Mathematical Logic 2016

Lecture 6: Soundness and Completeness of Tableaux and Resolution

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Where are we and where are we going?

We have seen

- propositional logic syntax and semantics
- normal forms
- proof methods tableaux and resolution

We will see

- soundness and completeness of the proof methods

Topic 6.1

Soundness

Why soundness theorems?

We need to show that

if

our proof method proves a theorem

then

it is a valid formula in the logic.

Structural induction in uniform notation

Since the uniform notation absorbs single negations, the original structural induction is not immediately applicable. We need the following theorem.

Theorem 6.1

Every propositional formula has a property Q if*

- *Base case: every atomic formula and its negation have property Q*
- *induction steps: if F has property Q so does $\neg\neg X$,*
  if α_1 and α_2 have property Q so does α, and
  if β_1 or β_2 have property Q so does β.

Proof Hint.

Induction hyp: F and $\neg F$ has property Q. Now apply the original structural induction to complete this proof. □

Exercise 6.1

Complete the above proof.

Note: *Technically, only those formulas that do not contain $\bot$, $\top$,$\oplus$ and $\Leftrightarrow$.

Satisfiable tableaux/resolution derivation

Definition 6.1 (Recall)

A set of formulas Σ is sat *if there is a model m s.t. for each $F \in \Sigma$, $m \models F$. We write $m \models \Sigma$.*

Definition 6.2

A branch ρ of a tableaux is sat *if the set of formulas that are labels of the nodes of the branch are satisfiable. If the model involved is m, we write $m \models \rho$.*

Definition 6.3

A tableaux T is sat *if there is a satisfiable branch in T. If the model involved is m, we write $m \models T$.*

Definition 6.4

A resolution derivation R is sat *if the set of clauses in R is sat. If the model involved is m, we write $m \models R$.*

Tableaux expansion preserves satisfiability

Theorem 6.2

If Σ is sat then a tableaux T for Σ is sat

Proof.

Let model $m \models \Sigma$.

base case: empty tableaux satisfies any model.

induction step: Assume $m \models T$. Let ρ be a branch of T s.t. $m \models \rho$.

Let T' be a tableaux obtained after application of an expansion rule.

- **case** ρ is not expanded in T': ρ is a branch of T' and $m \models T'$.
- **case** ρ is expanded using $F \in \Sigma$: $m \models F$, $m \models \rho F$, and $m \models T'$.
- **case** ρ is expanded using $F \in \rho$: Therefore, $m \models F$.
  case $F = \beta$: ρ is expanded into two branches $\rho\beta_1$ and $\rho\beta_2$. Due to semantics of β, $m \models \beta_1$ or $m \models \beta_2$. Therefore, $m \models \rho\beta_1$ or $m \models \rho\beta_2$.
  case $F = \alpha$: .... case $F = \neg\neg G$: .... $\square$

Exercise 6.2

a. Complete the above proof

b. Prove if Σ is sat then a resolution derivation R for Σ is sat

Tableaux method is sound

Theorem 6.3
If $\vdash_{pt} F$ *then* $\models F$.

Proof.
Let us suppose $\models F$ does not hold.
Therefore, for some model m, $m \models \neg F$.
Therefore, there is no closed tableaux for $\{\neg F\}$.
Therefore, $\vdash_{pt} F$ does not hold. $\square$

Exercise 6.3
Prove if $\vdash_{pr} F$ *then* $\models F$.

Topic 6.2

Completeness

Completeness

The completeness property says that

if

there is a valid formula in the logic

then

there exists a proof in the proof system

Stronger claim: finding the proof is decidable/semi-decidable

First we will see a general technique to prove completeness.

Topic 6.3

Model existence theorem

Hintikka set

Definition 6.5

A set M of formulas is called Hintikka set if

1. *for each $p \in \textbf{Vars}$, not both $p \in M$ and $\neg p \in M$*
2. *if $\neg\neg F \in M$ then $F \in M$*
3. *if $\alpha \in M$ then $\alpha_1 \in M$ and $\alpha_2 \in M$*
4. *if $\beta \in M$ then $\beta_1 \in M$ or $\beta_2 \in M$*

Due to 2-4, if $F \in M$ then some formulas in $sub(F)$ must be in M. Hintikka sets are the result of a downward saturation.

