Mathematical Logic 2016

Lecture 14: FOL Proof methods - tableaux and resolution

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Where are we and where are we going?

We have seen

- FOL syntax and semantics
- model existence theorem for FOL

We will see FOL proof systems

- tableaux
- resolution

Formulas to sentences

Theorem 14.1

A formula F is true in m iff $\forall x.F$ is true in m.

We turn the question of proving validity of formulas into proving of validity of sentences.

Exercise 14.1

Prove the above theorem.

Topic 14.1

Tableaux

Tableaux

Definition 14.1

Let $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ be a signature and Σ be a set of $\mathbf{S}$-sentences. A tableaux T for Σ is a finite labelled tree that is initially empty and expanded according to the following tableaux expansion rules.

1. *$F \in \Sigma$ labelled node is added as a child to a leaf or root if empty tree*
2. *Let v be a leaf of T.*
   - 2.1 *If an ancestor of v is labelled with F then*
     - 2.1.1 *$F = \neg\neg G$: a child is added to v with label G*
     - 2.1.2 *$F = \alpha$: a child and grand child to v added with labels α_1 and α_2*
     - 2.1.3 *$F = \beta$: two children to v are added with labels β_1 and β_2*
     - 2.1.4 *$F = \gamma$: a child to v is added with label $\gamma(t)$ for some $t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}}$*
     - 2.1.5 *$F = \delta$: a child to v is added with label $\delta(c)$ for $c \in$ **par** that has not occurred in T*
   - 2.2 *a child is added to v with label $t \approx t$ for some $t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}}$*
   - 2.3 *If two ancestors of v are labelled with $F(t)$ and $t \approx u$ then a child is added to v with label $F(u)$*

Example : Tableaux expansion

Example 14.1

Consider example $\Sigma = \{\neg(\forall x.\ (P(x) \vee Q(x)) \Rightarrow \exists x.P(x) \vee \forall x.Q(x))\}$

$$
\begin{array}{c}
\neg(\forall x.\ (P(x) \vee Q(x)) \Rightarrow \exists x.P(x) \vee \forall x.Q(x)) \\
\downarrow \\
\forall x.\ (P(x) \vee Q(x)) \\
\downarrow \\
\neg(\exists x.P(x) \vee \forall x.Q(x)) \\
\downarrow \\
\neg\exists x.P(x) \\
\downarrow \\
\neg\forall x.Q(x) \\
\downarrow \\
\neg Q(c) \\
\downarrow \\
\neg P(c)
\end{array}
$$

Recall: closed tableaux

Definition 14.2
A branch of a tableaux is closed if it contains two nodes with labels F and $\neg F$ for some sentence F.

Definition 14.3
A branch of a tableaux is atomically closed if it contains two nodes with labels F and $\neg F$ for $F \in A_{\mathbf{S}}$.

Definition 14.4
A tableaux is (atomically) closed if every branch of the tableaux is (atomically) closed.

Exercise 14.2
Prove that a closed tableaux can always be expanded to a atomically closed tableaux.

Example : closed tableaux

Example 14.2 $\neg(\forall x.\,(P(x) \vee Q(x)) \Rightarrow \exists x.P(x) \vee \forall x.Q(x))$

$$\downarrow$$

$$\forall x.\,(P(x) \vee Q(x))$$

$$\downarrow$$

$$\neg(\exists x.P(x) \vee \forall x.Q(x))$$

$$\downarrow$$

$$\neg\exists x.P(x)$$

$$\downarrow$$

$$\neg\forall x.Q(x)$$

$$\downarrow$$

$$\neg Q(c)$$

$$\downarrow$$

$$\neg P(c)$$

$$\downarrow$$

$$P(c) \vee Q(c)$$

$$\swarrow \quad \searrow$$

$$P(c) \qquad Q(c)$$

Proved by tableaux

Definition 14.5

For a set of sentences Σ. If there exists a closed tableaux then we call Σ tableaux inconsistent. Otherwise, tableaux consistent.

Definition 14.6

If $\Sigma \cup \{\neg F\}$ is tableaux inconsistent then we write $\Sigma \vdash_t F$.

Definition 14.7

We write $\vdash_t F$ to denote $\emptyset \vdash_t F$. We say F is a theorem of tableaux proof method and the closed tableaux is a tableaux proof of F.

Practice tableaux

Exercise 14.3

Prove the following formulas are theorems in tableaux proof method

1. $\forall x.\exists y.\forall z.\exists w.(R(x,y) \lor \neg R(w,z))$
2. $\forall x.\exists y.x \approx y$
3. $\forall x.\forall y.((x \approx y \land f(y) \approx g(y)) \Rightarrow (h(f(x)) \approx h(g(y))))$

Soundness of tableaux

Definition 14.8

A branch ρ of a tableaux T is sat if the set of formulas that are labels of the nodes of the branch is sat. T is sat if there is ρ in T that is sat.
If the model involved is m, we write $m \models \rho$ and $m \models T$.

