Mathematical Logic 2016

Lecture 12: Substitution, Herbrand model, and Hinttika set

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Compile date: 2016-09-10

Where are we and where are we going?

We have seen

- Syntax and semantics of FOL

We will see some theorems about FOL

- Substitution
- Herbrand model
- Hinttika theorem

Topic 12.1

Substitution

Substitution

Definition 12.1

A substitution σ is a map from $\mathbf{Vars} \to T_{\mathbf{S}}$. We will write $t\sigma$ to denote $\sigma(t)$.

Definition 12.2

We say σ has finite support if only finite variables do not map to themselves. σ with finite support is denoted by $[t_1/x_1, .., t_n/x_n]$ or $\{x_1 \mapsto t_1, .., x_n \mapsto t_n\}$.

Commentary: We have seen substitution in propositional logic. Now in FOL, substitution is not a simple matter. Before introducing the other aspects of FOL. Let us present substitution first.

Substitution on terms

Definition 12.3

For $t \in T_{\mathbf{S}}$, let the following naturally define $t\sigma$ as extension of σ.

- $c\sigma \triangleq c$
- $(f(t_1, \dots, t_n))\sigma \triangleq f(t_1\sigma, \dots, t_n\sigma)$

Example 12.1

Consider $\sigma = [f(x, y)/x, f(y, x)/y]$

- $x\sigma = f(x, y)$
- $f(x, y)\sigma = f(f(x, y), f(y, x))$
- $(f(x, y)\sigma)\sigma = ?$

Composition

Definition 12.4

Let σ_1 and σ_2 be substitutions. The composition $\sigma_1\sigma_2$ of the substitutions is defined as follows.

$$\text{For each } x \in \mathbf{Vars},\ x(\sigma_1\sigma_2) \triangleq (x\sigma_1)\sigma_2.$$

Theorem 12.1

For each $t \in T_{\mathbf{S}}$, $t(\sigma_1\sigma_2) = (t\sigma_1)\sigma_2$

Proof.

Proved by trivial structural induction. □

Theorem 12.2

$\sigma_1(\sigma_2\sigma_3) = (\sigma_1\sigma_2)\sigma_3$ *(associativity)*

Proof.

Consider variable x.

$(x\sigma_1)(\sigma_2\sigma_3) = ((x\sigma_1)\sigma_2)\sigma_3 = (x(\sigma_1\sigma_2))\sigma_3$ □

Subtituion on atoms

We further extend the substitution σ to atoms.

Definition 12.5

For $F \in A_{\mathbf{S}}$, $F\sigma$ is defined as follows.

- $\top\sigma \triangleq \top$
- $\bot\sigma \triangleq \bot$
- $P(t_1, \dots, t_n)\sigma \triangleq P(t_1\sigma, \dots, t_n\sigma)$
- $(t_1 \approx t_2)\sigma \triangleq t_1\sigma \approx t_2\sigma$

Theorem 12.3

For each $F \in A_{\mathbf{S}}$, $F(\sigma_1\sigma_2) = (F\sigma_1)\sigma_2$

Proof.

Proved by trivial structural induction. □

Substitution

Sometimes, we may need to remove variable x from the support of σ.

Definition 12.6

Let $\sigma_x = \sigma[x \mapsto x]$.

Example 12.2

Consider $\sigma = [f(x,y)/x, f(y,x)/y]$
$\sigma_x = [f(y,x)/y]$

Commentary: The need of the definition will be clear soon.

Substitution in formulas (Incorrect)

Now we extend the substitution σ to all the formulas.

Definition 12.7

For $F \in \mathbf{P_S}$, $F\sigma$ is defined as follows.

- $(\neg G)\sigma \triangleq \neg(G\sigma)$
- $(G \circ H)\sigma \triangleq (G\sigma) \circ (H\sigma)$ *for some binary operator* $\circ$
- $(\forall x.G)\sigma \triangleq \forall x.(G\sigma_x)$
- $(\exists x.G)\sigma \triangleq \exists x.(G\sigma_x)$

Example 12.3

- $(P(x) \Rightarrow \forall x.Q(x))[y/x] = (P(y) \Rightarrow \forall x.Q(x))$
- $(\exists y.\ x \not\approx y)[z/x] = (\exists y.\ z \not\approx y)$
- $(\exists y.\ x \not\approx y)[y/x] = (\exists y.\ y \not\approx y)$ ☹ *Undesirable!!!*

Some substitutions should be disallowed.

Commentary: The above näive definition of the substitution in formulas is incorrect. In the next slide, we present the correct definition.

Substitution in formulas(Correct)

Definition 12.8

σ is suitable wrt to formula G and variable x if for all $y \neq x$, if y occurs in G then x does not occur in $y\sigma$.

