Mathematical Logic 2016

Lecture 9: Binary Decision Diagrams (BDDs)

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Where are we and where are we going?

We have seen

- definition of propositional logic
- proof systems for the logic
- various properties of the proof systems
- subclasses of propositional logic

We will see a practical automated method for the SAT problem

- Binary decision diagrams (BDDs)

Proving satisfiability/unsatisfiability/validity/implication

We only need to implement a satisfiability(sat) checker and rest will follow.

- To prove unsatisfiability(unsat) of F, show sat checker fails to find satisfying assignment
- To prove validity of F, show $\neg F$ is unsat
- To prove implication $F_1 \Rightarrow F_2$, show $F_1 \wedge \neg F_2$ is unsat

Topic 9.1

Binary Decision Diagrams

Partial evaluation

Let us suppose a partial model m s.t. $\mathbf{Vars}(F) \not\subseteq dom(m)$.
We can assign meaning to $m(F)$, which we will denote with $F|_m$.

Definition 9.1

For formula F and a partial model $m = \{p_1 \mapsto b_1, ..\}$, Let

$$F|_{x_i \mapsto b_i} = \begin{cases} F[\top/x_i] & \text{if } b_i = 1 \\ F[\bot/x_i] & \text{if } b_i = 0. \end{cases}$$

The partial evaluation $F|m$ be $F|_{p_i \mapsto b_1}|_{p_i \mapsto b_1}| \dots$ after the simplifications involving $\top$ and $\bot$.

For short hand, we may write $F|_p$ for $F|_{p \mapsto 1}$ and $F|_{\neg p}$ for $F|_{p \mapsto 0}$.

Theorem 9.1

$(F|_p \wedge p) \vee (F|_{\neg p} \wedge \neg p) \equiv F$

Example : partial evaluation

Example 9.1

Consider $F = (p \vee q) \wedge r$

$$((p \vee q) \wedge r)[\top/p] = (\top \vee q) \wedge r \equiv \top \wedge r \equiv r$$

Therefore, $F|_p = r$

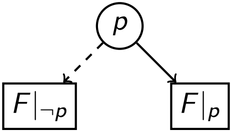
Exercise 9.1

Compute

- $F|_{\neg p}$
- $((p_1 \Leftrightarrow q_1) \wedge (p_2 \Leftrightarrow q_2))|_{p_1 \mapsto 0, p_2 \mapsto 0}$

Decision branch

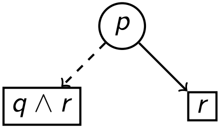
Due to the theorem in previous slide the following tree may be viewed as representing F.



Dashed arrows represent 0 decisions and solid arrows represent 1 decisions.

Example 9.2

Consider $(p \vee q) \wedge r$

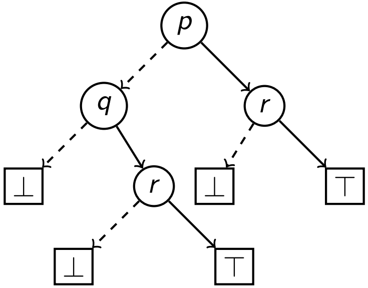


Decision tree

We may further expand $F|_{\neg p}$ and $F|_p$ until we are left with $\top$ and $\perp$ at the leaves. The obtained tree is called the decision tree for F.

Example 9.3

Consider $(p \vee q) \wedge r$



Binary decision diagram(BDD)

If two nodes represent same formula then we may rewire the incoming edges to only one of the nodes.

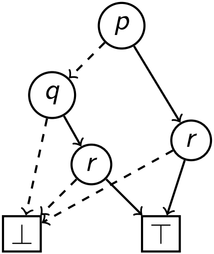
Definition 9.2

A BDD is a finite DAG s.t.

- *each internal node is labeled with a propositional variable var*
- *each internal node has a low (dashed) and high child (solid)*
- *there are exactly two leaves one is labelled with $\top$ and the other with $\perp$*

Example 9.4

The following is a BDD for $(p \vee q) \wedge r$



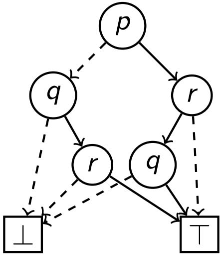
Ordered BDD (OBDD)

Definition 9.3

A BDD is ordered if there is an order $<$ over variables including $\top$ and $\bot$ s.t. for each node v, $v < low(v)$ and $v < high(v)$.

