Program verification 2016

Lecture 2: Symbolic methods

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Where are we and where are we going?

We have

- defined a simple language
- defined small step operation semantics of the language
- defined logical view of program statements
- defined strongest post and weakest pre
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We will

- Hoare logic
- labelled transition system
- we cover some methods that try/avoid to compute lfp

Topic 3.1

Hoare logic

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- Such a proof method has following steps
  - write (guess) a super set of reachable states
  - show it is actually a super set
  - show it does not intersect with error states

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- Computing a super set of the reachable states(lfp) that does not intersect with error states should be suffice for our goal
- Since we do not know how to compute lfp, we will first see a method of writing pen-paper proofs of program safety
- Such a proof method has following steps
  - write (guess) a super set of reachable states
  - show it is actually a super set
  - show it does not intersect with error states
- First such method was proposed by Tony Hoare
  - it is sometimes called axiomatic semantics

Hoare Triple

Definition 3.1

$$\{P\}\mathsf{c}\{Q\}$$

- $P : \Sigma(V)$, *usually called precondition*
- $\mathsf{c} : \mathcal{P}$
- $Q : \Sigma(V)$, *usually called postcondition*

Hoare Triple

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Definition 3.2

$\{P\}\mathtt{c}\{Q\}$ *is valid if all the executions of* $\mathtt{c}$ *that start from* P *end in* Q*, i.e.,*

$$\forall v, v'.\ v \models P \wedge ((v, \mathtt{c}), (v', \mathtt{skip})) \in T^* \Rightarrow v' \models Q.$$

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Example 3.1

Program

$$
\begin{array}{l}
\texttt{assume}(\top) \\
\texttt{r} := 1; \\
\texttt{i} := 1; \\
\texttt{while}(\texttt{i} < 3) \\
\{ \\
\quad \texttt{r} := \texttt{r} + \texttt{z}; \\
\quad \texttt{i} := \texttt{i} + 1 \\
\} \\
\texttt{assert}(\texttt{r} = 2\texttt{z} + 1)
\end{array}
$$

Hoare proof obligation/goal

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Here, a verification problem is proving validity of a Hoare triple.

Example 3.1

Program

```
assume(⊤)
r := 1;
i := 1;
while(i < 3)
{
  r := r + z;
  i := i + 1
}
assert(r = 2z + 1)
```

→

Hoare triple

$\{\top\}$
r := 1;
i := 1;
while(i < 3)
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Hoare Proof System

$$\frac{}{\{P\}\texttt{skip}\{P\}}(\text{skip rule})$$

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We may freely choose any of sp and wp for pre/post pairs for data statements.

Hoare Proof System (contd.)

$$\frac{\{P\}c_1\{Q\} \quad \{Q\}c_2\{R\}}{\{P\}c_1; c_2\{R\}}(\text{composition rule})$$

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Hoare Proof System (contd.)

$$\frac{\{P\}\mathtt{c}_1\{Q\} \quad \{Q\}\mathtt{c}_2\{R\}}{\{P\}\mathtt{c}_1;\mathtt{c}_2\{R\}}\text{(composition rule)}$$

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$$\frac{\{\mathtt{F} \wedge P\}\mathtt{c}_1\{Q\} \quad \{\neg\mathtt{F} \wedge P\}\mathtt{c}_2\{Q\}}{\{P\}\mathtt{if(F)\ c}_1\ \mathtt{else\ c}_2\{Q\}}\text{(if rule)}$$

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$$\frac{P_1 \Rightarrow P_2 \quad \{P_2\}\mathtt{c}\{Q_2\} \quad Q_2 \Rightarrow Q_1}{\{P_1\}\mathtt{c}\{Q_1\}}\text{(Consequence rule)}$$

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Non-mechanical step: invent `I` such that the while rule holds. `I` is called loop-invariant.

Example Hoare proof

Example 3.2

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$\mathtt{r} := 1;$

$\mathtt{i} := 1;$

$\mathtt{while}(\mathtt{i} < 3)$
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$\quad \mathtt{r} := \mathtt{r} + \mathtt{z}$

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Consider loop invariant: $\mathtt{I} = (i \leq 3 \wedge r = (i-1)z + 1)$

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Topic 3.2

Program as labeled transition system

A more convenient program model

- Simple language has many cases to write an algorithm

A more convenient program model

- Simple language has many cases to write an algorithm
- automata like program models allow more succinct description of verification methods

Program as labeled transition system (LTS)

Definition 3.3

A program P is a tuple $(V, L, \ell_0, \ell_e, E)$, where

- *V is a vector of variables,*
- *L be set of program locations,*
- *ℓ_0 is initial location,*
- *ℓ_e is error location, and*
- *$E \subseteq L \times \Sigma(V, V') \times L$ is a set of labeled transitions between locations.*

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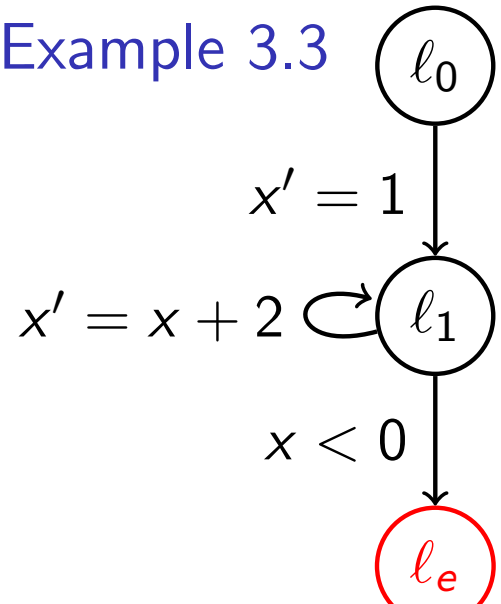

$$V = [\mathtt{x}]$$
$$L = \{\ell_0, \ell_1, \ell_e\}$$
$$E = \{(\ell_0, \mathtt{x}' = 1, \ell_1),$$
$$(\ell_1, \mathtt{x}' = \mathtt{x} + 2, \ell_1),$$
$$(\ell_1, \mathtt{x} < 0, \ell_e)\}$$

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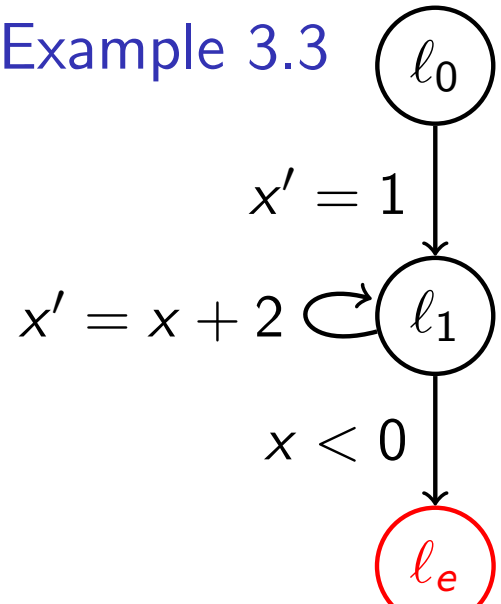

$V = [\mathtt{x}]$

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Notation:

If $e = (\ell, \rho(V, V'), \ell') \in E$, then

$e(V, V') \triangleq \rho(V, V')$,

$e(loc) \triangleq \ell$ and

$e(loc') \triangleq \ell'$

Guarded command

Definition 3.4 (Guarded command)

A guarded command is a pair of a formula in $\Sigma(V)$ and a sequence of update constraints (including havoc) of variables in V.

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Note: we may write transition formulas as guarded commands. Havoc encodes inputs.

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Example 3.4

Consider $V = [\mathtt{x}, \mathtt{y}]$. The formula represented by the guarded command $(\mathtt{x} > \mathtt{y}, [\mathtt{x} := \mathtt{x} + 1])$ is $\mathtt{x} > \mathtt{y} \wedge \mathtt{x}' = \mathtt{x} + 1 \wedge \mathtt{y}' = \mathtt{y}$.

Guarded command

Definition 3.4 (Guarded command)

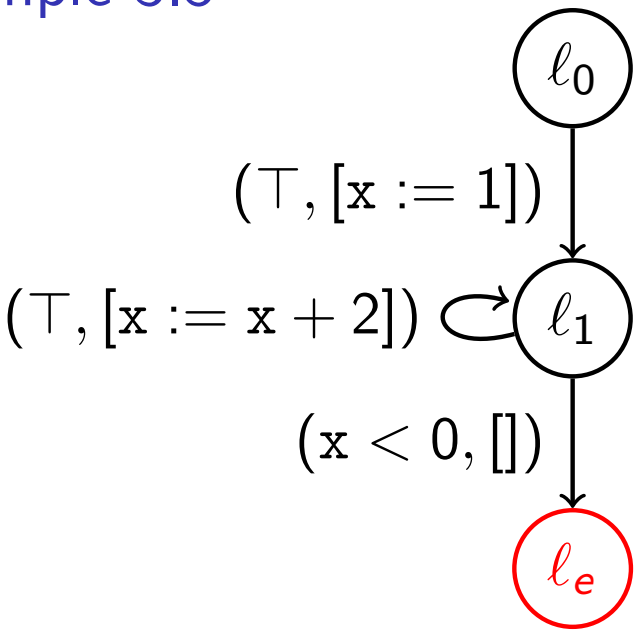
A guarded command is a pair of a formula in $\Sigma(V)$ and a sequence of update constraints (including havoc) of variables in V.

Note: we may write transition formulas as guarded commands. Havoc encodes inputs.

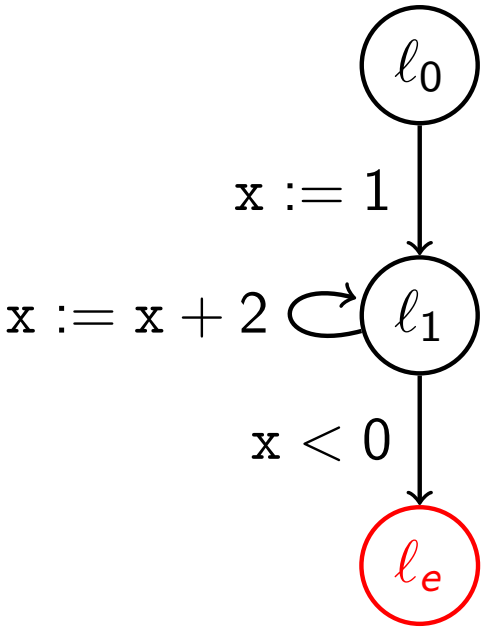
Example 3.4

Consider $V = [\mathrm{x}, \mathrm{y}]$. The formula represented by the guarded command $(\mathrm{x} > \mathrm{y}, [\mathrm{x} := \mathrm{x} + 1])$ is $\mathrm{x} > \mathrm{y} \wedge \mathrm{x}' = \mathrm{x} + 1 \wedge \mathrm{y}' = \mathrm{y}$.

