Mathematical Logic 2016

Lecture 13: Model existence theorem and its consequences

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Where are we and where are we going?

We have seen

- Syntax and semantics of FOL
- Herbrand model and Hinttika theorem

We will see

- Model existence theorem
- Compactness theorem
- Löwenheim-Skolem Theorem

Topic 13.1

Model existence theorem

Fresh symbols are needed

One often needs fresh symbols when instantiating existential quantifiers.

Example 13.1

Consider $\mathbf{S} = (\{a/0, b/0\}, \{P/1\})$.
Is the following formula sat?

$$P(a) \wedge P(b) \wedge \exists x.\, \neg P(x)$$

We need a new constant symbol c *that denotes a value s.t.* $\neg P(c)$ *is true.*
Note that a *and* b *can not do the job.*

Example 13.2

Consider $\mathbf{S} = (\{\}, \{P/1\})$. *Is the following formula sat?*

$$\forall x.\, P(x) \wedge \exists x.\, \neg P(x)$$

1. *Instantiate existential quantifier with a fresh symbol* c.
$$\neg P(c) \wedge \forall x.\, P(x) \wedge \exists x.\, \neg P(x)$$
2. *Instantiate universal quantifier with a term* c.
$$P(c) \wedge \neg P(c) \wedge \forall x.\, P(x) \wedge \exists x.\, \neg P(x)$$

Parameters

We need a supply of fresh symbols.

Let us define a signature extension that supplies the new constant symbols.

Definition 13.1

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a signature. Let* **par** *be an infinite countable set of constant symbols disjoint from* $\mathbf{S}$*. Let* $\mathbf{S}^{\mathbf{par}} = (\mathbf{F} \cup \mathbf{par}, \mathbf{R})$*.*

Definition 13.2

A set of formulas S *in* $\mathbf{S}^{\mathbf{par}}$ *is called unexhausted if there are infinitely many parameters in* **par** *that do not occur in* S*.*

Consistency property

Definition 13.3

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a signature and* $\mathcal{C}$ *be a collection of sets of sentences in* $\mathbf{S}^{\mathbf{par}}$*.* $\mathcal{C}$ *is a consistency property wrt to* $\mathbf{S}$ *if for each* $S \in \mathcal{C}$ *satisfies the following.*

1. *for each* $F \in A_{\mathbf{S}^{\mathbf{par}}}$*, either* $F \notin S$ *or* $\neg F \notin S$
2. *if* $\neg\neg F \in S$ *then* $\{F\} \cup S \in \mathcal{C}$
3. *if* $\alpha \in S$ *then* $\{\alpha_1, \alpha_2\} \cup S \in \mathcal{C}$
4. *if* $\beta \in S$ *then* $\{\beta_1\} \cup S \in \mathcal{C}$ *or* $\{\beta_2\} \cup S \in \mathcal{C}$
5. *if* $\gamma \in S$ *then* $\{\gamma(t)\} \cup S \in \mathcal{C}$ *for each* $t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}}$
6. *if* $\delta \in S$ *then* $\{\delta(c)\} \cup S \in \mathcal{C}$ *for some* $c \in \mathbf{par}$
7. $S \cup \{t \approx t\} \in \mathcal{C}$ *for each* $t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}}$
8. *if* $t_1 \approx u_1,..,t_n \approx u_n \in S$ *then* $S \cup \{f(\vec{t}) \approx f(\vec{u})\} \in \mathcal{C}$ *for each* $f/n \in \mathbf{F}$
9. *if* $t_1 \approx u_1,..,t_n \approx u_n, P(\vec{t}) \in S$ *then* $S \cup \{P(\vec{u})\} \in \mathcal{C}$ *for each* $P/n \in \mathbf{R} \cup \{\approx /2\}$

Model existence theorem

Theorem 13.1

Let $\mathcal{C}$ be a consistency property wrt to $\mathbf{S}$, S be a set of $\mathbf{S}$-sentences. If $S \in \mathcal{C}$ then S is sat.

Recall the proof in propositional case.

1. convert $\mathcal{C}$ into finite character
2. show limit exists in finite character
3. construct a monotonic sequence of elements of $\mathcal{C}$ starting from S
4. show its limit is a maximal element of $\mathcal{C}$
5. show the limit is a Hinittika set

Naturally things are more complicated here.

Recall: subset closed consistency property

Theorem 13.2

Every consistency property $\mathcal{C}$ can be extended to a consistency property that is subset closed.

Proof.

Let $\mathcal{C}^+ \triangleq \{S' | S' \subseteq S \text{ and } S \in \mathcal{C}\}$. We show $\mathcal{C}^+$ is consistency property.
Consider $S' \in \mathcal{C}^+$. By definition, there is $S \in \mathcal{C}$ s.t. $S' \subseteq S$.