Now we correctly extend the substitution σ to all formulas.

Definition 12.9

For $F \in \mathbf{P_S}$, $F\sigma$ is defined as follows.

- $(\neg G)\sigma \triangleq \neg(G\sigma)$
- $(G \circ H)\sigma \triangleq (G\sigma) \circ (H\sigma)$ *for some binary operator* $\circ$
- $(\forall x.G)\sigma \triangleq \forall x.(G\sigma_x)$, *where σ is suitable wrt G and x*
- $(\exists x.G)\sigma \triangleq \exists x.(G\sigma_x)$, *where σ is suitable wrt G and x*

It is not a true restriction. We will see later.

For short hand, we may write a formula as $F(x_1, \dots, x_k)$, where we say that $x_1, \dots, x_k$ are the variables that play a special role in the formula F. Let $F(t_1, \dots, t_n)$ be $F[t_1/x_1, \dots, t_n/x_n]$.

Substitution composition

Theorem 12.4

if $F\sigma_1$ and $(F\sigma_1)\sigma_2$ are defined then $(F\sigma_1)\sigma_2 = F(\sigma_1\sigma_2)$

Proof.

We prove it by induction. Non-quantifier cases are simple structural induction.

Assume $F = \forall x.G$

Since $F\sigma_1$ is defined, $G\sigma_{1x}$ is defined.

Since $(F\sigma_1)\sigma_2$ is defined, $(G\sigma_{1x})\sigma_{2x}$ is defined (why?).

By Ind. Hyp. $(G\sigma_{1x})\sigma_{2x} = G(\sigma_{1x}\sigma_{2x})$

claim: $G(\sigma_{1x}\sigma_{2x}) = G(\sigma_1\sigma_2)_x$

Choose $y \in FV(G)$ and $y \neq x$

$y(\sigma_{1x}\sigma_{2x})$

$= ((y\sigma_{1x})\sigma_{2x}) = ((y\sigma_1)\sigma_{2x})$

$= ((y\sigma_1)\sigma_2)$ $\qquad x \notin FV(y\sigma_1)$ (why?)

$= y(\sigma_1\sigma_2) = y(\sigma_1\sigma_2)_x$

$(\forall x.G\sigma_1)\sigma_2 = (\forall x.G(\sigma_{1x}\sigma_{2x})) = (\forall x.G(\sigma_1\sigma_2)_x) = F(\sigma_1\sigma_2)$ $\square$

Topic 12.2

Herbrand model

Importance of Hinttika set/theorem

We have seen the use of Hinttika theorem in propositional logic.

The theorem was used to prove completeness of the proof methods.

Now we will present the theorem and related concepts before presenting the proof methods.

Assignment as substitution

Theorem 12.5

Let t be a closed term, and m be a model of signature $\mathbf{S}$. Let x be a variable and assignment ν such that $m^\nu(x) = m^\nu(t)$. Then, for any $\mathbf{S}$-formula F

$$m, \nu \models F[t/x] \text{ iff } m, \nu \models F$$

Proof.

Simple structural induction. □

Exercise 12.1

a. Is substitution $F[t/x]$ defined?

b. Let $d \in D_m$, show $m, \nu[x \to d] \models F[t/x]$ iff $m, \nu \models F$

> Substitutions by closed terms may be viewed as assignment.
> Some $d \in D_m$ may not correspond to any closed term.

Herbrand model

Definition 12.10

A model m for a signature $\mathbf{S}$ *is Herbrand model if*

- $D_m = \hat{T}_{\mathbf{S}}$*, the set of closed terms and*
- $m(t) = t \in \hat{T}_{\mathbf{S}}$

By definition, all $d \in D_m$ correspond to some closed term, namely itself.

Example 12.4

Consider $\mathbf{S} = (\{\mathbf{0}/0, \mathbf{s}/1\}, \{>\})$ *and a Herbrand model m for* $\mathbf{S}$
$D_m = \{\mathbf{0}, \mathbf{s}(\mathbf{0}), \mathbf{s}(\mathbf{s}(\mathbf{0})), ...\}$

Exercise 12.2

Show Herbrand model is well defined.

Commentary: Since the models of functions and predicates are not explicitly given under the Herbrand model, we need to prove that the definition of the model induces a well defined model.

Properties of Herbrand model

In Herbrand model, assignments are instances of substitutions.

Theorem 12.6

Let m be a Herbrand model for signature $\mathbf{S}$. For $t \in T_{\mathbf{S}}$, $m^{\nu}(t) = m(t\nu)$

Theorem 12.7

Let m be a Herbrand model for signature $\mathbf{S}$. For $F \in \mathbf{P}_{\mathbf{S}}$,

$$m^{\nu} \models F \textit{ iff } m \models F\nu$$

Topic 12.3

Hinttika set

Uniform notation

Recall, we assumed that $\top$, $\bot$, $\Leftrightarrow$ and $\oplus$ are removed.

