Mathematical Logic 2016

Lecture 21: Gödel's incompleteness theorem I

Instructor: Ashutosh Gupta

TIFR, India

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Where are we and where are we going?

We have seen

- Definition of FO-theories
- an algorithm that decides quantifier free formulas for decidable theories

We will

- start showing that number theory is not axiomatizable and the endeavor will last 2 lectures
- define representability of relations

Gödel's incompleteness theorem

Theorem 21.1

$m_{\mathbb{N}}$ can not be axiomatized.

Proof structure.

1. Choose a subtheory of $m_{\mathbb{N}}$ s.t. that it can encode the resolution proofs in any subtheory of $m_{\mathbb{N}}$
2. This allows us to construct a sentence and show that this sentence is true in $m_{\mathbb{N}}$ but there is no axiomatization that can deduce its validity
3. Therefore, no axiomatization of $m_{\mathbb{N}}$

□

Topic 21.1

A special subtheory of $m_{\mathbb{N}}$

A special subtheory of $m_{\mathbb{N}}$

In the next slide, we will see a subtheory of $m_{\mathbb{N}}$.

We will see that this theory will be capable of saying something very strong about decidable sets.

Our choice of the theory is not minimum. Proofs at other places uses even fewer axioms.

The subtheory of number theory

Consider signature $\mathbf{S} = (\{0/0, s/1, +/2, \cdot/2, e/2\}, \{< /2\})$.

Consider a theory $\mathcal{T}_D = Cn(A_D)$, where A_D contains the following axioms

1. $\forall x.\ s(x) \not\approx 0$
2. $\forall x, y.\ s(x) \approx s(y) \Rightarrow x \approx y$
3. $\forall x, y.\ x < s(y) \Leftrightarrow (x < y \lor x \approx y)$
4. $\forall x.\ x \not< 0$
5. $\forall x, y.\ (x < y \lor x \approx y \lor y < x)$
6. $\forall x.\ x + 0 \approx x$
7. $\forall x, y.\ x + s(y) \approx s(x + y)$
8. $\forall x.\ x \cdot 0 \approx 0$
9. $\forall x, y.\ x \cdot s(y) \approx x \cdot y + x$
10. $\forall x.\ e(x, 0) \approx s(0)$
11. $\forall x, y.\ e(x, s(y)) \approx e(x, y) \cdot x$

We will refer to the axioms by their number.

These axioms are weak. Various natural claims can not be proven. e.g. $\forall x.\ x \not\approx 0 \Rightarrow \exists y.\ x \approx s(y)$

Clearly, $\mathcal{T}_D \subseteq \mathcal{T}_{\mathbb{N}} = Th(m_{\mathbb{N}})$

Exercise 21.1

Show if $n \neq m$,
$A_D \vdash s^m(0) \not\approx s^n(0)$

Counting up to a fixed number

Theorem 21.2

a. $A_D \vdash \forall x.\ x \not< 0$

b. For any $k \in \mathbb{N}$, $A_D \vdash \forall x.\ x < s^{k+1}(0) \Leftrightarrow x \approx s^0(0) \vee .. \vee x \approx s^k(0)$

Proof.

a. claim is axiom 4

b. We prove by induction over k.

base case:

Due to axiom 3, $x < s(0) \Leftrightarrow (x < 0 \vee x \approx 0)$

Due to axiom 4, $x < s(0) \Leftrightarrow x \approx 0$

induction step:

Due to induction hypothesis, $x < s^k(0) \Leftrightarrow (x \approx s^0(0) \vee .. \vee x \approx s^{k-1}(0))$

Due to axiom 3, $x < s^{k+1}(0) \Leftrightarrow (x < s^k(0) \vee x \approx s^k(0))$

After substitution, we obtain the result.

