} \text{-sentences} \mid \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \} \).

Claim: \( C \) is a consistency property.
Let \( S' \in C \). We need to satisfy the nine conditions.
Compactness

Theorem 13.9

Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If each finite subset of $S$ is sat then $S$ is sat.

Proof.

Let $C := \{ S' \subset \text{Sparsentences} | \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \}.$

Claim: $C$ is a consistency property.

Let $S' \in C$. We need to satisfy the nine conditions.

1. If $\{ F, \neg F \} \subset S'$, then $\{ F, \neg F \}$ is sat. contradiction. First cond. holds.
Compactness

Theorem 13.9

Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If each finite subset of $S$ is sat then $S$ is sat.

Proof.

Let $C := \{ S' \subset S_{\text{par}} \text{-sentences}| \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \}$. 

Claim: $C$ is a consistency property.

Let $S' \in C$. We need to satisfy the nine conditions.

1. If $\{F, \neg F\} \subseteq S'$, then $\{F, \neg F\}$ is sat. contradiction. First cond. holds.

3. Let $\alpha \in S'$. Consider a finite $T \subseteq \{\alpha_1, \alpha_2\} \cup S'$. 


Compactness

Theorem 13.9
Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences.
If each finite subset of $S$ is sat then $S$ is sat.

Proof.
Let $C := \{ S' \subseteq S^{\text{par}} \mid \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \}$.

Claim: $C$ is a consistency property.
Let $S' \in C$. We need to satisfy the nine conditions.

1. If $\{ F, \neg F \} \subseteq S'$, then $\{ F, \neg F \}$ is sat. contradiction. First cond. holds.

3. Let $\alpha \in S'$. Consider a finite $T \subseteq \{ \alpha_1, \alpha_2 \} \cup S'$.
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{ \alpha, \alpha_1, \alpha_2 \} \cup T'$.
Compactness

Theorem 13.9
Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences.
If each finite subset of $S$ is sat then $S$ is sat.

Proof.
Let $C := \{S' \subseteq S^{\text{par}} \text{-sentences}\mid \text{all finite subsets of } S' \text{ are sat and there are}
\text{infinitely many parameters in } \text{par} \text{ that do not occur in } S'\}.$

Claim: $C$ is a consistency property.
Let $S' \in C$. We need to satisfy the nine conditions.

1. If $\{F, \neg F\} \subseteq S'$, then $\{F, \neg F\}$ is sat. contradiction. First cond. holds.

3. Let $\alpha \in S'$. Consider a finite $T \subseteq \{\alpha_1, \alpha_2\} \cup S'$.
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\alpha, \alpha_1, \alpha_2\} \cup T'$.
   Since $T' \cup \{\alpha\} \subseteq S'$, $T' \cup \{\alpha\}$ is sat.
Compactness

Theorem 13.9

Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If each finite subset of $S$ is sat then $S$ is sat.

Proof.

Let $C := \{ S' \in S^\text{par}-\text{sentences} \mid \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \}$. 

Claim: $C$ is a consistency property.

Let $S' \in C$. We need to satisfy the nine conditions.

1. If $\{F, \neg F\} \subseteq S'$, then $\{F, \neg F\}$ is sat. contradiction. First cond. holds.

3. Let $\alpha \in S'$. Consider a finite $T \subseteq \{\alpha_1, \alpha_2\} \cup S'$.
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\alpha, \alpha_1, \alpha_2\} \cup T'$.
   Since $T' \cup \{\alpha\} \subseteq S'$, $T' \cup \{\alpha\}$ is sat.
   Therefore, $T' \cup \{\alpha, \alpha_1, \alpha_2\}$ is sat.
Compactness

Theorem 13.9
Let $S = (\mathbf{F}, \mathbf{R})$ be a signature and $S$ be a set of $\mathbf{S}$-sentences.
If each finite subset of $S$ is sat then $S$ is sat.

Proof.
Let $C := \{ S' \subseteq \mathbf{S}^{\text{par}} \text{-sentences} | \text{all finite subsets of } S' \text{ are sat and there are} \text{ infinitely many parameters in } \text{par} \text{ that do not occur in } S' \}.$

Claim: $C$ is a consistency property.
Let $S' \in C$. We need to satisfy the nine conditions.