Example 3.5



simplified view $\rightarrow$



Semantics

Consider program $P = (V, L, \ell_0, \ell_e, E)$.

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A path $\pi = e_1, \ldots, e_n$ in P is a sequence of transitions such that, for each $0 < i < n$, $e_i = (\ell_{i-1}, \_, \ell_i)$ and $e_{i+1} = (\ell_i, \_, \ell_{i+1})$.

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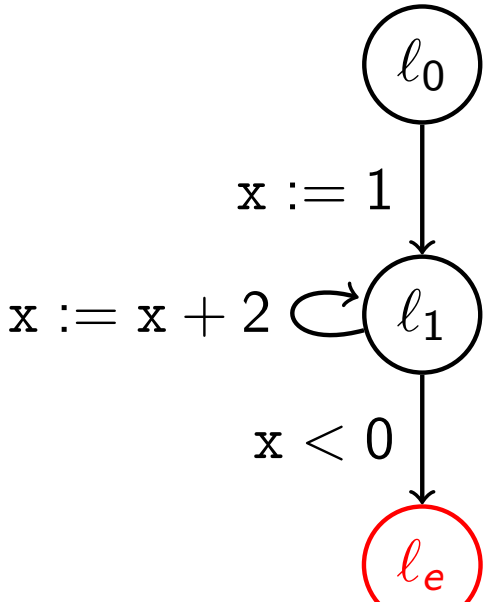

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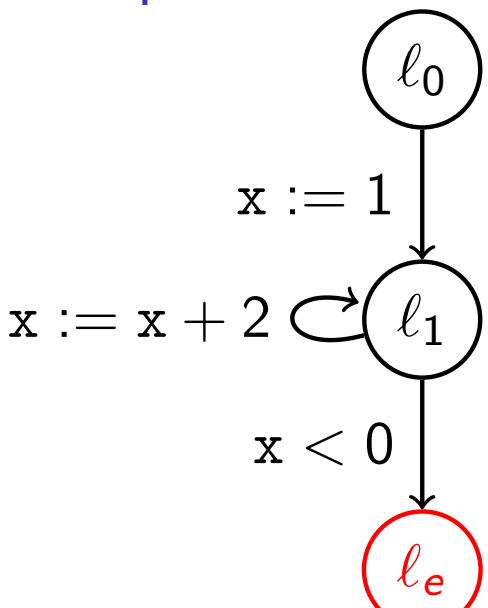

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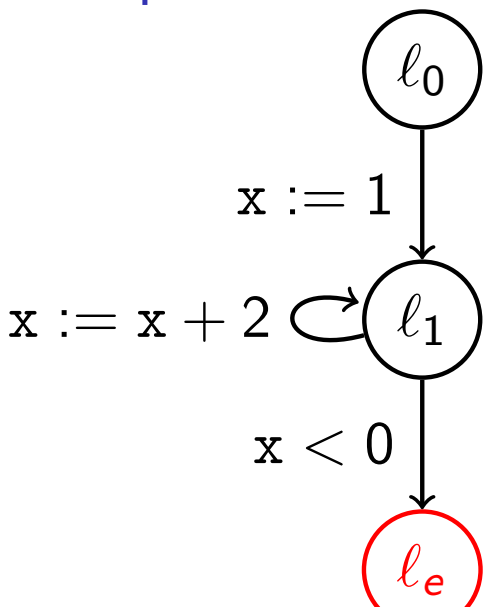

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Since F is unsat, there is no execution along the path

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Theorem 3.2

Show simple programming language is isomorphic to the labelled transition system.

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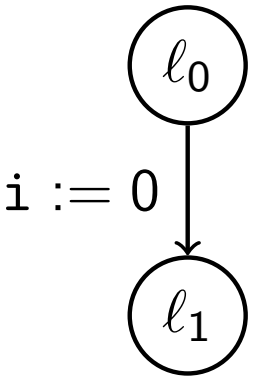

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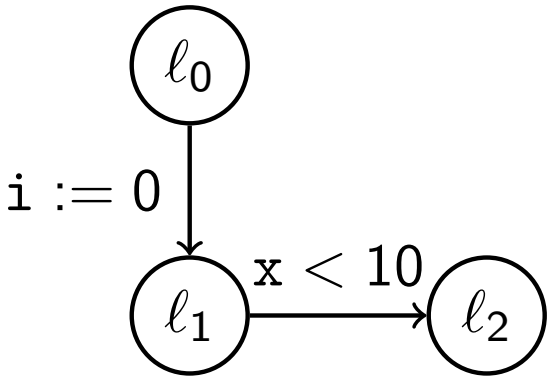

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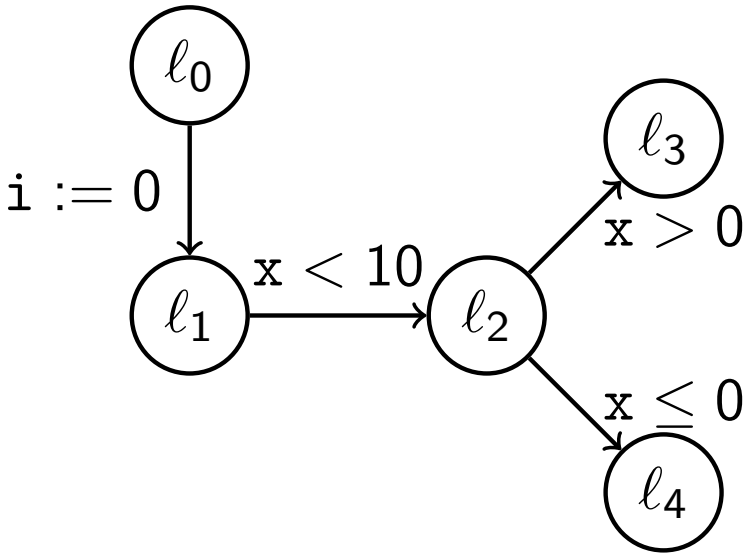

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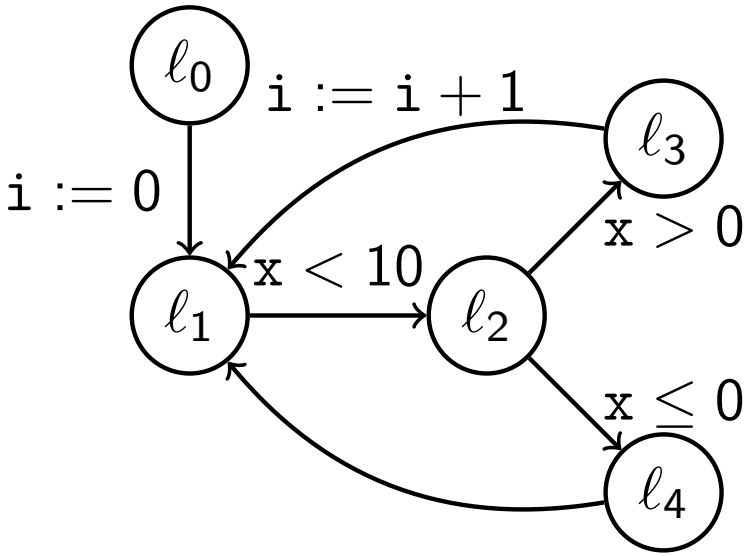

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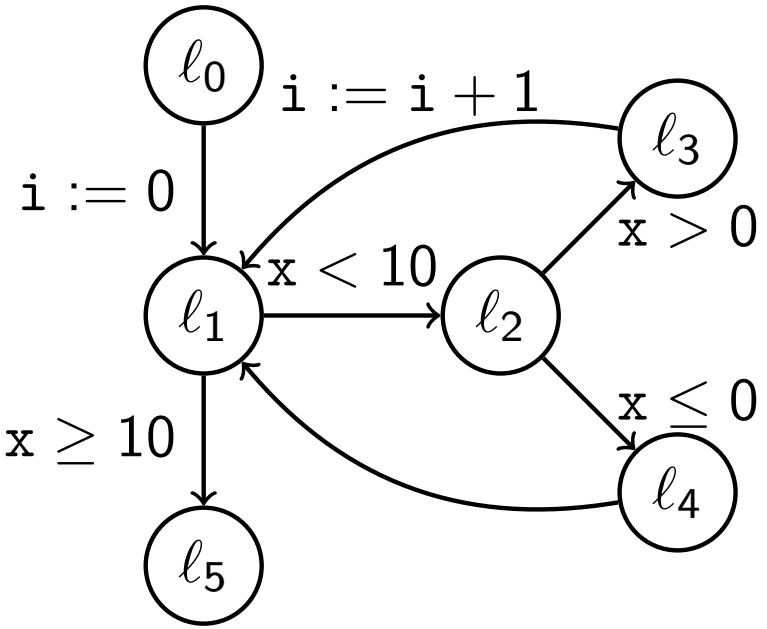

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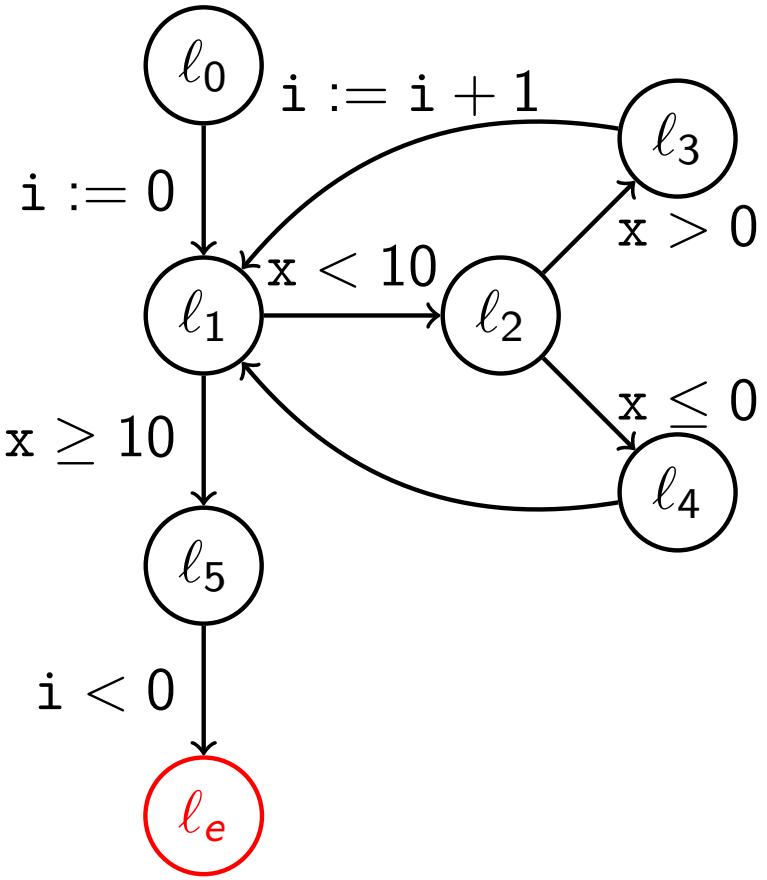