$x < s^{k+1}(0) \Leftrightarrow (x \approx s^0(0) \vee .. \vee x \approx s^{k-1}(0) \vee x \approx s^k(0))$ $\square$

Evaluating closed terms

Theorem 21.3

For every variable-free term t, there is a unique $n \in \mathbb{N}$ s.t. $A_D \vdash t \approx s^n(0)$

Proof.

Since $m_{\mathbb{N}} \models A_D$, for $m \neq n$, $A_D \not\vdash s^n(0) \approx s^m(0)$. Therefore, uniqueness.

We prove existence of n by induction over structure of t.

base case:

$A_D \vdash 0 \approx 0$

induction step:

Due to induction hyp., let $A_D \vdash t \approx s^k(0)$ and $A_D \vdash u \approx s^l(0)$.

case s: Due to congruence, $A_D \vdash s(t) \approx s^{k+1}(0)$.

case $+$: Due to congruence, $A_D \vdash t + u \approx s^k(0) + s^l(0)$.

After l applications of axiom 7, $A_D \vdash t + u \approx s^l(s^k(0) + 0)$.

After applying axiom 6, $A_D \vdash t + u \approx s^{k+l}(0)$.

Similarly the other construction cases. $\square$

Evaluating quantifier-free(QF) sentences

Theorem 21.4

For every QF sentence F, if $\models_{\mathcal{T}_{\mathbb{N}}} F$ then $A_D \vdash F$.

Proof.

base case:

Assume $\models_{\mathcal{T}_{\mathbb{N}}} t_1 \approx t_2$.
Therefore for some n, $\models_{\mathcal{T}_{\mathbb{N}}} t_1 \approx s^n(0)$ and $\models_{\mathcal{T}_{\mathbb{N}}} t_2 \approx s^n(0)$.
Therefore due to previous theorem, $A_D \vdash t_1 \approx s^n(0)$ and $A_D \vdash t_2 \approx s^n(0)$.
Therefore, $A_D \vdash t_1 \approx t_2$.

Similarly, if $\models_{\mathcal{T}_{\mathbb{N}}} t_1 \not\approx t_2$ then $A_D \vdash t_1 \not\approx t_2$ (why?).
Again similar argument for $<$ and $\not<$.

induction step:

Since F is QF, the induction trivially follows the boolean structure $\square$

Exercise 21.2

Show for every QF sentence F, either $A_D \vdash F$ or $A_D \vdash \neg F$

Notation:
vector notation for tuple of variables/values/terms

- Let $\vec{x} := x_1, .., x_n$
- Let $\vec{a} := a_1, .., a_n$
- Let $s^{\vec{a}}(0) := s^{a_1}(0), .., s^{a_n}(0)$

Existential formula

Definition 21.1

An existential formula is of the following form.

$$\exists \vec{x}.F(\vec{x})$$

where F is QF formula.

An existential sentence is an existential formula without free variables.

Theorem 21.5

Let $\exists \vec{x}.F(\vec{x})$ be an existential sentence. If $\mathcal{T}_{\mathbb{N}} \models \exists \vec{x}.F(\vec{x})$ then $A_D \vdash \exists \vec{x}.F(\vec{x})$.

Proof.

Since $\mathcal{T}_{\mathbb{N}} \models \exists \vec{x}.F(\vec{x})$, there are terms $s^{\vec{k}}(0)$ s.t. $\mathcal{T}_{\mathbb{N}} \models F(s^{\vec{k}}(0))_{\text{(why?)}}$.

Due to the previous theorem, $A_D \vdash F(s^{\vec{k}}(0))$.

Therefore, $A_D \vdash \exists \vec{x}.F(\vec{x})$. $\square$

Unlike the last theorem, the claim is not closed under negation
For universal formulas, the above theorem does not hold

What have we been proving?

We are showing

classes of formulas whose truth value

can be established

by systematic(non-deterministic) applications of axioms of A_D.

"systematic application of axioms" is another phrase for decision procedure.

Now, we will generalize the concept of the ability to establish truth.