In order to avoid writing many cases we will use a uniform notation.

Conjunctive		Disjunctive		Universal		Existential	
α	$\alpha_1 \quad \alpha_2$	β	$\beta_1 \quad \beta_2$	γ	$\gamma(t)$	δ	$\delta(t)$
$(F \wedge G)$	$F \quad G$	$\neg(F \wedge G)$	$\neg F \quad \neg G$	$\forall x F$	$F[t/x]$	$\exists x F$	$F[t/x]$
$\neg(F \vee G)$	$\neg F \quad \neg G$	$(F \vee G)$	$F \quad G$	$\neg\exists x.F$	$\neg F[t/x]$	$\neg\forall x.F$	$\neg F[t/x]$
$\neg(F \Rightarrow G)$	$F \quad \neg G$	$(F \Rightarrow G)$	$\neg F \quad G$				

A non-atomic formula can be from one the following five types.

- α
- β
- γ
- δ
- $\neg\neg F$

Hinttika set

Definition 12.11

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a signature. A set* M *of* **S**-*sentences is called Hintikka set if*

1. *for each* $F \in A_{\mathbf{S}}$*, not both* $F \in M$ *and* $\neg F \in M$
2. *if* $\neg\neg F \in M$ *then* $F \in M$
3. *if* $\alpha \in M$ *then* $\alpha_1 \in M$ *and* $\alpha_2 \in M$
4. *if* $\beta \in M$ *then* $\beta_1 \in M$ *or* $\beta_2 \in M$
5. *if* $\gamma \in M$ *then* $\gamma(t) \in M$ *for each* $t \in \hat{T}_{\mathbf{S}}$
6. *if* $\delta \in M$ *then* $\delta(t) \in M$ *for some* $t \in \hat{T}_{\mathbf{S}}$
7. $t \approx t \in M$ *for each* $t \in \hat{T}_{\mathbf{S}}$
8. *if* $t_1 \approx u_1, .., t_n \approx u_n \in M$ *then* $f(t_1, .., t_n) \approx f(u_1, .., u_n) \in M$ *for each* $f/n \in \mathbf{F}$
9. *if* $t_1 \approx u_1, .., t_n \approx u_n$, $P(t_1, .., t_n) \in M$ *then* $P(u_1, .., u_n) \in M$ *for each* $P/n \in \mathbf{R} \cup \{\approx/2\}$

1-6 break formulas into atoms.
7-9 create atoms with larger terms.

Equivalence relation over terms in Hinttika set

Let M be a Hinttika set.

Theorem 12.8

If $t \approx u \in M$ then $u \approx t \in M$.

Proof.

$t \approx t \in M$ due to rule 7.

Due to rule 9, since $t \approx u, t \approx t, t \approx t \in M$, $u \approx t \in M$ $\square$

Theorem 12.9

If $t \approx u \in M$ and $u \approx v \in M$ then $t \approx v \in M$.

Proof.

$t \approx t \in M$ due to rule 7.

Due to rule 9, since $t \approx t, u \approx v, t \approx u \in M$, $t \approx v \in M$ $\square$

Theorem 12.10

Equality atoms in M define a equivalence relation over terms.

Proof.

Due to rule 7, reflexive. Due to the above theorems, symmetric and transitive.

Example : Hinttika set

Example 12.5

Consider $\mathbf{S} = (\{\mathbf{0}/0, s/1\}, \{\})$ *and a set of sentences*
$M_0 = \{\forall x. s(s(x)) \approx x, \exists x. s(x) \not\approx x\}$.
We extend the set to a Hinttika set

M contains atoms that define two equivalence classes over terms.

$$M = M_0 \cup M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$$

Apply rule 5 (γ expansion rule),
$M_1 = \{s(s(\mathbf{0})) \approx \mathbf{0}, s(s(s(\mathbf{0}))) \approx s(\mathbf{0}), s(s(s(s(\mathbf{0})))) \approx s(s(\mathbf{0})), \dots\}$
Apply rule 6 (δ expansion rule), $M_2 = \{s(s(\mathbf{0})) \not\approx s(\mathbf{0})\}$
Apply rule 7, $M_3 = \{\mathbf{0} \approx \mathbf{0}, s(\mathbf{0}) \approx s(\mathbf{0}), s(s(\mathbf{0})) \approx s(s(\mathbf{0})), \dots\}$
Apply rule 9 to derive symmetry of $\approx/2$,
$M_4 = \{\mathbf{0} \approx s(s(\mathbf{0})), s(\mathbf{0}) \approx s(s(s(\mathbf{0}))), s(s(\mathbf{0})) \approx s(s(s(s(\mathbf{0})))), \dots\}$
Apply rule 9 to derive transitivity of $\approx/2$,
$M_5 = \{\underbrace{s(..s(}_{i}\mathbf{0})..) \approx \underbrace{s(..s(}_{j}\mathbf{0})..) | 2k = |i - j|, k > 1\}$

In most cases Hintikka sets are infinite.