Cut-points

Definition 3.10

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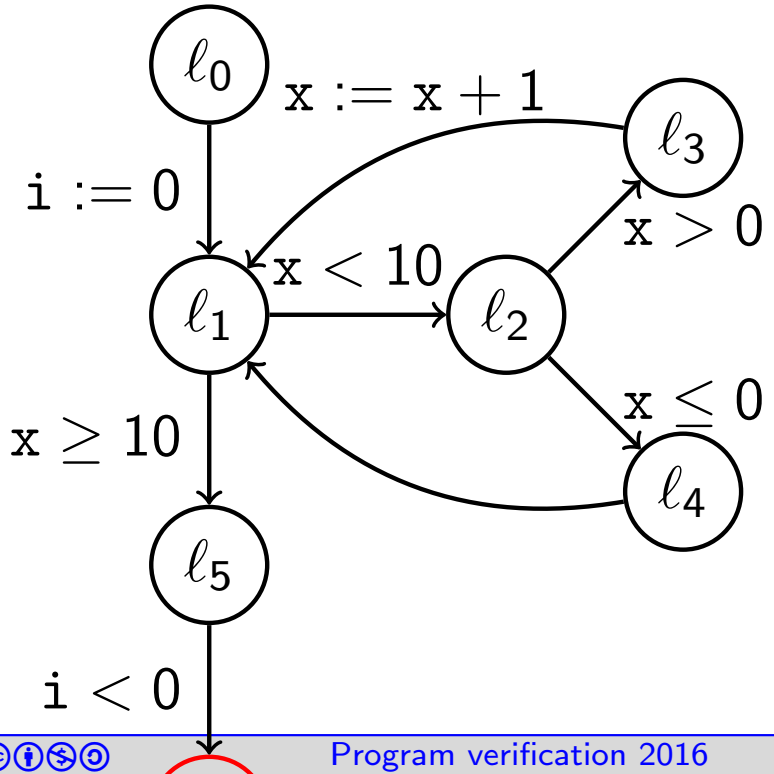
$\textsc{CutPoints}(P)$ in LTS loop heads in simple language

Example 3.8

Consider the following program P.



$\textsc{CutPoints}(P) = \{\ell_1\}$

Reminder: symbolic strongest post

$$sp : \Sigma(V) \times \Sigma(V, V') \to \Sigma(V)$$

We define symbolic post over labels of P as follows.

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For path $\pi = e_1, .., e_n$ of P, $sp((\ell, F), \pi) \triangleq sp(sp((\ell, F), e_1), e_2..e_n)$.

Topic 3.3

Loop invariants

Invariants

Definition 3.11

For P, a map $\mathrm{I} : L \to \Sigma(V)$ is called invariant map if, for each $\ell \in L$, all reachable states at ℓ satisfy $\mathrm{I}(\ell)$.

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Exercise 3.1

What is the algorithm for invariant checking?

Cut-point invariant maps

Let P be a program and $C = \textsc{CutPoints}(P) \cup \{\ell_0, \ell_e\}$.

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Proof.

Every path from ℓ_0 to ℓ_e can be segmented into loop free paths between cut-points. Therefore, no such path is feasible. □

Annotated verification: VCC demo

http://rise4fun.com/Vcc

Annotated verification: VCC demo

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Exercise 3.2

Complete the following program such that Vcc proves it correct

```
#include <vcc.h>
int main()
{
  int x, y;
  _(assume x > y +3 && x < 3000 )
  while( 0 < y ) _(invariant ....) {
    x = x + 1;
    y = y -1;
  }
  _(assert x >= y)
  return 0;
}
```

Annotated verification

- There are many tools like VCC that require user to write invariants at the loop heads and function boundaries
- Rest of the verification is done as discussed in earlier slides

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What if we want to compute the invariants automatically?

Topic 3.4

Concrete model checking - enumerate reachable states

Isn't enumeration impossible?

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- Explore the transition graph explicitly, light weight machinery
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- Explore the transition graph explicitly, light weight machinery
- If edge labels are guarded commands then finding next values are trivial
- After resolving non-determinism, concrete model checking reduces to program execution
- May be only finitely many states are reachable
- May be impossible to cover all states explicitly, but it may cover a portion of interest

Isn't enumeration impossible?

- Explore the transition graph explicitly, light weight machinery
- If edge labels are guarded commands then finding next values are trivial
- After resolving non-determinism, concrete model checking reduces to program execution
- May be only finitely many states are reachable
- May be impossible to cover all states explicitly, but it may cover a portion of interest
- Useful for learning design principles of computing reachable states

Concrete model checking

Algorithm 3.1: Concrete model checking

Input: $P = (V, L, \ell_0, \ell_e, E)$

Output: SAFE if P is safe, UNSAFE otherwise

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  if $(\ell, v) \notin reach$ **then**

    $reach := reach \cup \{(\ell, v)\}$;

    foreach $(\ell, F(V, V'), \ell') \in E$ **do**

      $worklist := worklist \cup \{(\ell', v') | F(v, v')\}$;

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if $(\ell_e, \_) \in reach$ **then**

  return UNSAFE

else

  return SAFE

Concrete model checking

Algorithm 3.1: Concrete model checking

Input: $P = (V, L, \ell_0, \ell_e, E)$
Output: SAFE if P is safe, UNSAFE otherwise

```
reach := ∅;
worklist := {(ℓ0, v) | v ∈ ℤ^|V|};
while worklist ≠ ∅ do
    choose (ℓ, v) ∈ worklist;
    worklist := worklist \ {(ℓ, v)};
    if (ℓ, v) ∉ reach then
        reach := reach ∪ {(ℓ, v)};
        foreach (ℓ, F(V, V'), ℓ') ∈ E do
            worklist := worklist ∪ {(ℓ', v') | F(v, v')};
if (ℓe, _) ∈ reach then
    return UNSAFE
else
    return SAFE
```

Exercise 3.3

Suggest improvements in the algorithm

Concrete model checking

Algorithm 3.1: Concrete model checking

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if (ℓe, _) ∈ reach then
    return UNSAFE
else
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```

the choice defines the nature of exploration

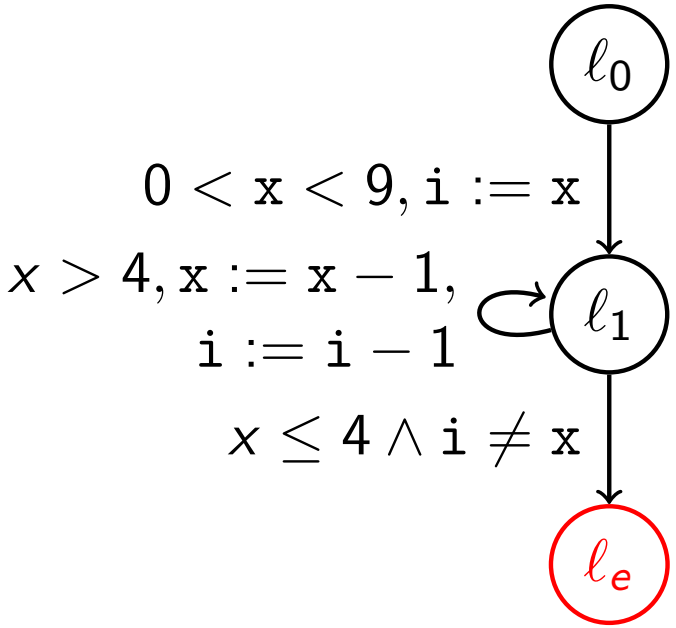
Exercise 3.3

Suggest improvements in the algorithm

Example: concrete model checking

Example 3.9

Consider the following example



Example: concrete model checking

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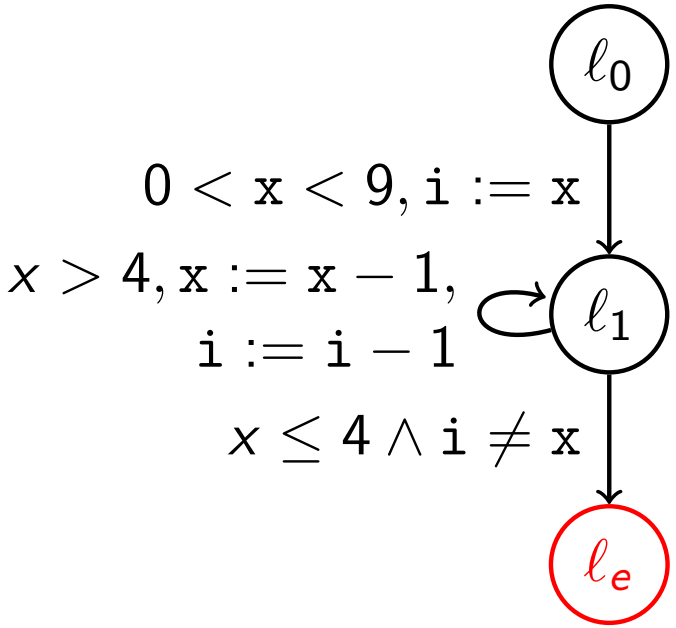

Let $V = [\mathtt{x}, \mathtt{i}]$

Example: concrete model checking

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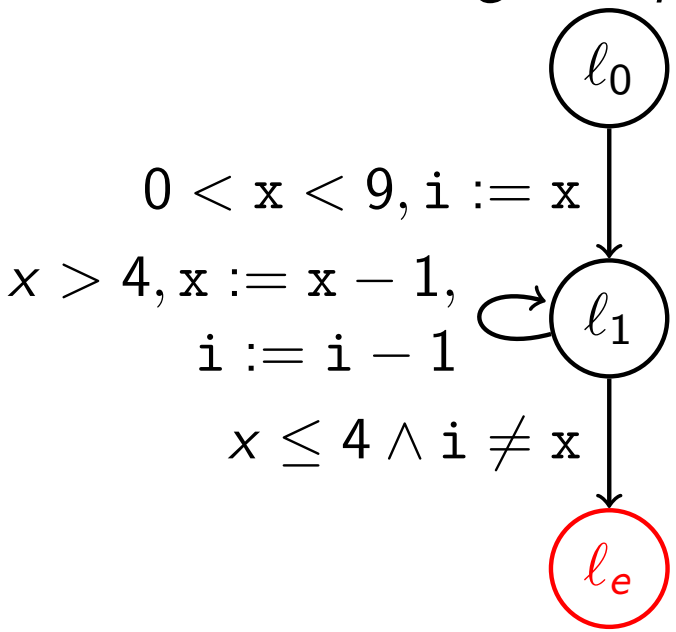