Topic 21.2

Representability

Definability in number theory

Definition 21.2

A relation $R \subseteq \mathbb{N}^n$, is defined by a formula $F(\vec{x})$ in $m_{\mathbb{N}}$ if

$$\vec{a} \in R \quad \textit{iff} \quad \models_{\mathcal{T}_{\mathbb{N}}} F(s^{\vec{a}}(0))$$

Exercise 21.3

Show the following relations are definable in number theory

- *divisibility relation*
- *set of prime numbers*
- *set of pairs of consecutive primes*

Exercise 21.4

a. Prove there are undefinable relations.

b. Give a relation that is not definable in number theory

Representability

Note that the definition is about some theory $\mathcal{T}$, not $\mathcal{T}_D$.

Definition 21.3

A formula F represents a relation $R \subseteq \mathbb{N}^n$ in theory $\mathcal{T}$ (with signature containing s and 0) if for each $\vec{a} \in \mathbb{N}^n$

$$\text{if } \vec{a} \in R \text{ then } F(s^{\vec{a}}(0)) \in \mathcal{T}$$
$$\text{if } \vec{a} \notin R \text{ then } \neg F(s^{\vec{a}}(0)) \in \mathcal{T}$$

Theorem 21.6

F represents R in $\mathcal{T}_{\mathbb{N}}$ iff F defines R in $m_{\mathbb{N}}$

Theorem 21.7

F represents R in $\mathcal{T}_D$ iff

$$\text{if } \vec{a} \in R \text{ then } A_D \vdash F(s^{\vec{a}}(0))$$
$$\text{if } \vec{a} \notin R \text{ then } A_D \vdash \neg F(s^{\vec{a}}(0))$$

Proof.

The above holds due to the completeness of FOL and the definition of axiomatizable theories. □

Definability vs. Representability

Definability in number theory

says that

a **S**-formula can describe a relation.

Representability in A_D

says that

A_D can deduce the membership of the relation.

We need to a bit more concertize the concept of deducible.

Numeralwise determined

Definition 21.4

Let $F(\vec{x})$ be a formula with only free variables $\vec{x}$ and $|\vec{x}| = n$.
$F(\vec{x})$ is numeralwise determined by A_D iff for every $\vec{a} \in \mathbb{N}^n$ either

$$A_D \vdash F(s^{\vec{a}}(0)) \qquad \textit{or} \qquad A_D \vdash \neg F(s^{\vec{a}}(0)).$$

Theorem 21.8

A formula $F(\vec{x})$ represents a relation R in $\mathcal{T}_D$ iff
$F(\vec{x})$ is numeralwise determined by $\mathcal{T}_D$ and $F(\vec{x})$ defines R in $m_{\mathbb{N}}$.

Proof.

Forward direction:

Since $F(\vec{x})$ represents R, $F(\vec{x})$ is numeralwise determined by $\mathcal{T}_D$.
Since $\mathcal{T}_D \subseteq \mathcal{T}_{\mathbb{N}}$, $F(\vec{x})$ defines R in $m_{\mathbb{N}}$.

Backward direction:

assume $\vec{a} \in R$.
Therefore, $\models_{m_{\mathbb{N}}} F(s^{\vec{a}}(0))$.
Since $m_{\mathbb{N}} \models A_D$, $A_D \not\vdash \neg F(s^{\vec{a}}(0))$.
Therefore, $A_D \vdash F(s^{\vec{a}}(0))$.

assume $\vec{a} \notin R$.
Therefore, $\models_{m_{\mathbb{N}}} \neg F(s^{\vec{a}}(0))$.
Therefore, $A_D \not\vdash F(s^{\vec{a}}(0))$.
Therefore, $A_D \vdash \neg F(s^{\vec{a}}(0))$. $\square$

Numeralwise determined

"Numeralwise determined by A_D" is a property of a formula.

"Representability in A_D" is a property of a relation.

The first is a means to achieve the later.