1. If $\{ F, \neg F \} \subseteq S'$, then $\{ F, \neg F \}$ is sat. contradiction. First cond. holds.

3. Let $\alpha \in S'$. Consider a finite $T \subseteq \{ \alpha_1, \alpha_2 \} \cup S'$.
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{ \alpha, \alpha_1, \alpha_2 \} \cup T'$.
   Since $T' \cup \{ \alpha \} \subseteq S'$, $T' \cup \{ \alpha \}$ is sat.
   Therefore, $T' \cup \{ \alpha, \alpha_1, \alpha_2 \}$ is sat.
   Therefore, $T$ is sat.
Compactness

Theorem 13.9
Let \( S = (F, R) \) be a signature and \( S \) be a set of \( S \)-sentences.
If each finite subset of \( S \) is sat then \( S \) is sat.

Proof.
Let \( C := \{ S' \subset \mathbf{S}^\text{par} \mid \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \} \).

Claim: \( C \) is a consistency property.
Let \( S' \in C \). We need to satisfy the nine conditions.

1. If \( \{ F, \neg F \} \subseteq S' \), then \( \{ F, \neg F \} \) is sat. contradiction. First cond. holds.

3. Let \( \alpha \in S' \). Consider a finite \( T \subseteq \{ \alpha_1, \alpha_2 \} \cup S' \).
   There is a finite \( T' \subseteq S' \) s.t. \( T \subseteq \{ \alpha, \alpha_1, \alpha_2 \} \cup T' \).
   Since \( T' \cup \{ \alpha \} \subseteq S' \), \( T' \cup \{ \alpha \} \) is sat.
   Therefore, \( T' \cup \{ \alpha, \alpha_1, \alpha_2 \} \) is sat.
   Therefore, \( T \) is sat.
   Therefore, every finite subset of \( \{ \alpha_1, \alpha_2 \} \cup S' \) is sat.
Compactness

Theorem 13.9
Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If each finite subset of $S$ is sat then $S$ is sat.

Proof.
Let $C := \{ S' \subset S^{\text{par}} \text{-sentences} \mid \text{all finite subsets of } S' \text{ are sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \}$. 

Claim: $C$ is a consistency property.
Let $S' \in C$. We need to satisfy the nine conditions.

1. If $\{ F, \neg F \} \subseteq S'$, then $\{ F, \neg F \}$ is sat. contradiction. First cond. holds.

3. Let $\alpha \in S'$. Consider a finite $T \subseteq \{ \alpha_1, \alpha_2 \} \cup S'$. There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{ \alpha, \alpha_1, \alpha_2 \} \cup T'$.
Since $T' \cup \{ \alpha \} \subseteq S'$, $T' \cup \{ \alpha \}$ is sat.
Therefore, $T' \cup \{ \alpha, \alpha_1, \alpha_2 \}$ is sat.
Therefore, $T$ is sat.
Therefore, every finite subset of $\{ \alpha_1, \alpha_2 \} \cup S'$ is sat.
Therefore, $\{ \alpha_1, \alpha_2 \} \cup S' \in C$
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Due to model existence theorem, $\mathcal{S}$ is sat.

Exercise 13.4

If $\Sigma | = F$ then there is a finite subset $\mathcal{S}$ of $\Sigma$ such that $\mathcal{S} | = F$. 

... similarly other cases are proven.
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$.
   Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?).
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$.

Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?). There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.
Compactness (contd.)

Exercise 13.3

Prove the \( \delta \) case.

Proof (contd.)

6. Let \( \delta \in S' \).

Consider a finite \( T \subseteq \{ \delta(c) \} \cup S' \) for fresh \( c \in \text{par} \) (why possible?).

There is a finite \( T' \subseteq S' \) s.t. \( T \subseteq \{ \delta, \delta(c) \} \cup T' \).