Initialization:
reach $= \emptyset$, *worklist* $= \{(\ell_0, v) | v \in \mathbb{Z}^2\}$

Let $V = [\mathtt{x}, \mathtt{i}]$

Example: concrete model checking

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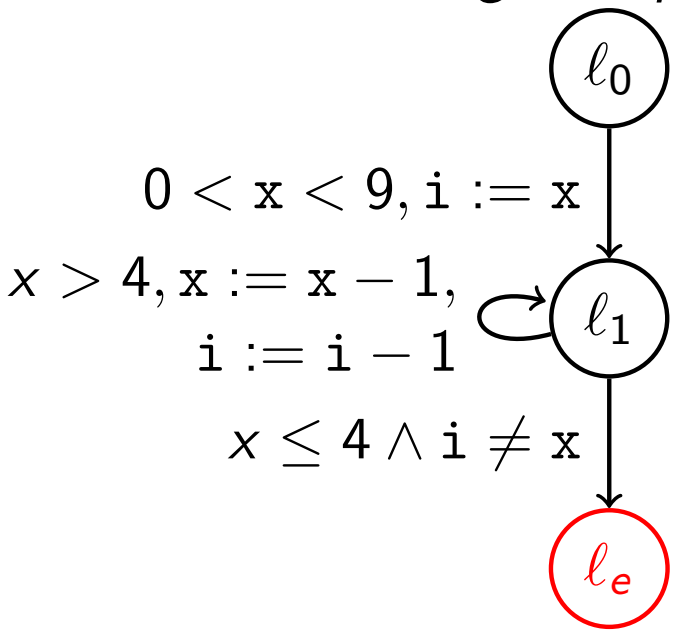

Initialization:
reach $= \emptyset$, *worklist* $= \{(\ell_0, v) | v \in \mathbb{Z}^2\}$

Choose a state:
Lets choose $(\ell_0, [8, 0])$

Let $V = [\mathtt{x}, \mathtt{i}]$

Example: concrete model checking

Example 3.9

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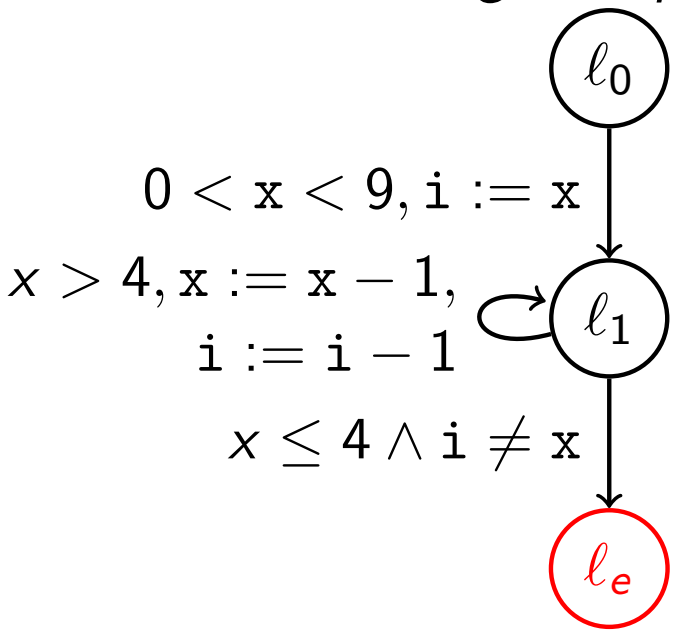

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Update worklist:
worklist $:=$ *worklist* $\setminus \{(\ell_0, [8, 8])\}$

Let $V = [\mathtt{x}, \mathtt{i}]$

Example: concrete model checking

Example 3.9

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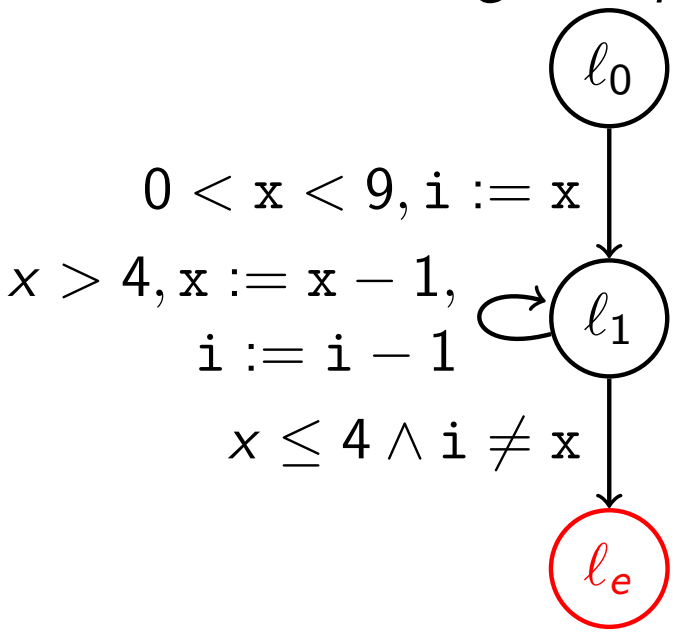

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reach $= \emptyset$, *worklist* $= \{(\ell_0, v) | v \in \mathbb{Z}^2\}$

Choose a state:
Lets choose $(\ell_0, [8, 0])$

Update worklist:
worklist $:=$ *worklist* $\setminus \{(\ell_0, [8, 8])\}$

Add successors in worklist if state not visited:
worklist $:=$ *worklist* $\cup \{(\ell_1, [8, 8])\}$
reach $:=$ *reach* $\cup \{(\ell_0, [8, 0])\}$

Example: concrete model checking

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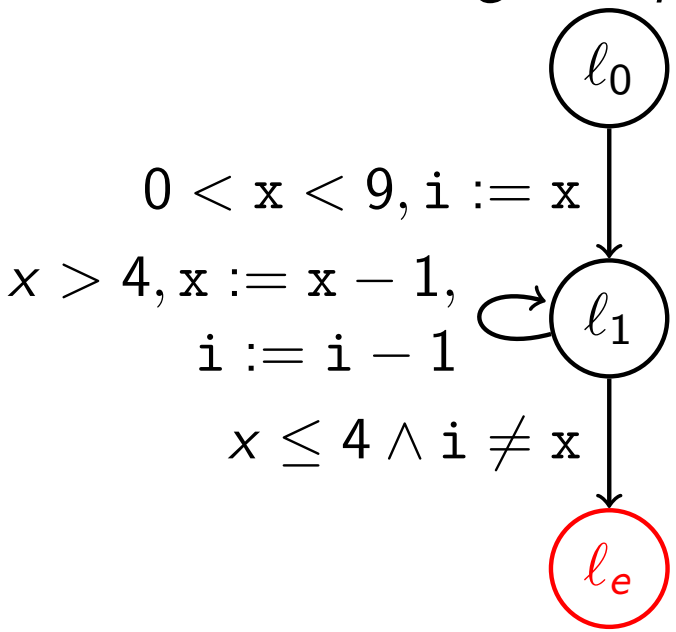

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Choose a state:
Lets choose $(\ell_0, [8, 0])$

Update worklist:
worklist $:=$ *worklist* $\setminus \{(\ell_0, [8, 8])\}$

Add successors in worklist if state not visited:
worklist $:=$ *worklist* $\cup \{(\ell_1, [8, 8])\}$
reach $:=$ *reach* $\cup \{(\ell_0, [8, 0])\}$

... go back to choosing a new state from worklist

Search strategies

- DFS
- BFS

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- BFS
- A^*
  - worklist is a priority queue,
  - weights are assigned to states based on estimate on possibility of reaching error

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  - worklist is a priority queue,
  - weights are assigned to states based on estimate on possibility of reaching error

Exercise 3.4

Describe A^* search strategy

Optimizations: exploiting structure

- Symmetry reduction
- Assume guarantee
- Partial order reduction (for concurrent systems)

Optimizations: reducing space

- hashed states - reach set contains hash of states (not sound)
- Stateless exploration - no reach set (redundant)

Trade-off among time, space, and soundness

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- hashed states - reach set contains hash of states (not sound)
- Stateless exploration - no reach set (redundant)

Trade-off among time, space, and soundness

Exercise 3.5

Write concrete model checking using hash tables

Proof and counterexample

Definition 3.17

A proof of a program is an object that allows one to check safety of the program using a low complexity (preferably linear) algorithm in the size of the object.

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Example 3.10

In our concrete model checking algorithm, reach set is the proof. The checker needs to find that no more states can be reached from reach.

Proof and counterexample

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A counterexample of a program is an execution that ends at ℓ_e.

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Example 3.10

In our concrete model checking algorithm, reach set is the proof. The checker needs to find that no more states can be reached from reach.

Definition 3.18

A counterexample of a program is an execution that ends at ℓ_e.

A verification method may produce three possible outcomes for a program

- proof
- counterexample
- unknown or non-termination

Enabling counterexample generation

Algorithm 3.2: Concrete model checking

Input: $P = (V, L, \ell_0, \ell_e, E)$
Output: SAFE if P is safe, UNSAFE otherwise

reach := $\emptyset$;
worklist := $\{(\ell_0, v) | v \in \mathbb{Z}^{|V|}\}$;
while *worklist* $\neq \emptyset$ **do**
    choose $(\ell, v) \in$ *worklist*;
    worklist := *worklist* $\setminus \{(\ell, v)\}$;
    if $(\ell, v) \notin$ *reach* **then**
        reach := *reach* $\cup \{(\ell, v)\}$;
        foreach v' *s.t.* $F(v, v')$ *is sat* $\wedge(\ell, F(V, V'), \ell') \in E$ **do**
            worklist := *worklist* $\cup \{(\ell', v')\}$;
if $(\ell_e, v) \in$ *reach* **then**
    return UNSAFE
else
    return SAFE

Exercise 3.6
add data structure to report counterexample

Enabling counterexample generation

Algorithm 3.2: Concrete model checking

Input: $P = (V, L, \ell_0, \ell_e, E)$
Output: SAFE if P is safe, UNSAFE otherwise

reach := $\emptyset$; *parents* := $\lambda x.\text{NAN}$;
worklist := $\{(\ell_0, v) | v \in \mathbb{Z}^{|V|}\}$;
while *worklist* $\neq \emptyset$ **do**
  choose $(\ell, v) \in$ *worklist*;
  worklist := *worklist* $\setminus \{(\ell, v)\}$;
  if $(\ell, v) \notin$ *reach* **then**
    reach := *reach* $\cup \{(\ell, v)\}$;
    foreach v' *s.t.* $F(v, v')$ *is sat* $\wedge(\ell, F(V, V'), \ell') \in E$ **do**
      worklist := *worklist* $\cup \{(\ell', v')\}$; *parents*$((\ell', v'))$:= (ℓ, v);
if $(\ell_e, v) \in$ *reach* **then**
  return UNSAFE*(traverseToInit(parents,* (ℓ_e, v)*))*
else
  return SAFE

Exercise 3.6
add data structure to report counterexample

Topic 3.5

Symbolic methods

Why symbolic?