The last theorem says, if we have the first property along with definability in $m_{\mathbb{N}}$ then we have achieved later.

Let us see which class of formulas are numeralwise determined.
We will drop "by A_d" in the following slides.

A class of numeralwise determined

Theorem 21.9

a. atomic formulas are numeralwise determined.

b. if F and G are numeralwise determined then $\neg F$, $F \circ G$ are numeralwise determined, where $\circ$ is binary boolean operator

Proof.

a. we have seen how to evaluate the variable-free formulas.

b. trivial $\square$

Exercise 21.5

Complete the above argument.

Bounded quantification is numeralwise determined

Theorem 21.10

If F is numeralwise determined then so are the following formulas

$$\forall x(x < y \Rightarrow F(x, y, \vec{z}))$$
$$\exists x(x < y \wedge F(x, y, \vec{z}))$$

Proof.

Consider $\exists x(x < y \wedge F(x, y, \vec{z}))$, where $|\vec{z}| = n$. Choose $(c, \vec{a}) \in \mathbb{N}^{n+1}$.

We need to show that either

$$A_D \vdash \exists x(x < s^c(0) \wedge F(x, s^c(0), s^{\vec{a}}(0)))$$

or

$$A_D \vdash \neg\exists x(x < s^c(0) \wedge F(x, s^c(0), s^{\vec{a}}(0))).$$

Pushing negation inside the later case we obtain
$A_D \vdash \forall x(x < s^c(0) \Rightarrow \neg F(x, s^c(0), s^{\vec{a}}(0)))$.

Bounded quantification is numeralwise determined(contd.)

Proof(contd.)

Let us suppose there is a $0 \leq a_0 < c$ s.t. $A_D \vdash F(s^{a_0}(0), s^c(0), s^{\vec{a}}(0))$.

Therefore, $A_D \vdash s^{a_0}(0) < s^c(0)$

$A_D \vdash s^{a_0}(0) < s^c(0) \land F(s^{a_0}(0), s^c(0), s^{\vec{a}}(0))$

Therefore, the first possibility occurs

$$A_D \vdash \exists x(x < s^c(0) \land F(x, s^c(0), s^{\vec{a}}(0)))$$

Bounded quantification is numeralwise determined(contd.)

Proof(contd.)

Now suppose for each $0 \leq a_0 < c$ s.t. $A_D \vdash \neg F(s^{a_0}(0), s^c(0), s^{\vec{a}}(0))$.

Since we know

$$A_D \vdash \neg\forall x.(x < s^{a_0}(0) \Rightarrow x \approx s^0(0) \vee .. \vee x \approx s^{a_0}(0))$$

Therefore, we can show the following

$$A_D \vdash \forall x(x < s^c(0) \Rightarrow \neg F(x, s^c(0), s^{\vec{a}}(0)))$$

The other formula is shown numeralwise determined similarly. □

Exercise 21.6

Write a resolution proof that proves the last formula given the top two.

Exercise 21.7

Consider $F(v_1) = s(0) < v_1 \wedge \forall x.\ (x < v_1 \Rightarrow \forall y(y < v_1 \Rightarrow x \cdot y \approx v_1))$.
Numeralwise determine $F(s(s(s(0))))$ *and* $F(s(s(s(s(0)))))$.

Some closure properties of representable relations

Theorem 21.11

The class of representable relations is closed under union, intersection, and negation.

Proof.

trivially due to theorem 21.9 □

Theorem 21.12

If $R \subseteq \mathbb{N}^{n+1}$ is representable then the following relations are also representable

$$\{(\vec{a}, b) | \textit{for each } c < b, (\vec{a}, c) \in R\}$$

and

$$\{(\vec{a}, b) | \textit{there is } c < b, (\vec{a}, c) \in R\}.$$

Proof.