Example 6.1

$\{(p \wedge (\neg q \Rightarrow \neg p)), p, (\neg q \Rightarrow \neg p)\}$ *is not a Hintikka set*
$\{(p \wedge (\neg q \Rightarrow \neg p)), p, (\neg q \Rightarrow \neg p), \neg\neg q, q\}$ *is a Hintikka set*
$\{(p \wedge (\neg q \Rightarrow \neg p)), p, (\neg q \Rightarrow \neg p), \neg p\}$ *is not a Hintikka set*

Exercise 6.4

Extend the following sets into Hintikka sets

- $\{(p \vee q), (\neg p \wedge \neg q)\}$
- $\{\neg(p \Rightarrow (q \Rightarrow p))\}$
- $\{\neg(\neg r \vee (r \Rightarrow s)) \vee (q \wedge (r \Rightarrow s))\}$

Hintikka's Theorem

Theorem 6.4

Every Hintikka set M is sat

Proof.

Hintikka sets explicate what syntax can possibly say about semantics!!

We construct a model m s.t. $m \models M$.
For each $p \in \mathbf{Vars}$,

1. if $p \in M$ then $m(p) := 1$,
2. if $\neg p \in M$ then $m(p) := 0$, and
3. assign $m(p)$ any value otherwise.

By the new structural induction we will show that for each $F \in M$, $m \models F$.
base: trivially due to the assignment
step: $F \in M$

- case $F = \neg\neg H$: Since $H \in M$, $m \models H$. Therefore $m \models \neg\neg H$
- case $F = \alpha$: Since $\alpha_1, \alpha_2 \in M$, $m \models \alpha_1$ and $m \models \alpha_2$.Therefore, $m \models \alpha$.
- case $F = \beta$: .... $\square$

Exercise 6.5

Show for a Hintikka set M, for each F either $F \notin M$ or $\neg F \notin M$.

Consistency property

Definition 6.6

Let $\mathcal{C}$ be a collection of sets of formulas. $\mathcal{C}$ is a consistency property if each $S \in \mathcal{C}$ satisfies the following.

1. *for each $p \in$ **Vars**, either $p \notin S$ or $\neg p \notin S$*
2. *if $\neg\neg F \in S$ then $\{F\} \cup S \in \mathcal{C}$*
3. *if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \cup S \in \mathcal{C}$*
4. *if $\beta \in S$ then $\{\beta_1\} \cup S \in \mathcal{C}$ or $\{\beta_2\} \cup S \in \mathcal{C}$*

Note that the above definition defines a collection of sets
The definition reads like Hinttika set but it is not.

Subset closed consistency property

Theorem 6.5

Every consistency property $\mathcal{C}$ can be extended to a consistency property that is subset closed.

Proof.

Let $\mathcal{C}^+ \triangleq \{S' | S' \subseteq S \text{ and } S \in \mathcal{C}\}$. We show $\mathcal{C}^+$ is consistency property.
Consider $S' \in \mathcal{C}^+$. By definition, there is $S \in \mathcal{C}$ s.t. $S' \subseteq S$.

1. Therefore, S' does not contain contradictory literals.
2. If $\neg\neg F \in S'$. Therefore, $\neg\neg F \in S$. Therefore, $\{F\} \cup S \in \mathcal{C}$. Therefore, $\{F\} \cup S' \in \mathcal{C}^+$.
3. ....

□

Exercise 6.6

Complete the above argument

Finite character

Definition 6.7

A consistency property $\mathcal{C}$ has finite character if

$$S \in \mathcal{C} \quad \textit{iff} \quad \textit{every finite subset of } S \textit{ is in } \mathcal{C}.$$

Theorem 6.6

if $\mathcal{C}$ is of finite character then $\mathcal{C}$ is subset closed.

Exercise 6.7

Prove the above theorem.

Commentary: Please note the peculiar use of "iff" in the above definition.

Extendable to finite character

Theorem 6.7

A subset closed consistency property $\mathcal{C}$ is extendable to one of finite character.

Proof.

claim: $\mathcal{C}^+ \triangleq \{S' |$ all finite subsets of S' are in $\mathcal{C}\}$ is consistency property.

Let $S' \in \mathcal{C}^+$. We have four conditions to satisfy.

1. If $\{p, \neg p\} \subseteq S'$, then $\{p, \neg p\} \in \mathcal{C}$. contradiction. First cond. holds.
2. case $\neg\neg F \in S'$: Consider finite set $T \subseteq S' \cup \{F\}$.
   Therefore, $(T - \{F\}) \subseteq S'$.
   Therefore, $\{\neg\neg F\} \cup (T - \{F\}) \subseteq S'$.
   Therefore, $\{\neg\neg F\} \cup (T - \{F\}) \in \mathcal{C}$.
   Since $\mathcal{C}$ is consistency property, $\{\neg\neg F\} \cup (T - \{F\}) \cup \{F\} \in \mathcal{C}$.
   Therefore, $\{\neg\neg F\} \cup T \cup \{F\} \in \mathcal{C}$.
   Since $\mathcal{C}$ is subset closed, $T \in \mathcal{C}$.
   Therefore, $S' \cup \{F\} \in \mathcal{C}^+$.
3. ....