To avoid, state explosion problem

Symbolic methods

Now, we cover some methods that try/avoid to compute lfp

- Symbolic model checking
- Constraint based invariant generation

Symbolic state

Definition 3.19

A symbolic state s of $P = (V, L, \ell_0, \ell_e, E)$ is a pair (ℓ, F), where

- $\ell \in L$
- *F is a formula over variables V in a given theory*

Symbolic model checking

Algorithm 3.3: Symbolic model checking

Input: $P = (V, L, \ell_0, \ell_e, E)$

Output: SAFE if P is safe, UNSAFE otherwise

Symbolic model checking

Algorithm 3.3: Symbolic model checking

Input: $P = (V, L, \ell_0, \ell_e, E)$

Output: SAFE if P is safe, UNSAFE otherwise

$reach : L \to \Sigma(V) \ := \ \lambda x.\bot;$

$worklist \ := \ \{(\ell_0, \top)\};$

Symbolic model checking

Algorithm 3.3: Symbolic model checking

Input: $P = (V, L, \ell_0, \ell_e, E)$

Output: SAFE if P is safe, UNSAFE otherwise

$reach : L \to \Sigma(V) \ := \ \lambda x.\bot$;

$worklist \ := \ \{(\ell_0, \top)\}$;

while $worklist \neq \emptyset$ **do**

  choose $(\ell, F) \in worklist$;

  $worklist \ := \ worklist \setminus \{(\ell, F)\}$;

Symbolic model checking

Algorithm 3.3: Symbolic model checking

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  $worklist \ := \ worklist \setminus \{(\ell, F)\}$;

  if $\neg(F \Rightarrow reach(\ell))$ *is sat* **then**

    $reach \ := \ reach[\ell \mapsto reach(\ell) \vee F]$;

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    foreach $(\ell, \rho(V, V'), \ell') \in E$ **do**

      $worklist \ := \ worklist \cup \{(\ell', sp(F, \rho))\}$;

Symbolic model checking

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        foreach $(\ell, \rho(V, V'), \ell') \in E$ **do**
            $worklist := worklist \cup \{(\ell', sp(F, \rho))\}$;

if $reach(\ell_e) \neq \bot$ **then**
    return UNSAFE
else
    return SAFE

Note: We need efficient implementations of various logical operators!

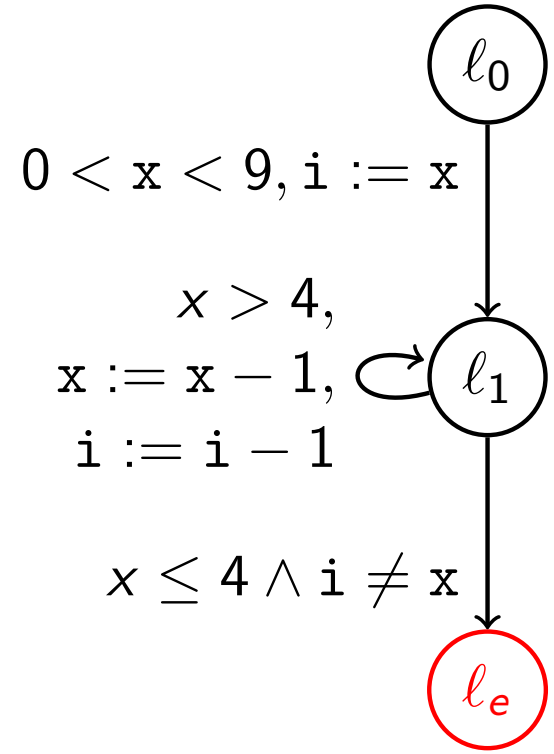
Exercise 3.7

Give a condition for definite termination?

Example: symbolic model checking

Example 3.11

Consider the following example



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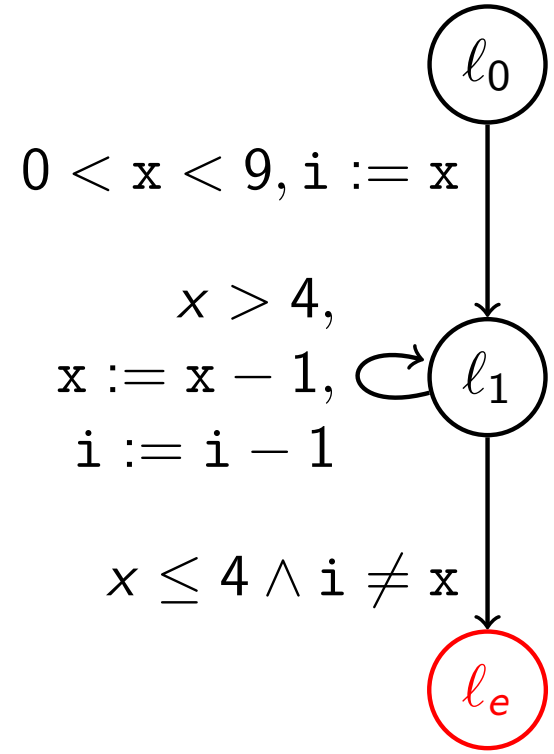

Let $V = [\mathtt{x}, \mathtt{i}]$

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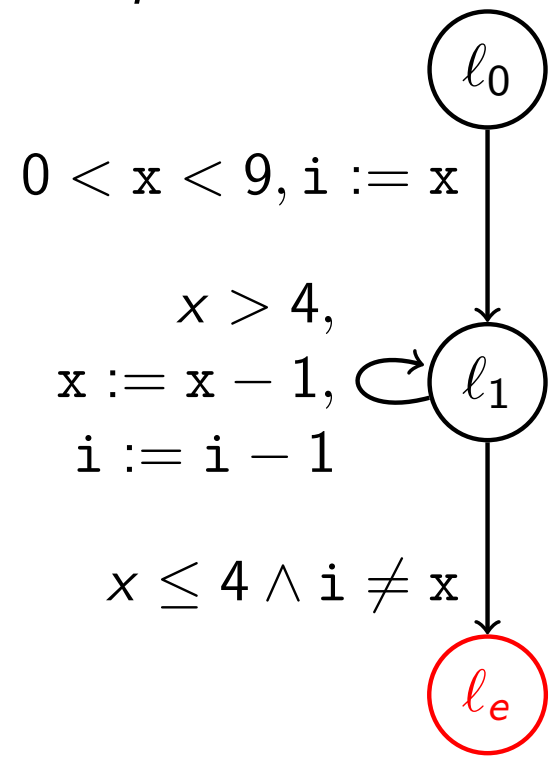

Init: $\mathit{reach} = \lambda x.\bot$, $\mathit{worklist} = \{(\ell_0, \top)\}$

Let $V = [\mathtt{x}, \mathtt{i}]$

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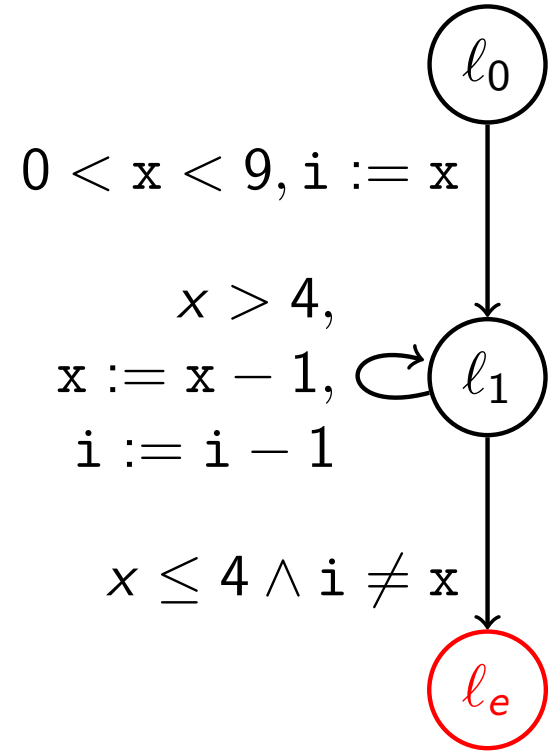

Init: $reach = \lambda x.\bot$, $worklist = \{(\ell_0, \top)\}$

Choose a state: $(\ell_0, \top)$ *(only choice)*

Let $V = [\mathtt{x}, \mathtt{i}]$

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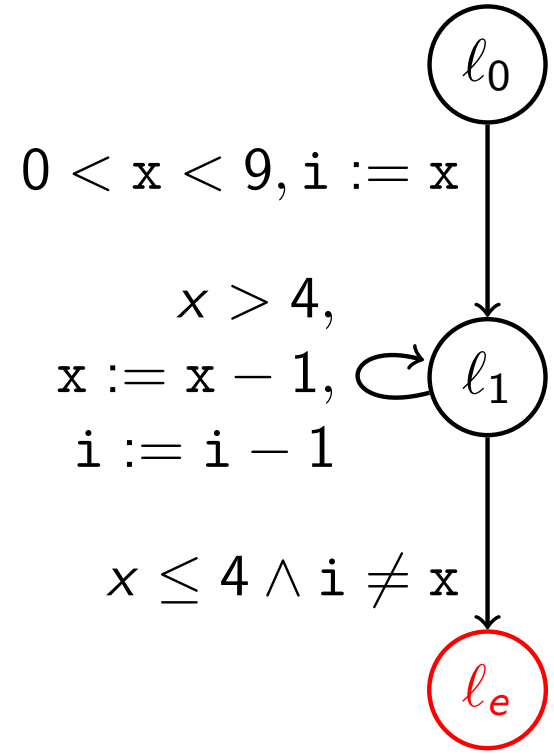

Init: $reach = \lambda x.\bot$, $worklist = \{(\ell_0, \top)\}$
Choose a state: $(\ell_0, \top)$ *(only choice)*
Update worklist: $worklist := \emptyset$

Let $V = [\mathtt{x}, \mathtt{i}]$

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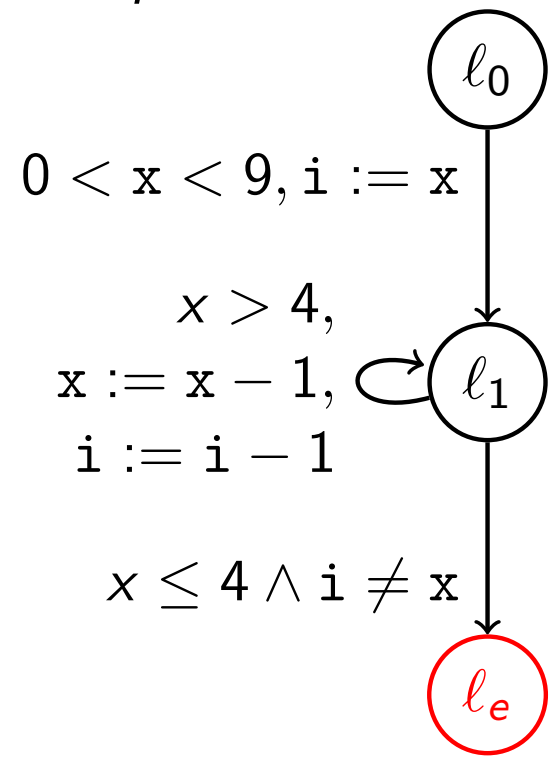