Theorem 14.2

If Σ is sat then a tableaux T for Σ is sat

Proof.

Let model $m \models \Sigma$. **base case:** empty T is sat in any model.
induction step: Assume $m \models T$. Let ρ be a branch of T s.t. $m \models \rho$.
Let T' be a tableaux obtained after application of an expansion rule.

- We will skip propositional cases.
- **case** ρ is expanded using $\gamma \in \rho$:
  Let t be the term that is used in the expansion. Let $d = m(t)$.
  Due the definition of universal quantification $m, \{x \mapsto d\} \models \gamma(x)$.
  Therefore, $m \models \gamma(t)$. Therefore, $m \models \rho\gamma(t)$.Therefore, $m \models T'$.

Soundness of tableaux (contd.)

Proof(contd.)

- **case** ρ is expanded using $\delta \in \rho$:
  Let c be the fresh constant that is used in the expansion.
  Due the def. of existential quantification, there is a $d \in D_m$ s.t.
  $m, \{x \mapsto d\} \models \delta(x)$.
  Let $m' := m[c \mapsto d]$.
  Since c does not occur in ρ, $m' \models \rho$. Therefore, $m' \models \delta(c)$.
  Therefore, $m' \models \rho\delta(c)$. Therefore, $m' \models T'$.
- **case** ρ is expanded using $t \approx t$: $m \models t \approx t$, $m \models \rho(t \approx t)$.
  Therefore, $m \models T'$.
- **case** ρ is expanded using $t \approx u, F(t) \in \rho$:
  Since $m \models F(t)$, $m, \{x \mapsto m(t)\} \models \delta(x)$.
  Since $m(t) = m(u)$, $m, \{x \mapsto m(u)\} \models \delta(x)$.
  Therefore, $m \models \delta(u)$.Therefore, $m \models \rho\delta(u)$. Therefore, $m \models T'$.

□

Completeness of tableaux

Theorem 14.3

The collection of sets of $\mathbf{S}^{\mathbf{par}}$-sentences that are tableaux consistent is a consistency property.*

Proof.

Let Σ be a tableaux consistent set. We need to show the 9 conditions hold.

1. If $\{F, \neg F\} \subseteq \Sigma$ then there is a closed tableaux. Therefore, $\{F, \neg F\} \not\subseteq \Sigma$

6. If $\delta \subseteq \Sigma$. Suppose $\{\delta(c)\} \cup \Sigma$ has a closed tableaux T.
   Then, we can construct a closed tableaux for Σ as follows.

$$\begin{array}{c} \delta \\ \downarrow \\ \delta(c) \\ \downarrow \\ T' \end{array}$$

   where T' is obtained by removing all the nodes with label $\delta(c)$ in T if the node was added due to the introduction rule. Contradiction.
   Therefore, $\{\delta(c)\} \cup \Sigma$ is tableaux consistent.

Exercise 14.4

Prove case 8.

Completeness of tableaux (contd.)

Proof(contd.)

4. If $\beta \subseteq \Sigma$.
   Suppose $\{\beta_1\} \cup \Sigma$ and $\{\beta_2\} \cup \Sigma$ have closed tableaux T_1 and T_2 respectively. Then, we can construct a closed tableaux for Σ as follows.

$$
\begin{array}{ccccc}
 & \swarrow & \beta & \searrow & \\
\beta_1 & & & & \beta_2 \\
\downarrow & & & & \downarrow \\
T_1' & & & & T_2'
\end{array}
$$

   where T_1' and T_2' are obtained by removing all the nodes with label β_1 and β_2 in T_1 and T_2 respectively if the node was added due to the introduction rule.
   Furthermore, we need to assume T_1 and T_2 used disjoint sets of parameters(why?).
   Contradiction.
   Either $\{\beta_1\} \cup \Sigma$ and $\{\beta_2\} \cup \Sigma$ are tableaux consistent.
5. ... other cases have similar proofs. $\square$

Tableaux sound and complete

Theorem 14.4

$\Sigma \vdash_t F$ *iff* $\Sigma \models F$

Exercise 14.5

Prove the above theorem.

Topic 14.2

Resolution

Proof rules

$$\textsc{RuleName}\ \frac{\text{Stuff-already-there}}{\text{Stuff-to-be-added}}\ \text{Conditions to be met}$$

Resolution derivation

Definition 14.9

A *resolution derivation* *R for a set of $\mathbf{S}$-sentences Σ is a finite sequence of clauses that are generated by the following resolution expansion rules.*

$$\textsc{Intro}\frac{}{\{F\}}F \in \Sigma \qquad \textsc{Db-Neg}\frac{\{\neg\neg F\} \cup C}{\{F\} \cup C} \qquad \alpha\text{-}\textsc{Rule}\frac{\{\alpha\} \cup C}{\begin{array}{c}\{\alpha_1\} \cup C\\ \{\alpha_2\} \cup C\end{array}}$$