Trivially follows from theorem 21.10. □

Topic 21.3

Representable functions

Function as relation and function as function

Definition 21.5

A function $f : \mathbb{N}^n \to \mathbb{N}$ may be viewed as a relation.

$$(\vec{a}, b) \in f \textit{ iff } f(a) = b$$

In the lhs, f is referred as relation.

A formula may represent a function (viewed as a relation).

However, we need further definitions since the above definition is not conducive for function composition, etc.

Representable function

Definition 21.6

Let $f : \mathbb{N}^n \to \mathbb{N}$ be a function.
A formula $F(\vec{x})$ with $|\vec{x} = n+1|$ functionally represents f if for every $\vec{a} \in \mathbb{N}^n$,

$$A_D \vdash \forall y.\ (F(s^{\vec{a}}(0), y) \Leftrightarrow y \approx s^{f(\vec{a})}(0))$$

Theorem 21.13

If $F(\vec{x})$ functionally represents f then $F(\vec{x})$ represents f as relation

Proof.

Since $F(\vec{x})$ functionally represents f, we have for each $\vec{a} \in \mathbb{N}^n$
$A_D \vdash \forall y.\ (F(s^{\vec{a}}(0), y) \Leftrightarrow y \approx s^{f(\vec{a})}(0))$.

For any $b \in \mathbb{N}$, $A_D \vdash (F(s^{\vec{a}}(0), s^b(0)) \Leftrightarrow s^b(0) \approx s^{f(\vec{a})}(0))$

If $(\vec{a}, b) \in f$, then rhs of $\Leftrightarrow$ is trivially true. Therefore, $A_D \vdash F(s^{\vec{a}}(0), s^b(0))$.

If $(\vec{a}, b) \notin f$, then the rhs is false by A_D. Therefore, $A_D \vdash \neg F(s^{\vec{a}}(0), s^b(0))$.

□

Converse of function representability

Theorem 21.14

If $F(\vec{x}, y)$ represents f as a relation, there is $F'(\vec{x}, y)$ that functionally represents f.

Proof.

Let $F'(\vec{x}, y) = F(\vec{x}, y) \land \forall z.\ (z < y \Rightarrow \neg F(\vec{x}, z))$.
For each $\vec{a} \in \mathbb{N}^n$, we show $A_D \vdash \forall y.\ (F'(s^{\vec{a}}(0), y) \Leftrightarrow y \approx s^{f(\vec{a})}(0))$.

Since $F(\vec{x}, y)$ represents f for each $b < f(\vec{a})$, $A_D \vdash \neg F(s^{\vec{a}}(0), s^b(0))$.
Since $A_D \vdash \forall x.\ (x < s^{f(\vec{a})}(0) \Rightarrow x \approx s^0(0) \lor .. \lor x \approx s^{f(\vec{a})-1}(0))$,

$$A_D \vdash \forall z.\ (z < s^{f(\vec{a})}(0) \Rightarrow \neg F(s^{\vec{a}}(0), z)) \qquad (*)$$

Therefore the following holds,

$$A_D \vdash F(s^{\vec{a}}(0), s^{f(\vec{a})}(0)) \land \forall z.\ (z < s^{f(\vec{a})}(0) \Rightarrow \neg F(s^{\vec{a}}(0), z)),$$

which is the backward implication in the desired formula.

Converse of function representability (contd.)

Proof(contd.)

Now we consider forward direction of the implication.
Lets assume lhs $F'(s^{\vec{a}}(0), y) = F(s^{\vec{a}}(0), y) \land \forall z. (z < y \Rightarrow \neg F(s^{\vec{a}}(0), z))$.

Due to equation (*), $A_D, F'(s^{\vec{a}}(0), y) \vdash \neg(y < s^{f(\vec{a})}(0))$.

Instantiate z by $s^{f(\vec{a})}(0)$, we obtain $A_D, F'(s^{\vec{a}}(0), y) \vdash \neg(s^{f(\vec{a})}(0) < y)$.