Example 9.5

The following BDD is not an ordered BDD



Exercise 9.2

a. *Convert the above BDD into a formula*

b. *Give an ordered BDD of the formula*

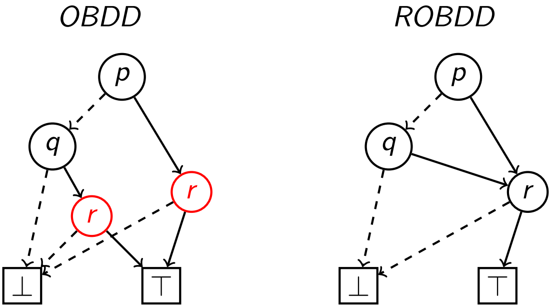
Reduced OBDD (ROBDD)

Definition 9.4

A OBDD is reduced if

- *for any nodes u and v, if $var(u) = var(v)$, $low(u) = low(v)$, $high(u) = high(v)$ then $u = v$*
- *for each node u, $low(u) \neq high(u)$*

Example 9.6



Converting to ROBDD

Any OBDD can be converted into ROBDD by iteratively applying the following transformations.

1. If there are nodes u and v s.t. $var(u) = var(v)$, $low(u) = low(v)$, $high(u) = high(v)$ then remove u and connect all the parents of u to v.
2. If there is node u s.t. $low(u) = high(u)$ then remove u and connect all the parents of u to $low(u)$.

Exercise 9.3

Prove the above iterations terminate.

Canonical ROBDD

Theorem 9.2

For a function $f : \mathcal{B}^n \rightarrow \mathcal{B}$ there is exactly one ROBDD u with ordering $p_1 < \cdots < p_n$ such that u represents $f(p_1, \ldots, p_n)$.

Proof.

We use the induction over the number of parameters.

base (n=0): There are only two functions $f() = 0$ and $f() = 1$, which are represented by nodes $\perp$ and $\top$ respectively.

step ($n > 0$): Assume, unique ROBDD for functions with n parameters. Consider a function $f : \mathcal{B}^{n+1} \rightarrow \mathcal{B}$.

Let $f_0(p_2, \ldots, p_{n+1}) = f(0, p_2, \ldots, p_{n+1})$ which is represented by ROBDD u_0.
Let $f_1(p_2, \ldots, p_{n+1}) = f(1, p_2, \ldots, p_{n+1})$ which is represented by ROBDD u_1.

...

Canonical ROBDD (cond.) II

Proof(contd.)

case $u_0 = u_1$:

Therefore, $f = f_0 = f_1$. Therefore, u_0 represents f.

Assume there is $u' \neq u_0$ that represents f.

Therefore, $var(u') = p_1$ (why?), $low(u') = high(u') = u_0$.

Therefore, u' is not a ROBDD. ...

Canonical ROBDD (cond.) III

Proof(contd.)

case $u_0 \neq u_1$:

Let u be with $var(u) = p_{1\text{(why?)}}$, $low(u) = u_0$, and $high(u) = u_1$.

Clearly, u is a ROBDD.

Assume there is $u' \neq u$ that represents f. Therefore, $var(u') = p_{1\text{(why?)}}$.

Due to induction hyp., $low(u') = u_0$, and $high(u') = u_1$.

Due to the reduced property, $u = u'$. $\square$

Satisfiablility via BDD

Build a ROBDD that represents F and unsat formulas have only one node $\perp$.

Benefits of ROBDD

- If intermediate ROBDDs are small then the satisfiability check will be efficient.
- Cost of computing ROBDDs vs sizes of BDDs
- Due to the canonicity property, ROBDD is used as a formula store
- Various operations on the ROBDDs are conducive to implementation

Issues with ROBDD

- BDDs are very sensitive to the variable ordering. There are formulas that have exponential size ROBDDs for some orderings
- There is no efficient way to detect good variable orderings

Exercise 9.4

Draw the ROBDD for

$$(x_1 \wedge x_2) \vee (x_3 \wedge x_4)$$

with the following ordering on variables $x_1 < x_3 < x_2 < x_4$.

Topic 9.2

Algorithms for BDDs

Algorithms for BDDs

Next we will present algorithms for BDDs to illustrate the convenience of the data structure.

Global data structures

The algorithms maintain the following two global data structures.

