Mathematical Logic 2016

Lecture 8: Low complexity subclasses of SAT

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Where are we and where are we going?

We have seen

- propositional logic
- proof methods for the logic
- soundness, completeness, and complexity of the methods

We will see

- Low complexity subclasses of SAT

Special classes of formulas

We will discuss the following subclasses whose SAT problems are polynomial

- 2-SAT
- Horn clauses
- XOR-SAT
- SLUR

Topic 8.1

2-SAT

2-SAT

Definition 8.1

A 2-sat formula is a CNF formula that has only binary clauses

We assume that unit clauses are replaced by clauses with repeated literals.

Example 8.1

- $(\neg p \lor q) \land (\neg q \lor r) \land (\neg r \lor p) \land (r \lor q)$ *is a 2-SAT formula*
- $(p \lor p) \land (\neg p \lor \neg p)$ *is a 2-SAT formula*

Implication graph

Definition 8.2

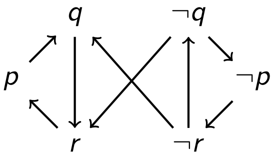
Let F be a 2-SAT formula s.t. $\mathbf{Vars}(F) = \{p_1, \dots, p_n\}$.
The implication graph (V, E) *for F is defined as follows.*

- $V = \{p_1, \dots, p_n, \neg p_1, \dots, \neg p_n\}$
- $E = \{(\bar{\ell_1}, \ell_2), (\bar{\ell_2}, \ell_1) | (\ell_1 \vee \ell_2) \in F\}$,

where $\bar{p} = \neg p$ *and* $\overline{\neg p} = p$.

Example 8.2

Consider $(\neg p \vee q) \wedge (\neg q \vee r) \wedge (\neg r \vee p) \wedge (r \vee q)$.



Exercise 8.1

Draw implication graphs of the following

1. $(p \vee q) \wedge (\neg p \vee \neg q)$
2. $(p \vee \neg q) \wedge (q \vee p) \wedge (\neg p \vee \neg r) \wedge (r \vee \neg p)$
3. $(p \vee p) \wedge (\neg p \vee \neg p)$
4. $(p \vee \neg p) \wedge (p \vee \neg p)$

Properties of implication graph

Consider a formula F and its implication graph (V, E).

Theorem 8.1

If there is a path from ℓ_1 to ℓ_2 in (V, E) then there is a path from $\bar{\ell_2}$ to $\overline{\ell_1}$.

Exercise 8.2

a. Prove the above theorem.

b. Does the above theorem imply

if there is a path from p to $\neg p$ in (V, E) then there is a path from $\neg p$ to p?

Theorem 8.2

For every strongly connected component(scc) $S \subseteq V$ in (V, E), there is another scc S^c, called complementary component, that has exactly the set of literals that are negation of the literals in S.

Proof.

Due to theorem 8.1. □

Properties of implication graph (contd.)

Theorem 8.3

For each model $m \models F$*, if there is a path from* ℓ_1 *to* ℓ_2 *in* (V, E) *then if* $m(\ell_1) = 1$ *then* $m(\ell_2) = 1$*.*

Theorem 8.4

For each model $m \models F$ *and each scc* S *in* (V, E)*,*
either for each $\ell \in S$ $m(\ell) = 1$ *or for each* $\ell \in S$ $m(\ell) = 0$*.*

Exercise 8.3

Prove the above theorems.

Reduced implication graph

Definition 8.3

The reduced implication DAG (V^R, E^R) *is a graph over scc's of* (V, E) *and is defined as follows.*

- $V^R = \{S | S \text{ is a scc in } (V,E)\}$
- $E^R = \{(S, S') | \text{there are } \ell \in S \text{ and } \ell' \in S' \text{ s.t. } (\ell, \ell') \in E\}$

Theorem 8.5

If $(S, S') \in E^R$ *then* $(S'^c, S^c) \in E^R$

Exercise 8.4

Prove the above theorem.

2-SAT satisfiablity

Theorem 8.6

A 2-SAT formula F is unsat iff there is a scc S in its implication graph (V, E) such that $\{p, \neg p\} \subseteq S$ for some p.

Proof.

Reverse direction

There must be a path that goes from p to $\neg p$.

$$p \longrightarrow \ell_1 \dashrightarrow \ell_n \longrightarrow \neg p$$

By applying resolution on corresponding clauses, we can derive $\neg p$.

$$\frac{(\neg p \vee \ell_1) \quad (\neg \ell_1 \vee \ell_2) \quad \dots \quad (\neg \ell_{n-1} \vee \ell_n) \quad (\neg \ell_n \vee \neg p)}{\neg p}$$

Similarly, we can derive p due to the path from $\neg p$ to p.

Therefore, we can derive empty clause.

F is unsat.

2-SAT satisfiablity(contd.)

Proof(contd.)

Fwd direction: Let us assume there is no such S.

We will construct a model of F as follows.