□

Exercise 6.8 *Write α and β cases.*

Limits in finite character

Theorem 6.8

Let consistency property $\mathcal{C}$ is of finite character. If $S_1, S_2, \dots$ is a sequence of members of $\mathcal{C}$ such that $S_1 \subseteq S_2 \subseteq \dots$. Then, $\bigcup_i S_i \in \mathcal{C}$.

Proof.

Consider finite set $\{F_1, \dots, F_k\} \subseteq \bigcup_i S_i$.
Let n_j be the smallest number s.t. $F_j \in S_{n_j}$.
Let $n = max(n_1, \dots, n_k)$.
Therefore, $\{F_1, \dots, F_k\} \subseteq S_n$
Since $\mathcal{C}$ is subset closed, $\{F_1, \dots, F_k\} \in \mathcal{C}$
Since $\mathcal{C}$ is of finite character, $\bigcup_i S_i \in \mathcal{C}$ $\square$

Model existence theorem

Theorem 6.9

Let $\mathcal{C}$ be a consistency property. If $S \in \mathcal{C}$, S is sat.

Proof.

Wlog, we assume $\mathcal{C}$ is of finite character(why?).

Let $F_1, F_2, \dots$ be enumeration of all the formulas in some order(why?).

Let us define a sequence $S_1, S_2, \dots$ as follows.

$$S_1 \triangleq S \qquad S_{n+1} \triangleq \begin{cases} S_n \cup \{F_n\} & S_n \cup \{F_n\} \in \mathcal{C} \\ S_n & \text{otherwise} \end{cases}$$

Since S_n are in $\mathcal{C}$ and $\mathcal{C}$ is of finite character, $\bigcup_n S_n \in \mathcal{C}$.

Let $M \triangleq \bigcup_n S_n$. ...

Model existence theorem (contd. I)

Proof(contd.)

claim: M is maximal in $\mathcal{C}$.

Assume M is not maximal and there is $M' \in \mathcal{C}$ such that $M \subset M'$.

There is F_n such that $F_n \in M'$ and $F_n \notin M$.

By def. of M, $S_n \cup \{F_n\} \notin \mathcal{C}$.

Since $S_n \cup \{F_n\} \subseteq M'$ and $\mathcal{C}$ is subset closed, $S_n \cup \{F_n\} \in \mathcal{C}$. Contradiction.

...

Model existence theorem (contd. II)

Proof(contd.)

claim: M is a Hinttika set.

If $\alpha \in M$ then $\{\alpha_1, \alpha_2\} \cup M \in \mathcal{C}$. Since M is maximal, $\{\alpha_1, \alpha_2\} \subseteq M$.

Other conditions hold similarly.

Since M is a Hinttika set, M is sat. Since $S \subseteq M$, S is sat. $\square$

Compactness theorem

Theorem 6.10

For a set of formulas S, if every finite subset of S is sat, then S is sat

Proof.

Let $\mathcal{C} \triangleq \{S' | \text{all finite subsets of } S' \text{ are sat }\}$.
claim: $\mathcal{C}$ is a consistency property.
Let $S' \in \mathcal{C}$. We need to satisfy the four conditions.

1. If $\{p, \neg p\} \subseteq S'$, then $\{p, \neg p\}$ is sat. contradiction. First cond. holds.
2. Let $\alpha \in S'$.
   We need to show that every finite subset of $\{\alpha_1, \alpha_2\} \cup S'$ is sat(why?).

...

Compactness theorem(contd.)

Evidence of unsatisfiablity is always a finite subset.

Proof(contd.)

Consider a finite $T \subseteq \{\alpha_1, \alpha_2\} \cup S'$.
There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\alpha, \alpha_1, \alpha_2\} \cup T'$.
Since $T' \cup \{\alpha\} \subseteq S'$, $T' \cup \{\alpha\}$ is sat.
Therefore, $T' \cup \{\alpha, \alpha_1, \alpha_2\}$ is sat.
Therefore, T is sat.
Therefore, every finite subset of $\{\alpha_1, \alpha_2\} \cup S'$ is sat.
Therefore, $\{\alpha_1, \alpha_2\} \cup S' \in \mathcal{C}$

.... similarly other cases are proven.
Due to model existence theorem, S is sat. $\square$

Exercise 6.9

Write down the β case.

Topic 6.4

Completeness of Tableaux and Resolution

Tableaux completeness

Theorem 6.11

The collection of sets of formulas that are tableaux consistent is a consistency property.

Proof.