$store = (Nodes, low, high, var) := (\{\bot, \top\}, \lambda x.\texttt{null}, \lambda x.\texttt{null}, \lambda x.\texttt{null})$

$reverseMap : (\mathbf{Vars} \times Nodes \times Nodes) \to Nodes := \lambda x.\texttt{null}$

Constructing a BDD node

Algorithm 9.1: $\textsc{MakeNode}(p, u_0, u_1)$

Input: $p \in$ **Vars**, $u_0, u_1 \in Nodes$

if $u_0 = u_1$ **then** **return** u_0 ;

if $reverseMap.exists(p, u_0, u_1)$ **then** **return** $reverseMap.lookup(p, u_0, u_1)$;

$u := store.add(p, u_0, u_1)$; $reverseMap.add((p, u_0, u_1), u)$;

Constructing BDDs from a formula

Algorithm 9.2: $\textsc{BuildROBDD}(F, p_1 < \dots < p_n)$

Input: $F(p_1, \dots, p_n) \in \mathbf{P}$, $p_1 < \dots < p_n$: an ordering over variables of F

if $n = 0$ **then**

  if $F \equiv \bot$ **then** **return** $\bot$; **else** **return** $\top$;

$u_0 := \textsc{BuildROBDD}(F|_{\neg p_1}, p_2 < \dots < p_n)$;

$u_1 := \textsc{BuildROBDD}(F|_{p_1}, p_2 < \dots < p_n)$;

return $\textsc{MakeNode}(p_1, u_0, u_1)$

Conjunction of BDDs

Algorithm 9.3: $\textsc{ConjBDDs}(u, v)$

Input: u and v ROBDDs with same variable ordering

if $u = \bot$ *or* $v = \top$ **then return** u;

if $u = \top$ *or* $v = \bot$ **then return** v;

$u_0 := low(u); u_1 := high(u); p_u := var(u)$;

$v_0 := low(v); v_1 := high(v); p_v := var(v)$;

if $p_u = p_v$ **then**

  return $\textsc{MakeNode}(p_u, \textsc{ConjBDDs}(u_0, v_0), \textsc{ConjBDDs}(u_1, v_1))$

if $p_u < p_v$ **then**

  return $\textsc{MakeNode}(p_u, \textsc{ConjBDDs}(u_0, v), \textsc{ConjBDDs}(u_1, v))$

if $p_u > p_v$ **then**

  return $\textsc{MakeNode}(p_u, \textsc{ConjBDDs}(u, v_0), \textsc{ConjBDDs}(u, v_1))$

Exercise 9.5

a. *Write an algorithm for computing disjunction of BDDs*

b. *Write an algorithm for computing not of BDDs.*

Restriction on a value

Algorithm 9.4: RESTRICT(u, p, b)

Input: u ROBDD with same variable ordering, variable p, $b \in \mathcal{B}$

$u_0 := low(u); u_1 := high(u); p_u := var(u);$

if $p_u = p$ *and* $b = 0$ **then**

  return RESTRICT(u_0, p, b)

if $p_u = p$ *and* $b = 1$ **then**

  return RESTRICT(u_1, p, b)

if $p_u < p$ **then**

  return MAKENODE(p_u,RESTRICT(u_0, p, b),RESTRICT(u_1, p, b))

if $p_u > p$ **then**

  return u

Impact of BDDs

- In 90s, BDDs revolutionized hardware verification
- Later other methods were found that are much faster and the fall of BDD was marked by the following paper,

  A. Biere,A. Cimatti,E. Clarke,Y. Zhu,
  Symbolic Model Checking without BDDs, TACAS 1999
- However, BDDs are still the heart of various software packages

Topic 9.3

Problems

Take and of the BDDs

Exercise 9.6

Construct ROBDD of the following formula for the order $p < q < r < s$.

$$F = (p \vee (q \oplus r) \vee (p \vee s))$$

Let u be the ROBDD node that represents F.
Give the output of $\textsc{Restrict}(u_F, p, b)$

Variable reordering

Exercise 9.7

Let u be an ROBDD with variable ordering $p_1 < ... < p_n$.
Give an algorithm to transforming u into a ROBDD with ordering $p_1 < .. < p_{i-1} < p_{i+1} < p_i < p_{i+2} < .. < p_n$.

BDD-XOR

Exercise 9.8

Write an algorithm for computing xor of BDDs

BDD model counting

Exercise 9.9

a. Give an algorithm for counting models for a given ROBDD.

b. Does this algorithm works for any BDD?

End of Lecture 9