Init: reach $= \lambda x.\bot$, *worklist* $= \{(\ell_0, \top)\}$

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Update worklist: worklist $:= \emptyset$

Add successors in worklist:

Let $V = [\mathtt{x}, \mathtt{i}]$

Example: symbolic model checking

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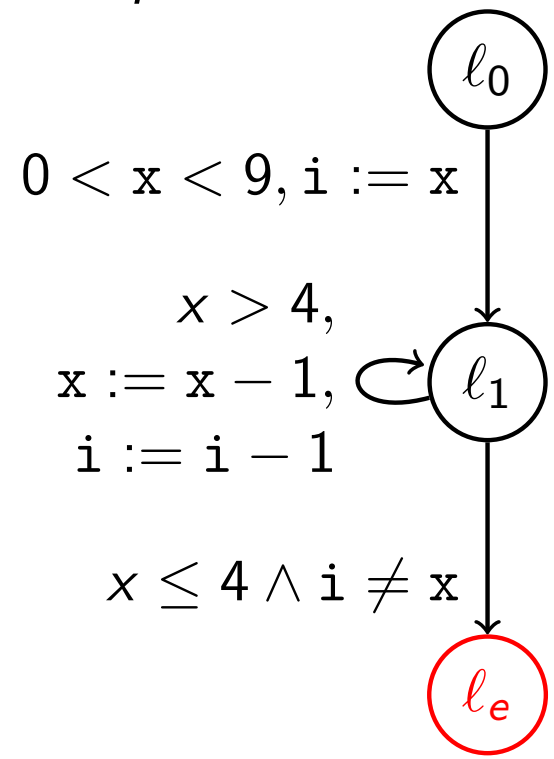

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Since $\neg(\top \Rightarrow reach(\ell_0))$ *is sat,*

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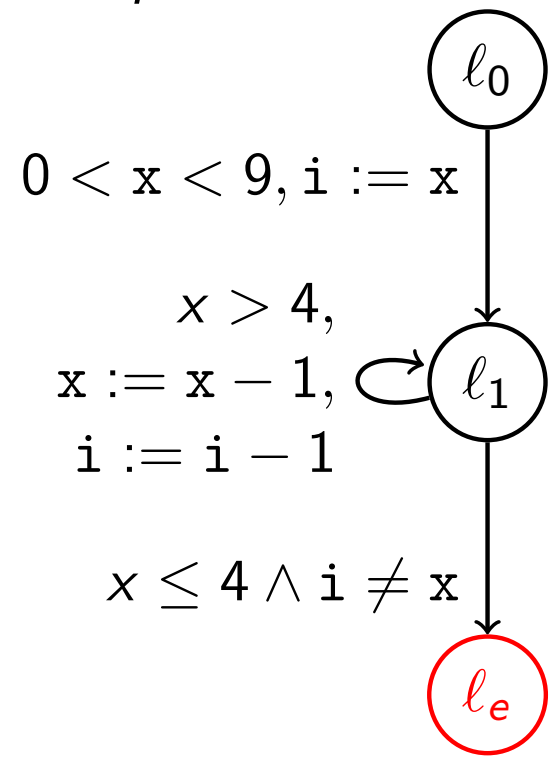

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Init: $reach = \lambda x.\bot$, $worklist = \{(\ell_0, \top)\}$
Choose a state: $(\ell_0, \top)$ *(only choice)*
Update worklist: $worklist := \emptyset$
Add successors in worklist:
Since $\neg(\top \Rightarrow reach(\ell_0))$ *is sat,*
$worklist := worklist \cup \{(\ell_1, 0 < x = i < 9)\}$

Example: symbolic model checking

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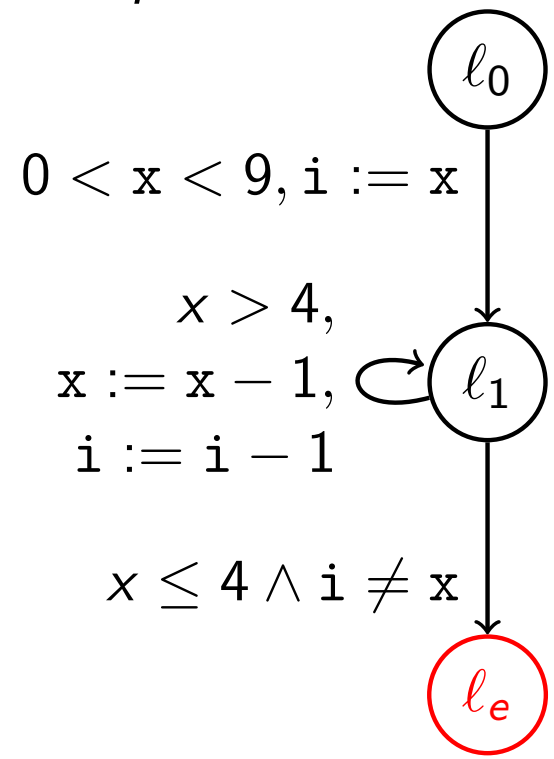

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$worklist := worklist \cup \{(\ell_1, 0 < x = i < 9)\}$
$reach(\ell_0) := reach(\ell_0) \vee \top := \top$

Example: symbolic model checking

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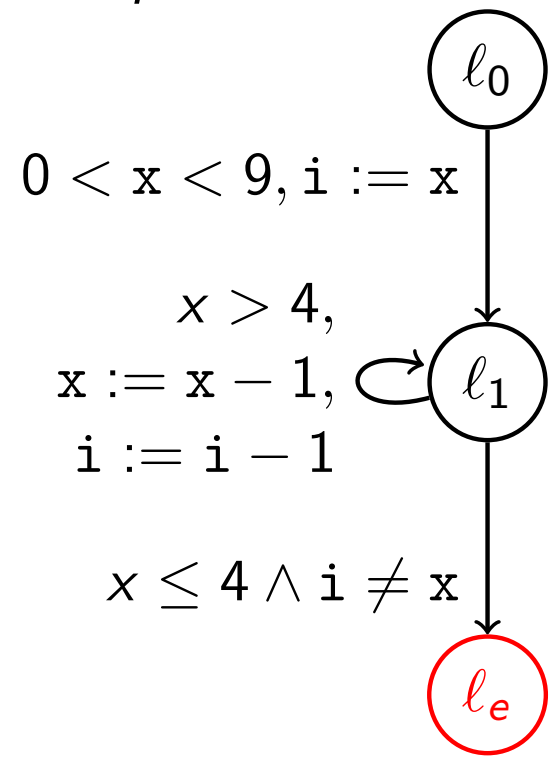

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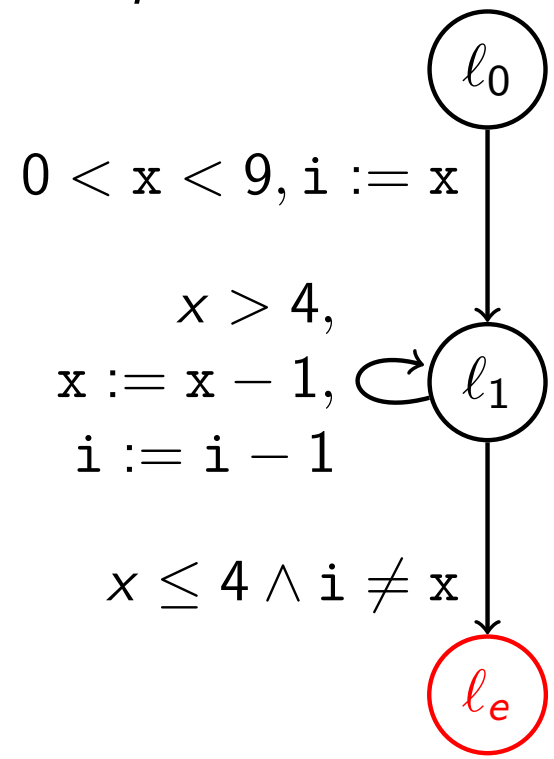
Choose a state: $(\ell_0, \top)$ *(only choice)*
Update worklist: $worklist := \emptyset$
Add successors in worklist:
Since $\neg(\top \Rightarrow reach(\ell_0))$ *is sat,*
$worklist := worklist \cup \{(\ell_1, 0 < x = i < 9)\}$
$reach(\ell_0) := reach(\ell_0) \vee \top := \top$
Again choose a state: $\{(\ell_1, 0 < x = i < 9)\}$

Example: symbolic model checking

Example 3.11

Consider the following example



Let $V = [\mathtt{x}, \mathtt{i}]$

Init: $reach = \lambda x.\bot$, $worklist = \{(\ell_0, \top)\}$
Choose a state: $(\ell_0, \top)$ *(only choice)*
Update worklist: $worklist := \emptyset$
Add successors in worklist:
Since $\neg(\top \Rightarrow reach(\ell_0))$ *is sat,*
$worklist := worklist \cup \{(\ell_1, 0 < x = i < 9)\}$
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Example: symbolic model checking

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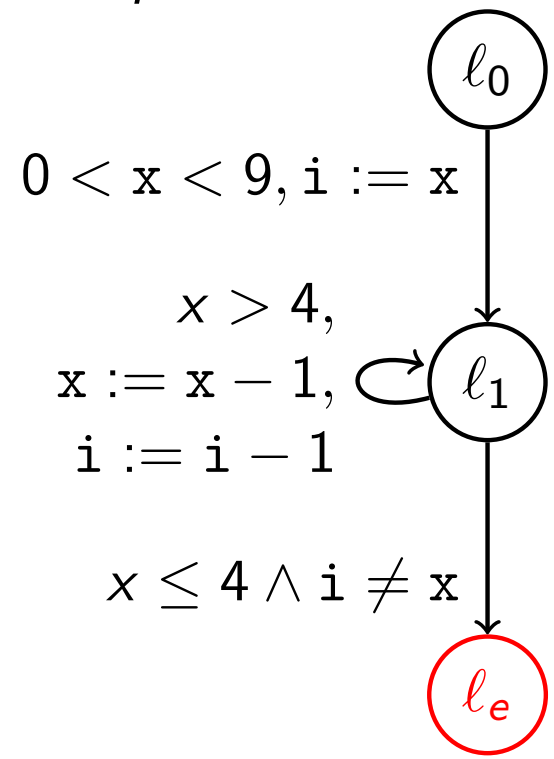