Due to axiom 5, $A_D, F'(s^{\vec{a}}(0), y) \vdash y \approx s^{f(\vec{a})}(0)$.

Therefore, $A_D \vdash \forall y. F'(s^{\vec{a}}(0), y) \Rightarrow y \approx s^{f(\vec{a})}(0)$. $\square$

Exercise 21.8

Why F could not represent f functionally and we need to construct F'?

Commentary: Note that y appeared as a free variable in the left hand side of $\vdash$.

Representing composition of functions

Theorem 21.15

If $g : \mathbb{N}^n \to \mathbb{N}$ and $h_i : \mathbb{N}^\ell \to \mathbb{N}$ are functionally representable then the following composition is also functionally representable.

$$f(\vec{a}) = g(h_1(\vec{a}), .., h_n(\vec{a}))$$

Proof.

Let $G(\vec{x}, y)$ functionally represent g and $H_i(\vec{z}, y)$ functionally represent h_i.

We show $F(\vec{z}, y) \triangleq \forall x_1. (H_1(\vec{z}, x_1) \Rightarrow ..(\forall x_n. H_n(\vec{z}, x_n) \Rightarrow G(\vec{x}, y))..)$ functionally represents f, i.e., for every $\vec{a}$,

$$A_D \vdash \forall y. (F(s^{\vec{a}}(0), y) \Leftrightarrow y \approx s^{g(h_1(\vec{a}),..,h_n(\vec{a}))}(0)).$$

So, we have

$$\forall y. (H_1(s^{\vec{a}}(0), y) \Leftrightarrow y \approx s^{h_1(\vec{a})}(0))$$

$$\vdots$$

$$\forall y. (H_n(s^{\vec{a}}(0), y) \Leftrightarrow y \approx s^{h_n(\vec{a})}(0))$$

$$\forall y. (G(s^{h_1(\vec{a})}(0), .., s^{h_n(\vec{a})}(0), y) \Leftrightarrow y \approx s^{g(h_1(\vec{a}),..,h_n(\vec{a})))}(0))$$

Representing composition of functions(contd.)

Proof(contd.)

forward direction:

We assume $A_D \vdash \forall x_1..x_n.\ (H_1(s^{\vec{a}}(0), x_1) \Rightarrow ..(H_n(s^{\vec{a}}(0), x_n) \Rightarrow G(\vec{x}, y))..)$.

Since we can instantiate x_1 to x_n with any term, let $x_i = s^{h_i(\vec{a})}(0)$.

We obtain, $A_D \vdash (H_1(s^{\vec{a}}(0), s^{h_1(\vec{a})}(0)) \Rightarrow ..(H_n(s^{\vec{a}}(0), s^{h_n(\vec{a})}(0)) \Rightarrow G(s^{h_1(\vec{a})}(0), .., s^{h_n(\vec{a})}(0), y))..)$.

Since lhs's are true (why?), we obtain $A_D \vdash G(s^{h_1(\vec{a})}(0), .., s^{h_n(\vec{a})}(0), y)$.

Due to assumptions, we obtain $y \approx s^{g(h_1(\vec{a}), .., h_n(\vec{a}))}(0)$.

Representing composition of functions(contd.)

Proof(contd.)

backward direction:

We need to show

$$A_D \vdash \forall x_1..x_n.\ (H_1(s^{\vec{a}}(0), x_1) \Rightarrow ..(H_n(s^{\vec{a}}(0), x_n) \Rightarrow G(\vec{x}, s^{g(h_1(\vec{a}),..,h_n(\vec{a}))}(0))))..)$$

If any of $x_i \not\approx s^{h_i(\vec{a})}(0)$, then the lhs chain is false (why?).
Therefore, matrix of the formula is trivially provable.