1. Therefore, S' does not contain contradictory literals.
2. If $\neg\neg F \in S'$. Therefore, $\neg\neg F \in S$. Therefore, $\{F\} \cup S \in \mathcal{C}$. Therefore, $\{F\} \cup S' \in \mathcal{C}^+$.
3. .... (trivially extends to all 9 cases) $\square$

Recall: finite character

Definition 13.4

A consistency property $\mathcal{C}$ has finite character if $S \in \mathcal{C}$ iff every finite subset of S is in $\mathcal{C}$.

Theorem 13.3

if $\mathcal{C}$ is of finite character then $\mathcal{C}$ is subset closed.

Theorem 13.4

Let consistency property $\mathcal{C}$ is of finite character. If $S_1, S_2, \ldots$ is sequence of members of $\mathcal{C}$ such that $S_1 \subseteq S_2 \subseteq \ldots$. Then, $\bigcup_i S_i \in \mathcal{C}$.

Proofs of the above theorems were given in lecture 6.

Extendable to finite character(not true!!)

Theorem 13.5 (Wrong theorem)

A subset closed consistency property $\mathcal{C}$ is extendable to one of finite character.

(counter case).

$\mathcal{C}^+ \triangleq \{S' |$all finite subsets of $S'$are in $\mathcal{C}\}$ is consistency property. Let $S' \in \mathcal{C}^+$.

6. case $\delta \in S'$: Consider finite set $T \subseteq S' \cup \{\delta(c)\}$ for some $c \in$ **par**.
   Therefore, $(T - \{\delta(c)\}) \subseteq S'$.
   Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \subseteq S'$. Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \in \mathcal{C}$.
   Since $\mathcal{C}$ is consistency property, $\{\delta\} \cup (T - \{\delta(c)\}) \cup \{\delta(c')\} \in \mathcal{C}$.
   ✗Therefore, $\{\delta\} \cup T \cup \{\delta(c)\} \in \mathcal{C}$.
   Since $\mathcal{C}$ is subset closed, $T \in \mathcal{C}$. Therefore, $S' \cup \{\delta(c)\} \in \mathcal{C}^+$.

$\square$

Commentary: We need to show that every finite subset of $S' \cup \{\delta(c)\}$ is in $\mathcal{C}$

Expanded consistency property

Definition 13.5

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a signature. Let* $\mathcal{C}$ *be a collection of sets of sentences in signature* $\mathbf{S}^{\mathbf{par}}$. $\mathcal{C}$ *is a expanded consistency property wrt to* $\mathbf{S}$ *if for each* $S \in \mathcal{C}$ *satisfies the following.*

1. *for each* $F \in A_{\mathbf{S}^{\mathbf{par}}}$, *either* $F \notin S$ *or* $\neg F \notin S$
2. *if* $\neg\neg F \in S$ *then* $\{F\} \cup S \in \mathcal{C}$
3. *if* $\alpha \in S$ *then* $\{\alpha_1, \alpha_2\} \cup S \in \mathcal{C}$
4. *if* $\beta \in S$ *then* $\{\beta_1\} \cup S \in \mathcal{C}$ *or* $\{\beta_2\} \cup S \in \mathcal{C}$
5. *if* $\gamma \in S$ *then* $\{\gamma(t)\} \cup S \in \mathcal{C}$ *for each* $t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}}$
6. *if* $\delta \in S$ *then* $\{\delta(c)\} \cup S \in \mathcal{C}$ *for each* $c \in \mathbf{par}$ *and not occurring in* S
7. $S \cup \{t \approx t\} \in \mathcal{C}$ *for each* $t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}}$
8. *if* $t_1 \approx u_1, .., t_n \approx u_n \in S$ *then* $S \cup \{f(t_1, .., t_n) \approx f(u_1, .., u_n)\} \in \mathcal{C}$ *for each* $f/n \in \mathbf{F}$
9. *if* $t_1 \approx u_1, .., t_n \approx u_n$, $P(t_1, .., t_n) \in H$ *then* $S \cup \{P(u_1, .., u_n)\} \in \mathcal{C}$ *for each* $P/n \in \mathbf{R} \cup \{\approx /2\}$

Converting to extended consistency property

Definition 13.6

A parameter substitution π *is* **par** $\to$ **par**. *For a formula* F, $F\pi$ *is a formula obtained by replacing* c *by* $\pi(c)$ *in* F *for each* $c \in$ **par**.
The substitution naturally extends to a set of formulas.

Theorem 13.6

For subset-closed consistency property $\mathcal{C}$, *let* $\mathcal{C}^+ \triangleq \{S \mid \text{there is } \pi \text{ s.t. } S\pi \in \mathcal{C}\}$.