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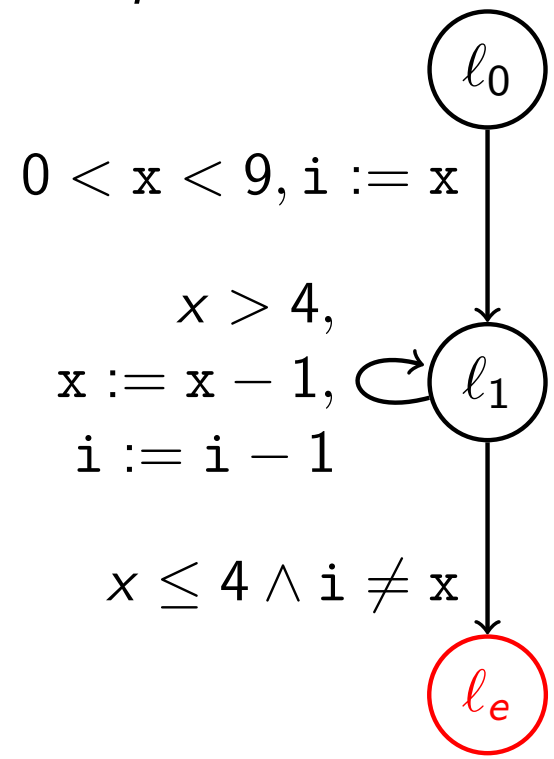

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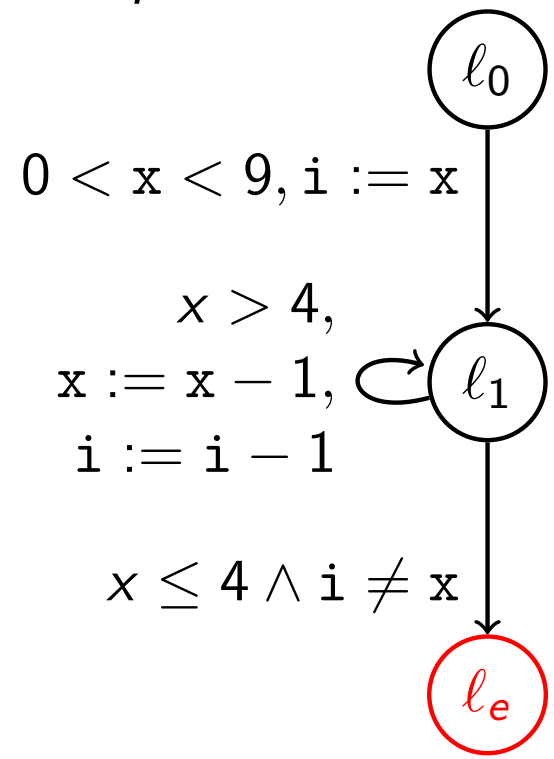

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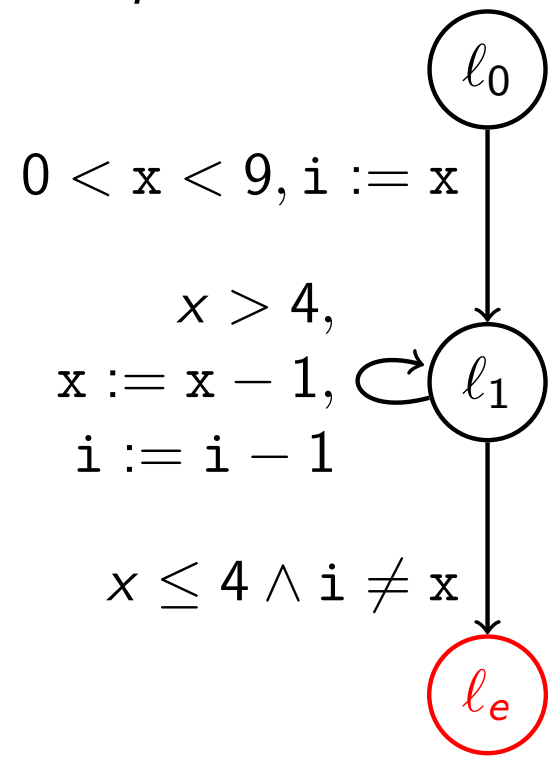

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Exercise 3.8

complete the run of the algorithm

Proof generation

If the symbolic model checker terminates with the answer SAFE, then it must also report a proof of the safety, which is the *reach* map.

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$$(\ell, \rho(V, V'), \ell') \in E \quad \{reach(\ell)\}\rho(V, V')\{reach(\ell')\}$$

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Exercise 3.9

Describe the construction for the above translation

Topic 3.6

Constraint based invariant generation

Invariant generation using constraint solving

Invariant generation: find a safe inductive invariant map I

Invariant generation using constraint solving

Invariant generation: find a safe inductive invariant map I

- This is our first method that computes the fixed point automatically without resorting to some kind of enumeration

Templates

Let $L = \{l_0, \ldots, l_n, l_e\}$,
Let $V = \{x_1, \ldots, x_m\}$

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Let $V = \{x_1, \ldots, x_m\}$

We assume the following templates for each invariant in the invariant map.

$$\begin{aligned} \mathrm{I}(l_0) &= 0 \leq 0 \\ \forall i \in 1..n.\ \mathrm{I}(l_i) &= (p_{i1}x_1 + \ldots p_{im}x_m \leq p_{i0}) \\ \mathrm{I}(l_e) &= 0 \leq -1 \end{aligned}$$

p_{ij} are called parameters to the templates and they define a set of candidate invariants.

Constraint generation

A safe inductive invariant map I must satisfy for all $(l_i, \rho, l_{i'}) \in E$

$$sp(\mathrm{I}(l_i), \rho) \Rightarrow \mathrm{I}(l_{i'}).$$

The above condition translates to

$$\forall V, V'.\ (p_{i1}x_1 + \ldots p_{im}x_m \leq p_{i0}) \wedge \rho(V, V') \Rightarrow (p_{i'1}x'_1 + \ldots p_{i'm}x'_m \leq p_{i'0})$$

Our goal is to find p_{ij}s such that the above constraints are satisfied. Unfortunately there is quantifier alternation in the constraints. Therefore, they are hard to solve.

Constraint solving using Farkas lemma

If all ρs are linear constraints then we can use Farkas lemma to turn the validity question into a "conjunctive satisfiablity question"

Lemma 3.1

For a rational matrix A, vectors a and b, and constant c,
$\forall X.\ AX \leq b \Rightarrow aX \leq c$ *iff*
$\exists \lambda \geq 0.\ \lambda^T A = a$ *and* $\lambda^T b \leq c$

Application of farkas lemma

Consider $(l_i, (AV + A'V \leq b), l_{i'}) \in E$

After applying Farkas lemma on

$$\forall V, V'.\ (p_{i1}x_1 + \ldots p_{im}x_m \leq p_{i0}) \wedge \rho(V, V') \Rightarrow (p_{i'1}x'_1 + \ldots p_{i'm}x'_m \leq p_{i'0}),$$

we obtain

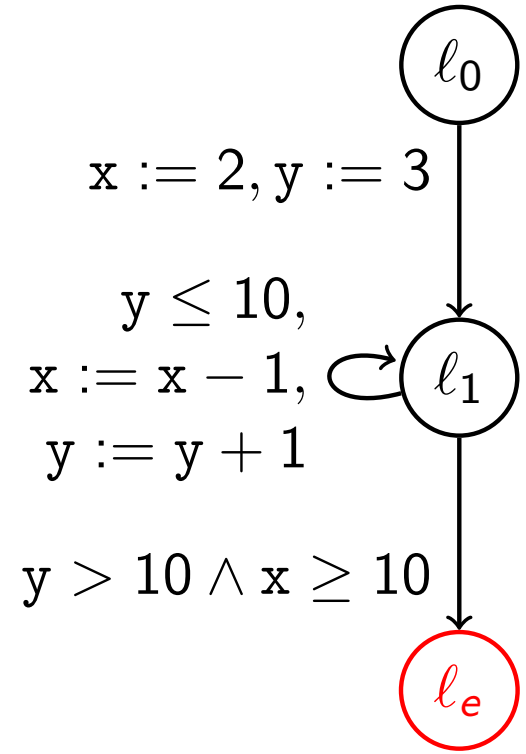
$$\exists \lambda_0, \lambda.\ (\lambda_0[p_{i1}, \ldots, p_{im}] + \lambda^T A) = 0 \wedge \lambda^T A' = [p_{i'1}, \ldots, p_{i'm}] \wedge \lambda_0 p_{i0} + \lambda^T b \leq p_{i'0}$$

All the variables p_{ij}s and λs are existentially quantified, which can be solved by a quadratic constraints solver.

Example: invariant generation

Example 3.12

Consider the following example



Example: invariant generation

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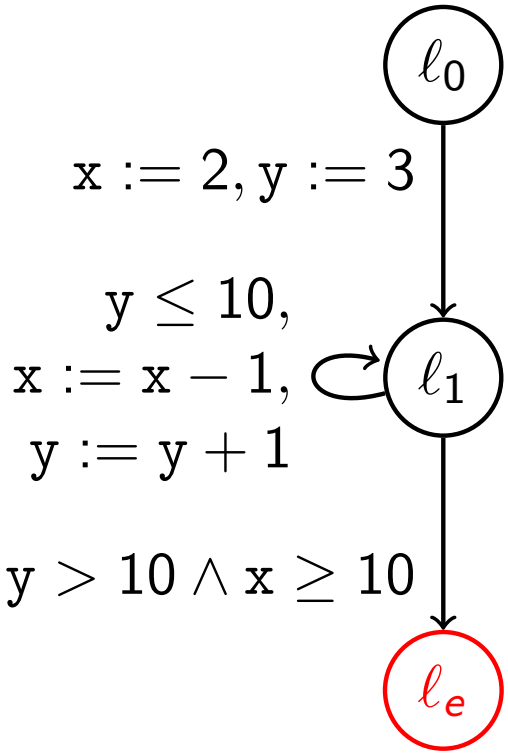