If all of $x_i \approx s^{h_i(\vec{a})}(0)$ then we need to prove
$A_D \vdash G(s^{h_1(\vec{a})}(0), .., s^{h_n(\vec{a})}(0), s^{g(h_1(\vec{a}),..,h_n(\vec{a}))}(0))$.
Again, it is provable due to assumptions. $\square$

Topic 21.4

More representable function and relations

Representing least zero

Theorem 21.16

Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be a representable function.
Then the following function f is also representable.

$$f(\vec{a}) = min\{b|g(\vec{a}, b) = 0\} = \underbrace{\mu b(g(\vec{a}, b) = 0)}_{\textit{new notation}}$$

Proof.

Let $G(\vec{x}, y, z)$ represents g (relationally).

Consider the formula $F(\vec{x}, y)$.

$$F(\vec{x}, y) \triangleq G(\vec{x}, y, 0) \land (\forall z. \, (z < y) \Rightarrow (\neg G(\vec{x}, z, 0)))$$

Since $F(\vec{x}, y)$ is numerically determined and F defines f in $m_{\mathbb{N}}$, F represents f. ∎

Characteristic functions

Theorem 21.17

If a relation R is representable iff its characteristic function K_R is representable.

$$K_R(\vec{a}) = \begin{cases} 1 & \vec{a} \in R \\ 0 & \vec{a} \notin R \end{cases}$$

Proof.

Forward direction:

Let $F(\vec{x})$ represents R. Then the $F'(\vec{x}, y)$ represents K_R, which defined as follows.

$$F'(\vec{x}, y) \triangleq (F(\vec{x}) \land y \approx s(0)) \lor (\neg F(\vec{x}) \land y \approx 0)$$

Backward direction:

Let $F(\vec{x}, y)$ represents K_R. Then, $F(\vec{x}, s(0))$ represents R. $\square$

Notation: extended least-zero functions

Theorem 21.18

Let $R \subseteq \mathbb{N}^{n+1}$ be a representable relation s.t. for every $\vec{a}$ there is b s.t. $(\vec{a}, b) \in R$. Then, the following function is representable.

$$f(\vec{a}) = \min\{b | (\vec{a}, b) \in R\} = \underbrace{\mu b((\vec{a}, b) \in R)}_{\textit{new notation}}$$

Proof.

$$f(\vec{a}) = \mu b(K_{\bar{R}}(\vec{a}, b) = 0)$$

□

Composition relation

Theorem 21.19

If R is a representable relation and $f_1, .., f_n$ are representable functions, then

$$R' = \{\vec{a} | (f_1(\vec{a}), .., f_n(\vec{a})) \in R\}.$$

is representable.

Proof.

Since R is representable, K_R is representable.
Therefore, the following composition is representable.

$$K_{R'} = K_R(f_1(\vec{a}), .., f_n(\vec{a}))$$

Therefore, R' is representable. □

Exercise 21.9

Suppose R is representable.
Show $\{(x, y) | (y, x, x) \in R\}$ is representable.

Non-strict bounded quantification

Theorem 21.20

If $R \subseteq \mathbb{N}^n$ is representable then the following relations are also representable

$$R_\forall = \{(\vec{a}, b) | \textit{for each } c \leq b, (\vec{a}, c) \in R\}$$

and

$$R_\exists = \{(\vec{a}, b) | \textit{there is } c \leq b, (\vec{a}, c) \in R\}.$$

Proof.

Let $R'_\forall = \{(\vec{a}, b) | \text{for each } c < b, (\vec{a}, c) \in R\}$, which is representable due to the bounded quantification theorem.

Since $R_\forall = \{(\vec{a}, b) | (\vec{a}, s(b)) \in R'_\forall\}$, due to previous theorem $R_\forall$ is representable.

Similarly, $R_\exists$ is representable. □

Are you confused/bored by now?

Review the content before the next lecture and try to grasp the following ideas

- representable = numeralwise determined + number theory definable
- representability means decidability
- The relations we have shown to be representable have clear sequence of instructions for applying deductions
- In other words, numeralwise determined == executable

Next, we use the power of representable relations

End of Lecture 21