Proof rules

$$\beta\text{-}\textsc{Rule}\frac{\{\beta\} \cup C}{\{\beta_1, \beta_2\} \cup C} \qquad \textsc{Res}\frac{\{\neg F\} \cup C \quad \{F\} \cup D}{C \cup D}$$

$$\gamma\text{-}\textsc{Rule}\frac{\{\gamma\} \cup C}{\{\gamma(t)\} \cup C}t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}} \qquad \delta\text{-}\textsc{Rule}\frac{\{\delta\} \cup C}{\{\delta(c)\} \cup C}\textit{fresh } c \in \mathbf{par}$$

$$\textsc{Ref}\frac{}{\{t \approx t\}}t \in \hat{T}_{\mathbf{S}^{\mathbf{par}}} \qquad \textsc{Replace}\frac{\{t \approx u\} \cup C \quad \{F(t)\} \cup D}{\{F(u)\} \cup C \cup D}$$

Recall: closed resolution

Definition 14.10

A resolution derivation is closed if it contains empty clause.

Definition 14.11

A resolution derivation is atomically closed if every resolution is applied using an atomic sentence.

Exercise 14.6

Prove that a closed resolution can always be transformed into a atomically closed resolution.

Example : closed resolution derivation

Example 14.3

1. $\{\neg(\forall x.\,(P(x) \vee Q(x)) \Rightarrow \exists x.P(x) \vee \forall x.Q(x))\}$
2. $\{\forall x.\,(P(x) \vee Q(x))\}$
3. $\{\neg(\exists x.P(x) \vee \forall x.Q(x))\}$ *//applying* α-Rule *on 1*
4. $\{\neg\exists x.P(x)\}$
5. $\{\neg\forall x.Q(x)\}$ *//applying* α-Rule *on 3*
6. $\{\neg Q(c)\}$ *//applying* δ-Rule *on 5*
7. $\{\neg P(c)\}$ *//applying* γ-Rule *on 4*
8. $\{P(c) \vee Q(c)\}$ *//applying* γ-Rule *on 2*
9. $\{P(c), Q(c)\}$ *//applying* β-Rule *on 8*
10. $\{P(c)\}$ *//applying* Res *on 9 and 6*
11. $\{\}$ *//applying* Res *on 7 and 10*

Proved by resolution

Definition 14.12
For a set of sentences Σ. If there exists a closed resolution then we call Σ resolution inconsistent. Otherwise, resolution consistent.

Definition 14.13
If $\Sigma \cup \{\neg F\}$ is resolution inconsistent then we write $\Sigma \vdash_r F$.

Definition 14.14
We write $\vdash_r F$ to denote $\emptyset \vdash_r F$. We say F is a theorem of resolution proof method and the closed tableaux is a resolution proof of F.

Practice resolution

Exercise 14.7

Prove the following formulas are theorems in resolution proof method

1. $\forall x.\exists y.\forall z.\exists w.(R(x,y) \lor \neg R(w,z))$
2. $\forall x.\exists y.x \approx y$
3. $\forall x.\forall y.((x \approx y \land f(y) \approx g(y)) \Rightarrow (h(f(x)) \approx h(g(y))))$

Soundness of resolution

Definition 14.15

A resolution derivation R is sat if the set of clauses in R is sat. If the model involved is m, we write $m \models R$.

Theorem 14.5

If Σ is sat then a resolution derivation R for Σ is sat.

Proof.

Proof is very much like Tableaux. □

Exercise 14.8

Prove that models are preserved after application of the $\textsc{Res}$ *rule.*

Completeness of resolution

Theorem 14.6

The collection of sets of $\mathbf{S^{par}}$-sentences that are resolution consistent is a consistency property.

Proof.

Again the same proof as propositional case.
Only added complication is the variable renaming similar to Tableaux. □

Theorem 14.7

$\Sigma \vdash_r F$ *iff* $\Sigma \models F$

Proof.

Due to soundness and completeness of resolution. □

Topic 14.3

Metatheorems

A confusion!

You may have noticed
that
we have theorems on the slides
and
there are theorems and proofs within the proof systems.

Theorems and Metatheorems

We are dealing with two level of mathematics.

We say theorems on the slides are metatheorems and others are theorems.

Theorems vs. Metatheorems

Metatheorems are due to our intuition and
theorems are due to the formal structure that is built using the intuition.

They look similar but they are different WORLDS.
As we will progress the worlds will collide.

Topic 14.4

Problems

Proof via resolution

Exercise 14.9

Consider the following formulas

$$\begin{aligned}\Sigma = \{\, & \forall x, y, z.\ (z \in x \Leftrightarrow z \in y) \Rightarrow x \approx y,\\ & \forall x, y.\ (x \subseteq y \Leftrightarrow \forall z.\ (z \in x \Rightarrow z \in y)),\\ & \forall x, y, z.\ (z \in x - y \Leftrightarrow (z \in x \wedge z \notin y))\}.\end{aligned}$$

Prove the following using resolution proof system

$$\Sigma \models \forall x, y.\ x \subseteq y \Rightarrow \exists z.(y - z \approx x)$$

End of Lecture 14