Since \( T' \cup \{ \delta \} \subseteq S' \), \( T' \cup \{ \delta \} \) is sat.
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$.
   Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?).
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.
   Since $T' \cup \{\delta\} \subseteq S'$, $T' \cup \{\delta\}$ is sat.
   Therefore, $T' \cup \{\delta, \delta(c)\}$ is sat.
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$.
   Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?). There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.
   Since $T' \cup \{\delta\} \subseteq S'$, $T' \cup \{\delta\}$ is sat.
   Therefore, $T' \cup \{\delta, \delta(c)\}$ is sat.
   Therefore, $T$ is sat.
Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$. 

Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?). 

There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.  

Since $T' \cup \{\delta\} \subseteq S'$, $T' \cup \{\delta\}$ is sat.  

Therefore, $T' \cup \{\delta, \delta(c)\}$ is sat.  

Therefore, $T$ is sat.  

Therefore, every finite subset of $\{\delta(c)\} \cup S'$ is sat.
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$.

   Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?).
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.
   Since $T' \cup \{\delta\} \subseteq S'$, $T' \cup \{\delta\}$ is sat.
   Therefore, $T' \cup \{\delta, \delta(c)\}$ is sat.
   Therefore, $T$ is sat.
   Therefore, every finite subset of $\{\delta(c)\} \cup S'$ is sat.
   Therefore, $\{\alpha_1, \alpha_2\} \cup S' \in C$

7. .... similarly other cases are proven.
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$.

Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?).
There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.
Since $T' \cup \{\delta\} \subseteq S'$, $T' \cup \{\delta\}$ is sat.
Therefore, $T' \cup \{\delta, \delta(c)\}$ is sat.
Therefore, $T$ is sat.
Therefore, every finite subset of $\{\delta(c)\} \cup S'$ is sat.
Therefore, $\{\alpha_1, \alpha_2\} \cup S' \in C$

7. .... similarly other cases are proven.

Due to model existence theorem, $S$ is sat.

\[\square\]
Compactness (contd.)

Exercise 13.3

Prove the $\delta$ case.

Proof (contd.)

6. Let $\delta \in S'$.
   Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in \text{par}$ (why possible?).
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.
   Since $T' \cup \{\delta\} \subseteq S'$, $T' \cup \{\delta\}$ is sat.
   Therefore, $T' \cup \{\delta, \delta(c)\}$ is sat.
   Therefore, $T$ is sat.
   Therefore, every finite subset of $\{\delta(c)\} \cup S'$ is sat.
   Therefore, $\{\alpha_1, \alpha_2\} \cup S' \in \mathbb{C}$

7. .... similarly other cases are proven.

Due to model existence theorem, $S$ is sat.

Exercise 13.4

If $\Sigma \models F$ then there is a finite subset $S$ of $\Sigma$ such that $S \models F$
Impossibility of encoding finite models

Theorem 13.10

Let $\mathcal{S} = (F, R)$ be a signature and $S$ be a set of $\mathcal{S}$-sentences. If $S$ is sat in arbitrary large finite models then $S$ is true in an infinite model.
Theorem 13.10

Let \( S = (F, R) \) be a signature and \( S \) be a set of \( S \)-sentences. If \( S \) is sat in arbitrary large finite models then \( S \) is true in an infinite model.

Proof.

Let \( E/2 \) be a predicate symbol that is not in \( S \). Let \( S' = (F, R \cup \{E/2\}) \).
Impossibility of encoding finite models

Theorem 13.10
Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If $S$ is sat in arbitrary large finite models then $S$ is true in an infinite model.

Proof.
Let $E/2$ be a predicate symbol that is not in $S$. Let $S' = (F, R \cup \{E/2\})$.

As we have seen, let $F_i$ be a $S'$-sentence only using predicate $E$ that is false in models with domain smaller than $i$, and sometimes true in larger models.
Impossibility of encoding finite models

Theorem 13.10
Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If $S$ is sat in arbitrary large finite models then $S$ is true in an infinite model.

Proof.
Let $E/2$ be a predicate symbol that is not in $S$. Let $S' = (F, R \cup \{E/2\})$.

As we have seen, let $F_i$ be a $S'$-sentence only using predicate $E$ that is false in models with domain smaller than $i$, and sometimes true in larger models.