1. Initially all literals are unassigned.
2. if(some scc in V^R is unassigned)
3. Let $S \in V^R$ be an unassigned scc whose all children are assigned 1.
4. Assign literals of S to 1. Consequently, S^c is assigned 0.
5. goto 2.

We need to show that at step 3 we always find S.

claim: at step 3, there is a node whose all children are assigned.

Choose an unassigned node and descend down if there is unassigned child.

Since the DAG is finite, termination.

claim: an unassigned node can not have a child that is assigned 0.

If S is assigned 1, all its children are already 1. Therefore, all the parents of S^c are already assigned 0(due to theorem 8.5). □

Exercise 8.5 *Show the above procedure produces a satisfying model.*

2-SAT is polynomial

Theorem 8.7

A 2-SAT satisfiability problem can be solved in linear time.

Proof.

Due to the previous theorem, 2-SAT satisfiability problem is polynomial. □

Practice 2-SAT solving

Exercise 8.6

Find a satisfying assignment of the following formula

1. $(\neg x \vee \neg y) \wedge (\neg y \vee \neg z) \wedge (\neg z \vee \neg x) \wedge (x \vee \neg w) \wedge (y \vee \neg w) \wedge (z \vee \neg w)$
2. $(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge$
   $(p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (p_3 \vee p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6)$

Topic 8.2

Horn Clauses

Horn clauses

Definition 8.4

A Horn clause is a clause that has the following form

$$\neg p_1 \vee \cdots \vee \neg p_n \vee q,$$

*where $p_1, \ldots, p_n \in$ **Vars**, and $q \in$ **Vars** $\cup \{\bot\}$.*
A Horn formula is a set of Horn clauses, which is interpreted as conjunction of the Horn clauses.
The clauses with $\bot$ literals are called goal clauses and others are called implication clauses.

Example 8.3

The following set is a Horn formula

$$\begin{aligned}\{&p, \quad \neg q \vee \neg r \vee \neg t \vee p,\\ &\neg p \vee q, \quad \neg p \vee \neg r \vee t,\\ &\neg p \vee \neg q \vee t, \quad \neg r \vee \bot,\\ &\neg p \vee \neg q \vee \neg t \vee \bot\}\end{aligned}$$

Implication view of the horn clauses

We may view a Horn clause

$$\neg p_1 \lor \cdots \lor \neg p_n \lor q$$

as

$$p_1 \land \cdots \land p_n \Rightarrow q.$$

Example 8.4

The following is an implication view of a Horn formula

$$\begin{aligned}\{&\top \Rightarrow p, \quad q \land r \land t \Rightarrow p,\\ &p \Rightarrow q, \quad p \land r \Rightarrow t,\\ &\neg p \land q \Rightarrow t, \quad r \Rightarrow \bot,\\ &p \land q \land t \Rightarrow \bot\}\end{aligned}$$

Note $\top \Rightarrow p$ means p, which is a Horn clause without negative literals

Horn Satisfiability

Algorithm 8.1: $\textsc{HornSAT}$(Hs,Gs)

Input: *Hs*: implication clauses, *Gs* : goal clauses

Output: model/unsat

$m = \lambda x.0$;

while $m \not\models (p_1 \wedge .. \wedge p_n \Rightarrow p) \in Hs$ **do**

  $m \triangleq m[p \mapsto 1]$;

  if $m \not\models (q_1 \wedge .. \wedge q_k \Rightarrow \bot) \in Gs$ **then** **return** *unsat* ;

return m

Exercise 8.7

Solve

$$\{\top \Rightarrow p, \quad q \wedge r \wedge t \Rightarrow p, \quad p \Rightarrow q, \quad p \wedge r \Rightarrow t,$$
$$\neg p \wedge q \Rightarrow t, \quad r \Rightarrow \bot, \quad p \wedge q \wedge t \Rightarrow \bot\}$$

Recognizing Horn clauses

Sometimes a set of clauses are not immediately recognizable as Horn clause.

We may convert a CNF into a Horn formula by flipping the negation sign for some variables. Such CNF are called Horn clause renameable.

Definition 8.5

Let F be a CNF formula and m be a model. Let $flip(F, m)$ denote the formula obtained by flipping the variables that are assigned 1 in m.

Example 8.5

$flip((p \vee \neg q \vee \neg s), [p \mapsto 1, q \mapsto 0, s \mapsto 1, ...]) = (\neg p \vee \neg q \vee s)$

cc by-nc-sa

Renaming Horn clauses

Theorem 8.8

A CNF formula $F = \{C_1, \dots, C_n\}$, where $C_i = \{\ell_{i1}, \dots, \ell_{i|C_i|}\}$ is Horn clause renameable iff the following 2-SAT formula is satisfiable.[*]

$$G = \{\ell_{ij} \vee \ell_{ik} | i \in 1..n \text{ and } 1 \leq j < k \leq |C_i|\}$$

Proof.