1. $\mathcal{C}^+$ *extends* $\mathcal{C}$ *and subset closed*
2. $\mathcal{C}^+$ *is expanded consistency property*

Proof.

Part 1 can be easily proved. $\square$

Exercise 13.1

Show if $F\pi$ *is sat then* F *is sat.*

Commentary: The definition allows π to be many-one. This may lead to mapping of many constants to the same constants.

Converting to extended consistency property(contd.)

Proof for part 2.

For part 2 we can easily check that conditions 2-5 and 7-8 holds.
Consider $S \in \mathcal{C}^+$.

1. Choose closed atom F.
   Assume $\{F, \neg F\} \in S$.
   There is a π s.t. $S\pi \in \mathcal{C}$.
   Since $\{F\pi, (\neg F)\pi\} \subseteq S\pi$, $\{F\pi, \neg(F\pi)\} \subseteq S\pi$. Contradiction.

6. case $\delta \in S$:
   Choose $c \in$ **par** s.t. c does not occur in S.
   Since there is a π s.t. $S\pi \in \mathcal{C}$, there is a $c' \in$ **par** s.t.
   $S\pi \cup \{\delta\pi(c')\} \in \mathcal{C}$.
   Therefore, $S\pi \cup \{\delta(c)(\pi[c \mapsto c'])\} \in \mathcal{C}$.
   Therefore, $(S \cup \{\delta(c)\})(\pi[c \mapsto c']) \in \mathcal{C}$.
   Therefore, $(S \cup \{\delta(c)\}) \in \mathcal{C}^+$. $\square$

Extension to finite character

Theorem 13.7

A subset-closed expanded consistency property $\mathcal{C}$ is extendable to one of finite character.

Proof.

$\mathcal{C}^+ \triangleq \{S' | \text{all finite subsets of } S' \text{are in } \mathcal{C}\}$ is an expanded consistency property.
Let $S' \in \mathcal{C}^+$

6. case $\delta \in S'$:
   Consider finite set $T \subseteq S' \cup \{\delta(c)\}$ for some fresh $c \in$ **par** wrt S'.
   Therefore, $(T - \{\delta(c)\}) \subseteq S'$.
   Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \subseteq S'$. Therefore, $\{\delta\} \cup (T - \{\delta(c)\}) \in \mathcal{C}$.
   Since c does not occur in $\{\delta\} \cup (T - \{\delta(c)\})$ and $\mathcal{C}$ is expanded consistency property, $\{\delta\} \cup (T - \{\delta(c)\}) \cup \{\delta(c)\} \in \mathcal{C}$.
   Therefore, $\{\delta\} \cup T \cup \{\delta(c)\} \in \mathcal{C}$.
   Since $\mathcal{C}$ is subset closed, $T \in \mathcal{C}$. Therefore, $S' \cup \{\delta(c)\} \in \mathcal{C}^+$.

Other cases are similarly proven. □

Exercise 13.2 *Prove case 8.*

Model existence theorem

Theorem 13.8

Let $\mathcal{C}$ be a consistency property wrt to $\mathbf{S}$. If $S \in \mathcal{C}$ then S is sat.

Proof.

Wlog, we assume $\mathcal{C}$ is of finite character and expanded (why?).
Let $F_1, F_2, ..$ be an enumeration of all the sentences of $\mathbf{S}^{\mathbf{par}}$ in an order(why?).

Let us define a sequence $S_1, S_2, \dots$ as follows.

$$S_1 = S \qquad S_{n+1} = \begin{cases} S_n \cup \{F_n, \delta(c)\} & S_n \cup \{F_n\} \in \mathcal{C} \text{ and } F_n = \delta \\ S_n \cup \{F_n\} & S_n \cup \{F_n\} \in \mathcal{C} \text{ and } F_n \neq \delta \\ S_n & \text{otherwise} \end{cases}$$

where c is a fresh parameter wrt $S_n \cup \{F_n\}$.

Since S_n are in $\mathcal{C}$ and $\mathcal{C}$ is of finite character, $\bigcup_n S_n \in \mathcal{C}$. Let $M \triangleq \bigcup_n S_n$.

Exercise 13.3

Why we need the special case for δ?

Model existence theorem(contd.)

Proof.

Claim: M is maximal in $\mathcal{C}$. (same argument as in propositional logic)

Assume $M' \in \mathcal{C}$ s.t.$M \subset M'$. There is F_n such that $F_n \in M'$ and $F_n \notin M$.

By def. of M, $S_n \cup \{F_n\} \notin \mathcal{C}$.

Since $S_n \cup \{F_n\} \subseteq M'$ and $\mathcal{C}$ is subset closed, $S_n \cup \{F_n\} \in \mathcal{C}$. Contradiction.