Let $S' = S \cup \{F_1, F_2, F_3, \ldots \}$. 
Impossibility of encoding finite models

Theorem 13.10

Let $S = (F, R)$ be a signature and $S$ be a set of $S$-sentences. If $S$ is sat in arbitrary large finite models then $S$ is true in an infinite model.

Proof.

Let $E/2$ be a predicate symbol that is not in $S$. Let $S' = (F, R \cup \{E/2\})$. As we have seen,

let $F_i$ be a $S'$-sentence only using predicate $E$ that is false in models with domain smaller than $i$, and sometimes true in larger models.

Let $S' = S \cup \{F_1, F_2, F_3, \ldots \}$.

By construction, $S'$ cannot be satisfied by a finite model.
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**claim:** $S'$ is sat.

Let $L$ be a finite subset of $S'$. Let $k$ be the largest number s.t. $F_k \in L$. 

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By construction, $S'$ cannot be satisfied by a finite model.

claim: $S'$ is sat.

Let $L$ be a finite subset of $S'$. Let $k$ be the largest number s.t. $F_k \in L$.

Since $S$ is sat in arbitrary large finite models and $S$ does not mention $E$, $L$ is sat in a model larger than $k$. 
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By construction, $S'$ cannot be satisfied by a finite model.

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Due to compactness, $S'$ is sat.
Impossibility of encoding finite models

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let \( F_i \) be a \( S' \)-sentence only using predicate \( E \) that is false in models with domain smaller than \( i \), and sometimes true in larger models.
Let \( S' = S \cup \{F_1, F_2, F_3, \ldots \} \).
By construction, \( S' \) cannot be satisfied by a finite model.

claim: \( S' \) is sat.
Let \( L \) be a finite subset of \( S' \). Let \( k \) be the largest number s.t. \( F_k \in L \).
Since \( S \) is sat in arbitrary large finite models and \( S \) does not mention \( E \), \( L \) is sat in a model larger than \( k \).
Due to compactness, \( S' \) is sat.

Therefore, \( S' \) has infinite model.
Löwenheim-Skolem Theorem

Theorem 13.11

Let $S = (F, R)$ be a countable signature and $S$ be a set of $S$-sentences. If $S$ is sat then $S$ is true in a countable model.

Proof.

Let $C := \{ S' \subset S^{par} \, | \, S'$ is sat and there are infinitely many parameters in par that do not occur in $S' \}$. 
Löwenheim-Skolem Theorem

Theorem 13.11

Let $S = (F, R)$ be a countable signature and $S$ be a set of $S$-sentences. If $S$ is sat then $S$ is true in a countable model.

Proof.

Let $C := \{ S' \subset S^{\text{par}} \mid S' \text{ is sat and there are infinitely many parameters in } \text{par} \text{ that do not occur in } S' \}$. We can easily show $C$ is a consistency property.
Löwenheim-Skolem Theorem

Theorem 13.11
Let $S = (F, R)$ be a countable signature and $S$ be a set of $S$-sentences. If $S$ is sat then $S$ is true in a countable model.

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We can easily show $C$ is a consistency property.

Since $S \in C$, we can construct a Herbrand model of $S$ wrt $S^{\text{par}}$ that is countable.
Löwenheim-Skolem Theorem

Theorem 13.11

Let $S = (F, R)$ be a countable signature and $S$ be a set of $S$-sentences. If $S$ is sat then $S$ is true in a countable model.

Proof.

Let $\mathcal{C} := \{ S' \subseteq S^{\text{par}} \text{-sentences} | S'$ is sat and there are infinitely many parameters in $\text{par}$ that do not occur in $S'$ $\}$.

We can easily show $\mathcal{C}$ is a consistency property.

Since $S \in \mathcal{C}$, we can construct a Herbrand model of $S$ wrt $S^{\text{par}}$ that is countable.

Remark:

For every satisfiable set of first order sentences, we have a countable model therefore real numbers can not be axiomatized using formulas in FOL.

Actually the story is more complicated. Check out “skolem’s paradox”!

End of Lecture 13