Forward direction: there is a model m such that $flip(F, m)$ is a Horn formula

claim: $m \models G$

consider a clause $\ell_{ij} \vee \ell_{ik} \in G$

case $\ell_{ij} = p, \ell_{ik} = q$: one of them must flip,i.e.,$m(p) = 1$ or $m(q) = 1$

case $\ell_{ij} = \neg p, \ell_{ik} = \neg q$: at least one must not flip, i.e., not $m(p) = m(q) = 1$

case $\ell_{ij} = \neg p, \ell_{ik} = q$: if p flips then q must,i.e., if $m(p) = 1$ then $m(q) = 1$

In all the three cases $m \models \ell_{ij} \vee \ell_{ik}$.

[*] H. Lewis. Renaming a Set of Clauses as a Horn Set. J. of the ACM, 25:134-135, 1978.

Renaming Horn clauses(contd.)

Proof(contd.)

Reverse direction: Let $m \models G$. Let $F' = \mathit{flip}(F, m)$.
claim: F' is a Horn formula
Suppose F' is not a Horn formula
Then, there are positive literals ℓ'_{ij} and ℓ'_{ik} in F'.
Therefore, $m \not\models \ell_{ij} \vee \ell_{ik}$(why?). Contradiction. □

Exercise 8.8

What is the complexity of checking if a formula is Horn clause renameable?

Exercise 8.9

Can you improve the above complexity?

Topic 8.3

XOR SAT

XOR-SAT

Definition 8.6
A formula is XOR-SAT if it is a conjunction of xors of literals.

Example 8.6
$(p \oplus r \oplus s) \land (q \oplus \neg r \oplus s) \land (p \oplus q \oplus \neg s) \land (p \oplus \neg q \oplus \neg r)$
is a XOR-SAT formula.

Solving XOR-SAT

Since xors are negation of equality, we may eliminate variables via substitution.

Theorem 8.9

For a variable, p, xor formula G, and XOR-SAT formula F,
$(p \oplus G) \wedge F$ *is sat iff* $F[\neg G/p]$ *is sat*

Exercise 8.10

Prove the above theorem.

Example : solving XOR-SAT

Example 8.7

$(p \oplus r \oplus s) \wedge (q \oplus \neg r \oplus s) \wedge (p \oplus q \oplus \neg s) \wedge (p \oplus \neg q \oplus \neg r)$

Eliminate p:

Due to the first xor: $p \Leftrightarrow \neg r \oplus s$

After substitution: $(q \oplus \neg r \oplus s) \wedge (\neg r \oplus s \oplus q \oplus \neg s) \wedge (\neg r \oplus s \oplus \neg q \oplus \neg r)$

Simplification: $(q \oplus \neg r \oplus s) \wedge (\neg r \oplus \neg q) \wedge (s \oplus \neg q)$

Eliminate r:

Due to the second xor: $r \Leftrightarrow \neg q$

After substitution: $(q \oplus \neg\neg q \oplus s) \wedge (s \oplus \neg q)$

Simplification: $s \wedge (s \oplus \neg q)$

Eliminate q:

Due to the second xor: $q \Leftrightarrow s$

After substitution: s

Solution:

$m(s) = 1$ $\qquad m(q) = m(s) = 1$

$m(r) = m(\neg q) = 0$ $\qquad m(p) = m(\neg r \oplus s) = 0$

Exercise: XOR-SAT

Exercise 8.11

Find a satisfying assignment of the following formula

- $(p \oplus r \oplus s) \wedge (q \oplus r \oplus s) \wedge (\neg p \oplus q \oplus \neg s) \wedge (p \oplus \neg q \oplus \neg r)$

Topic 8.4

Single lookahead unit resolution

Single lookahead unit resolution(SLUR)

This subclass is defined using the following algorithm.

Algorithm 8.2: $\mathrm{SLUR}(F)$

Input: F: CNF formula
Output: model/unknown

$m = \lambda x.0;$
while *m is partial* **do**
  Choose an unassigned variable p in m ;
  Apply unit clause propagation and extend $m[p \mapsto 1]$ to m';
  if $m' \not\models F$ **then**
    Apply unit clause propagation and extend $m[p \mapsto 0]$ to m';
    if $m' \not\models F$ **then** **return** *unknown* ;
  $m := m'$;
return m

Definition 8.7

F belongs to SLUR class if $\mathrm{SLUR}(F)$ *can never return* unknown*.*

SLUR recognition

There is no efficient way to recognize a SLUR formula.

However, SLUR contains several interesting sub-classes proposed earlier.

- Extended horn
- CC balanced

Topic 8.5

Problems

Unsat XOR-sat

Exercise 8.12

Give an unsat XOR-sat formula that has only xors more than 3 arguments.

k-BRLR class

Exercise 8.13

A CNF formula is in k-BRLR if all consequence derived from it using resolution have at most k literals. What is the complexity of checking satisfiability of formulas in k-BRLR?

Another class

Exercise 8.14

Consider a CNF formula F such that every clause in F has at least two literals and for each variable p there is a model of F with p is true and a model with p is false. Give a linear time algorithm to find a satisfying assignment of F.

End of Lecture 8