Let Σ be a tableaux consistent set. We need to show the 4 conditions hold.

1. If $\{p, \neg p\} \subseteq \Sigma$ then there is a closed tableaux. Therefore, $\{p, \neg p\} \not\subseteq \Sigma$
2. If $\neg\neg F \in \Sigma$. Suppose $\{F\} \cup \Sigma$ has a closed tableaux T.
   Then, we can construct a closed tableaux for Σ as follows.
   $$\begin{array}{c} \neg\neg F \\ \downarrow \\ F \\ \downarrow \\ T'' \end{array}$$
   where T'' is obtained by removing all the nodes with label F in T if the node was added due to the introduction rule. Contradiction.
   Therefore, $\{F\} \cup \Sigma$ is a tableaux consistent set.
3. ... other cases have similar proofs. □

Exercise 6.10 *Complete the above proof*

Resolution completeness

Theorem 6.12

The collection of sets of formulas that are resolution consistent is a consistency property.

Proof.

Let Σ be a resolution consistent set.

1. If $\{p, \neg p\} \subseteq \Sigma$ then there is a closed derivation. Therefore, $\{p, \neg p\} \not\subseteq \Sigma$
2. case $\neg\neg F \in \Sigma$:
   Suppose $\{F\} \cup \Sigma$ has a closed derivation R.
   Then, we can construct a closed derivation for Σ as follows.

$$\begin{array}{c} \{\neg\neg F\} \\ \{F\} \\ R' \end{array}$$

   where R' is obtained by deleting occurrences of $\{F\}$ in R. Contradiction.
   Therefore, $\{F\} \cup \Sigma$ is a resolution consistent set.
3. case $\alpha \subseteq \Sigma$: similarly as above

...

Resolution completeness (contd.) I

Proof(contd.)

Constructing a closed derivation for Σ is not straight forward.

4. case $\beta \in \Sigma$:
   Assume $\{\beta_1\} \cup \Sigma$ and $\{\beta_2\} \cup \Sigma$ have closed derivations R_1 and R_2.

   We define derivation R_1' as follows:
   - $R_1' :=$ replace $\{\beta_1\}$ clauses by $\{\beta_1, \beta_2\}$ in R_1
   - Repeat $i \in 1..|R_1'|$, $R_1' :=$ repair ith clause C in R_1' as follows. If any antecedent of C is extended by β_2 then apply the expansion rule again and obtain a replacement, which is either C or $C \cup \{\beta_2\}$(why?).
   - $R_1' :=$ remove $\{\beta_1, \beta_2\}$ in R_1'

...

Resolution completeness (contd.) II

Proof(contd.)

case R'_1 has $\{\}$:

$$\begin{array}{c}\{\beta\}\\ \{\beta_1,\beta_2\}\\ R'_1\end{array}$$

is closed derivation of Σ.
Contradiction.

case R'_1 has $\{\beta_2\}$:
$R'_2 :=$ remove $\{\beta_2\}$ in R_2.

$$\begin{array}{c}\{\beta\}\\ \{\beta_1,\beta_2\}\\ R'_1\\ R'_2\end{array}$$

is closed derivation of Σ. Contradiction.

$\{\beta_1\} \cup \Sigma$ or $\{\beta_2\} \cup \Sigma$ is resolution consistent. □

Exercise 6.11

In which step of the above proof, the resolution rule plays a role?

Topic 6.5

Proofs are Enumerable

Implication is effectively enumerable.

Theorem 6.13

If Σ is a finite set of formulas, then $\Sigma \models F$ is decidable.

Proof.

Due to truth tables. $\square$

Theorem 6.14

If Σ is effectively enumerable, then $\Sigma \models F$ is semi-decidable.

Proof.

Due to compactness theorem, if $\Sigma \models F$ there is a finite set Σ_0 such that $\Sigma_0 \models F$.
Since Σ is effectively enumerable, let $G_1, G_2, ...$ be the enumeration of Σ.
Let $S_n \triangleq \{G_1, \dots, G_n\}$
There must be a $\Sigma_0 \subseteq S_{k\text{(why?)}}$.
Therefore, $S_k \models F$.
We may enumerate S_n and check $S_n \models F$, which is decidable.
Therefore, eventually we will say yes if $\Sigma \models F$. $\square$

Topic 6.6

Problems

Other basis of operators

Exercise 6.12

Assume $\perp$ and $\Rightarrow$ are the only fundamental operators in a propositional formalism.

a. Show the formalism is as expressive as the propositional logic

b. Define corresponding Hinttika set in the formalism

c. Prove that every Hinttika set is SAT, using the new definition.

(Hint: you may need to invent a uniform notation for the formalism).

End of Lecture 6