Hinttika theorem

Theorem 12.11

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a signature with nonempty closed terms.*
Every Hintikka set M *wrt to* $\mathbf{S}$ *is sat.*

Proof.

First, let us suppose $\approx$ is a predicate like any other predicate.

Let us construct a Herbrand model m'.

For each $f \in \mathbf{F}$, $f_{m'}$ is already defined since m' is a Herbrand model.
For any $P/n \in \mathbf{R} \cup \{\approx /2\}$, let $P_{m'} \triangleq \{(t_1, .., t_n) | P(t_1, .., t_n) \in M\}$.

Due to theorem 12.10, $\approx_{m'}$ defines equivalence relation over $D_{m'}$.

Commentary: The meaning of $\approx$ is already in FOL semantics. $\approx_{m'}$ must always be the identity relation. In the above construction this may not be the case. Therefore, we need to construct a suitable model.

Hinttika Lemma (contd.)

Proof(contd.)

Now we define the model m that we are looking for.

Let D_m be the equivalence classes due to $\approx_{m'}$.
Let $[t]$ denote the equivalence class that contains t.

For each $f \in \mathbf{F}$, $f_m([t_1], .., [t_n]) := [f_{m'}(t_1, \dots, t_n)]$.
For each $P \in \mathbf{F}$, $P_m := \{([t_1], .., [t_n]) | (t_1, .., t_n) \in P_{m'}\}$.

claim: f_m is well defined
Consider $[t_1]$,..,$[t_n]$.
Choose $u_1 \in [t_1]$,..,$u_n \in [t_n]$.
Due to rule 8, $[f_{m'}(t_1, \dots, t_n)] = [f_{m'}(u_1, \dots, u_n)]$.
Therefore, f_m is well defined.

Exercise 12.3

Do we need to prove that P_m is well defined?

Hinttika Lemma (contd.)

Proof(contd.)

By induction, we show m is a model of the Hinttika set.

base:

$t_1 \approx t_2 \in M$: Therefore, $[t_1] = [t_2]$. Therefore, $m \models t_1 \approx t_2$.

$t_1 \not\approx t_2 \in M$: Since $t_1 \approx t_2 \not\in M$, $[t_1] \neq [t_2]$. Therefore, $m \models t_1 \not\approx t_2$.

$P(t_1, .., t_n) \in M$: Therefore, $([t_1], .., [t_n]) \in P_m$. Therefore, $m \models P(t_1, .., t_n)$.

$\neg P(t_1, .., t_n) \in M$:
Consider $(t'_1, .., t'_n) \in [t_1] \times .. \times [t_n]$.
Assume $P(t'_1, .., t'_n) \in M$. Thus, $P(t_1, .., t_n) \in M$ due to rule 9.Contradiction.
Therefore, $P(t'_1, .., t'_n) \not\in M$.
Therefore, $([t_1], .., [t_n]) \not\in P_m$. Therefore, $m \models \neg P(t_1, .., t_n)$.

Hinttika Lemma (contd.)

Proof(contd.)

induction step:

We need to consider the five cases for the non-atomic FOL.

The arguments for cases $\neg\neg F$, α, and β are similar to PL. Let us consider γ.

case $\gamma \in M$:

claim: for each $[t] \in D_m$, $m, \{x \mapsto [t]\} \models \gamma(x)$

Choose $[t] \in D_m$. (every class is non-empty)

By def. of m, $m(t) = [t]$.

Due to rule 5, $\gamma(t) \in M$.

Due to induction hyp., $m \models \gamma(t)$.

Therefore, $m, \{x \mapsto [t]\} \models \gamma(x)$

Therefore $m \models \gamma$. $\square$

Exercise 12.4

Write the case for δ.

Topic 12.4

Problem

Hinttika set

Exercise 12.5

Let $\mathbf{S} = (\{c_1/0, c_2/0, ...\}, \{E\})$*.*

Extend the following set of sentences to be Hinttika set.

- $\{\forall x.\neg E(x,x) \land \exists x \exists y.E(x,y)\}$

Midterm advice

Live as if you were to die tomorrow.
Learn as if you were to live forever.

-Gandhi

End of Lecture 12