Let $V = [\mathrm{x}, \mathrm{y}]$

Example: invariant generation

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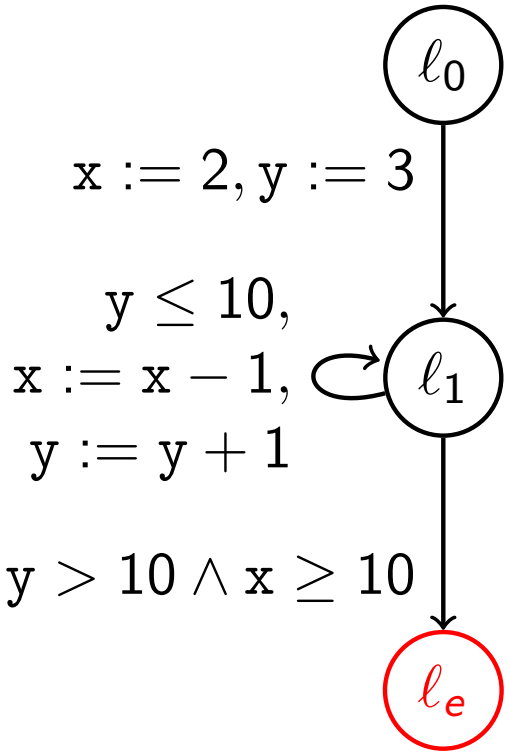

We assume the following invariant template at ℓ_1:
$\mathrm{I}(\ell_1) = (p_1\mathrm{x} + p_2\mathrm{y} \leq p_0)$

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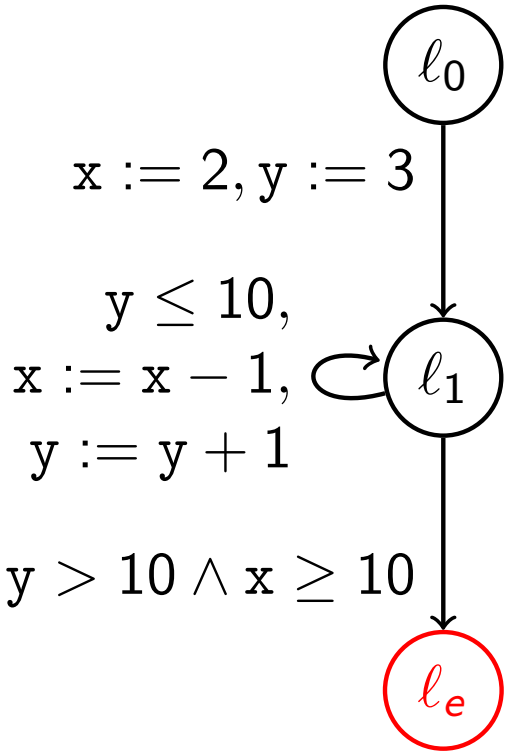

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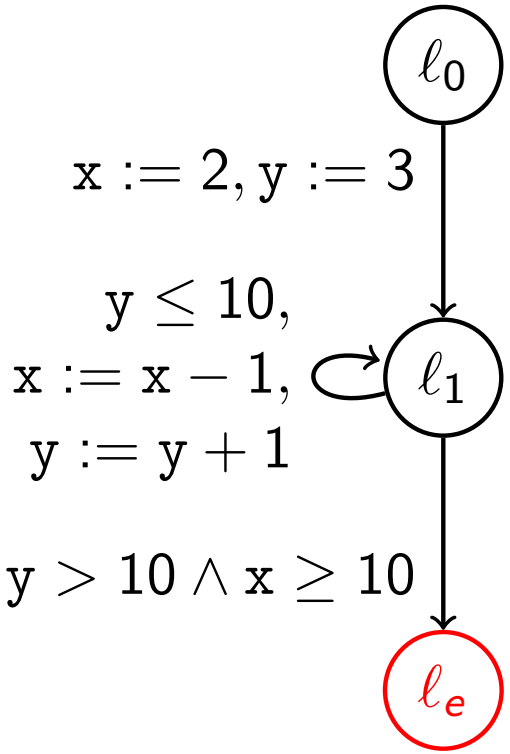

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$\forall \mathrm{x}', \mathrm{y}'.\ \mathrm{x}' = 2 \wedge \mathrm{y}' = 3 \Rightarrow (p_1\mathrm{x}' + p_2\mathrm{y}' \leq p_0)$

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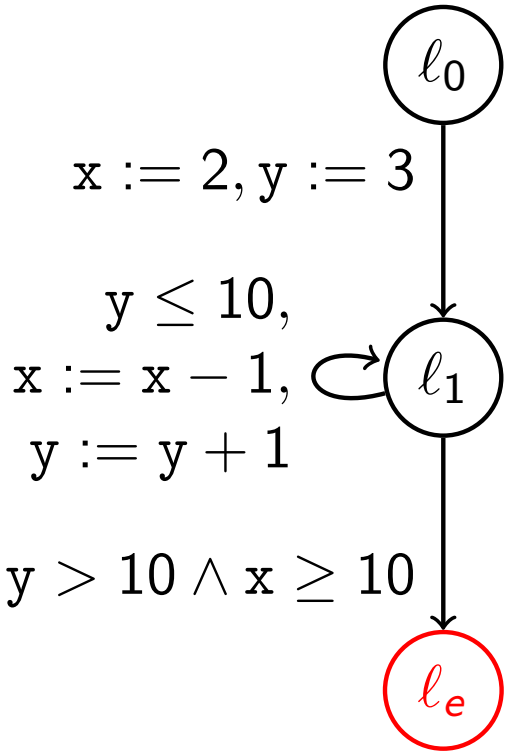

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For ℓ_1 to ℓ_1,
$\forall \mathrm{x}, \mathrm{y}, \mathrm{x}', \mathrm{y}'.\ (p_1\mathrm{x} + p_2\mathrm{y} \leq p_0) \wedge \mathrm{y} \leq 10 \wedge \mathrm{x}' = \mathrm{x} - 1 \wedge \mathrm{y}' = \mathrm{y} + 1 \Rightarrow (p_1\mathrm{x}' + p_2\mathrm{y}' \leq p_0)$

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Example: invariant generation

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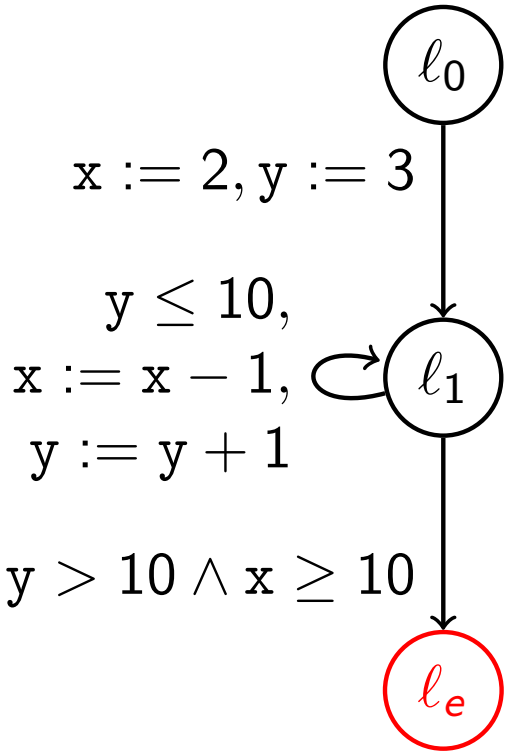

Let $V = [\mathrm{x}, \mathrm{y}]$

We assume the following invariant template at ℓ_1*:*
$\mathrm{I}(\ell_1) = (p_1\mathrm{x} + p_2\mathrm{y} \leq p_0)$

We generate the following constraints for program transitions:

For ℓ_0 *to* ℓ_1*,*
$\forall \mathrm{x}', \mathrm{y}'.\ \mathrm{x}' = 2 \wedge \mathrm{y}' = 3 \Rightarrow (p_1\mathrm{x}' + p_2\mathrm{y}' \leq p_0)$

For ℓ_1 *to* ℓ_1*,*
$\forall \mathrm{x}, \mathrm{y}, \mathrm{x}', \mathrm{y}'.\ (p_1\mathrm{x} + p_2\mathrm{y} \leq p_0) \wedge \mathrm{y} \leq 10 \wedge \mathrm{x}' = \mathrm{x} - 1 \wedge \mathrm{y}' = \mathrm{y} + 1 \Rightarrow (p_1\mathrm{x}' + p_2\mathrm{y}' \leq p_0)$

For ℓ_1 *to* ℓ_e*,*
$\forall \mathrm{x}, \mathrm{y}.\ (p_1\mathrm{x} + p_2\mathrm{y} \leq p_0) \wedge \mathrm{y} > 10 \wedge \mathrm{x} \geq 10 \Rightarrow \bot$

Example: invariant generation(contd.)

Now consider the second constraint:

$\forall \mathrm{x}, \mathrm{y}, \mathrm{x}', \mathrm{y}'.$

$(p_1\mathrm{x} + p_2\mathrm{y} \leq p_0) \wedge \mathrm{y} \leq 10 \wedge \mathrm{x}' = \mathrm{x} - 1 \wedge \mathrm{y}' = \mathrm{y} + 1 \Rightarrow (p_1\mathrm{x}' + p_2\mathrm{y}' \leq p_0)$

Example: invariant generation(contd.)

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Matrix view of the transition relation $\mathrm{y} \leq 10 \wedge \mathrm{x}' = \mathrm{x} - 1 \wedge \mathrm{y}' = \mathrm{y} + 1$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{x}' \\ \mathrm{y}' \end{bmatrix} \leq \begin{bmatrix} 10 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Example: invariant generation(contd.)

Applying farkas lemma on the constraint, we obtain

$$\begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & p_1 & p_2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{bmatrix} \begin{bmatrix} p_0 \\ 10 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} p_0 \end{bmatrix}$$

Exercise 3.10

Apply farkas lemma on the other two implications

$\forall \mathrm{x}', \mathrm{y}'.\ \mathrm{x}' = 2 \wedge \mathrm{y}' = 3 \Rightarrow (p_1\mathrm{x}' + p_2\mathrm{y}' \leq p_0)$

$\forall \mathrm{x}, \mathrm{y}.\ (p_1\mathrm{x} + p_2\mathrm{y} \leq p_0) \wedge \mathrm{y} > 10 \wedge \mathrm{x} \geq 10 \Rightarrow \bot$

Does this method work?

- Quadratic constraint solving does not scale
- For small tricky problems, this method may prove to be useful

Topic 3.7

Assignment

Problem 1

1. (1) Prove the following Hoare triple is valid

```
{true}
assume( n > 1);
i = n;
x = 0;
while(i > 0) {
  x = x + i;
  i = i - 1;
}
{ 2x = n*(n+1) }
```

Problem 2

2. (1) Fill the annotations to prove following program correct via Vcc

```
#include <vcc.h>
int main()
{
  int x = 0, y = 2;
 _(assume 1==1 )
  while( x < 3 ) _(invariant ... ) {
    x = x + 1;
    y = 3;
   }
  _(assert y == 3)
  return 0;
}
```

Problem 3

3. (2) extend your tool in the last assignment in the following ways
   - define classes for
     - locations,
     - variables,
     - guarded commands,
     - transitions (give names to the transitions), and
     - programs
   - encode the program in example 3.12 using the class
   - Write a function that computes path constraints for a given path
   - Read path from command line as space separated transition names and output the path constraints