Claim: M is a Hinttika set.

If $\alpha \in M$ then $\{\alpha_1, \alpha_2\} \cup M \in \mathcal{C}$. Since M is maximal, $\{\alpha_1, \alpha_2\} \subseteq M$.

Other conditions hold similarly,except δ case.

Since M is a Hinttika set, M is sat. Since $S \subseteq M$, S is sat. □

Topic 13.2

Consequences of model existence theorem

Compactness

Theorem 13.9

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a signature and* S *be a set of* $\mathbf{S}$*-sentences.*
If each finite subset of S *is sat then* S *is sat.*

Proof.

Let $\mathcal{C} \triangleq \{S' \subset \mathbf{S^{par}}\text{-sentences} |$ all finite subsets of S' are sat and S' is unexhausted $\}$.

Claim: $\mathcal{C}$ is a consistency property.

Let $S' \in \mathcal{C}$. We need to satisfy the nine conditions.

1. If $\{F, \neg F\} \subseteq S'$, then $\{F, \neg F\}$ is sat. contradiction. First cond. holds.
3. Let $\alpha \in S'$. Consider a finite $T \subseteq \{\alpha_1, \alpha_2\} \cup S'$.
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\alpha, \alpha_1, \alpha_2\} \cup T'$.
   Since $T' \cup \{\alpha\} \subseteq S'$, $T' \cup \{\alpha\}$ is sat.
   Therefore, $T' \cup \{\alpha, \alpha_1, \alpha_2\}$ is sat.
   Therefore, T is sat.
   Therefore, every finite subset of $\{\alpha_1, \alpha_2\} \cup S'$ is sat.
   Therefore, $\{\alpha_1, \alpha_2\} \cup S' \in \mathcal{C}$

Compactness (contd.)

Exercise 13.4

Prove the δ case.

Proof(contd.)

6. Let $\delta \in S'$.
   Consider a finite $T \subseteq \{\delta(c)\} \cup S'$ for fresh $c \in$ **par** (why possible?).
   There is a finite $T' \subseteq S'$ s.t. $T \subseteq \{\delta, \delta(c)\} \cup T'$.
   Since $T' \cup \{\delta\} \subseteq S'$, $T' \cup \{\delta\}$ is sat.
   Therefore, $T' \cup \{\delta, \delta(c)\}$ is sat.
   Therefore, T is sat.
   Therefore, every finite subset of $\{\delta(c)\} \cup S'$ is sat.
   Therefore, $\{\delta(c)\} \cup S' \in \mathcal{C}$
7. .... similarly other cases are proven.

Due to model existence theorem, S is sat. □

Exercise 13.5

If $\Sigma \models F$ then there is a finite subset S of Σ such that $S \models F$

Impossibility of encoding finite models

Theorem 13.10

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a signature and* S *be a set of* $\mathbf{S}$*-sentences. If* S *is sat in arbitrary large finite models then* S *is true in an infinite model.*

Proof.

Let $E/2$ be a predicate symbol that is not in $\mathbf{S}$. Let $\mathbf{S}' \triangleq (\mathbf{F}, \mathbf{R} \cup \{E/2\})$.
As we have seen, let F_i be a $\mathbf{S}'$-sentence only using predicate E that is false in models with domain smaller than i and sometimes true in larger models.

Let $S' \triangleq S \cup \{F_1, F_2, F_3, \dots\}$. By construction, S' is false in any finite model.

claim: S' is sat.
Let L be a finite subset of S' and k be the largest number s.t. $F_k \in L$.
Since S is sat in arbitrary large finite models and S does not mention E, L is sat in a model larger than k(why?).
Due to compactness, S' is sat.

Therefore, S' is true in an infinite model. □

Löwenheim-Skolem Theorem

Theorem 13.11

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ *be a countable signature and* S *be a set of* $\mathbf{S}$*-sentences.*
If S *is sat then* S *is true in a countable model.*

Proof.

Let $\mathcal{C} \triangleq \{S' \subset \mathbf{S}^{\mathbf{par}}\text{-sentences} |\ S' \text{ is sat and unexhausted } \}$.

We can show $\mathcal{C}$ is a consistency property.

Since $S \in \mathcal{C}$, there is a Herbrand model of S wrt $\mathbf{S}^{\mathbf{par}}$, which is countable. □

Remark:

For each satisfiable set of first order sentences, we have a countable model therefore real numbers can not be axiomatized using formulas in FOL.

Actually the story is more complicated. Check out "skolem's paradox"!

Topic 13.3

Problems

Finite paths

Exercise 13.6

A graph is connected if there is a finite path between any two nodes of the graph. Using compactness theorem, prove no first order formula can express connectedness of a graph.

End of Lecture 13