Program verification 2016

Lecture 5: Abstract Model checking

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Where are we?

- SMT solving
- Lattice theory
- Abstract interpretation

Lecture plan

- Model checking with finite abstraction
- Predicate abstraction
- Abstract reachability graph
- Model checking vs. Abstract interpretation
- Abstract counterexample
- Abstraction refinement
- Counter example guided abstraction refinement

Topic 5.1

Model checking

Abstract program

Definition 5.1

Let us consider a finite abstraction D and a program $P = (V, L, \ell_0, \ell_e, E)$. An abstract program $P^\# = \textsc{Abstract}(P, D)$ is $(V, L, \ell_0, \ell_e, E^\#)$ where $E^\#$ is defined as follows. If $(\ell, \rho, \ell') \in E$ then $(\ell, \rho^\#, \ell') \in E^\#$, where

$$\rho^\# = \{\gamma(d) \times \gamma(d') | d' = sp^\#(d, \rho)\}.$$

We assume D and P allow $\rho^\#$ to be easily representable in a computer.

Theorem 5.1

$\forall d \in D \exists d' \in D.\ sp(\gamma(d), \rho^\#) = \gamma(d')$

Theorem 5.2

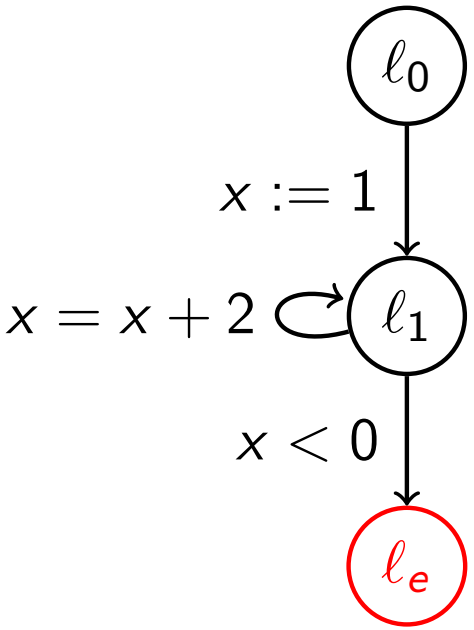
If $P^\#$ is safe then P is safe.

Example: abstract program

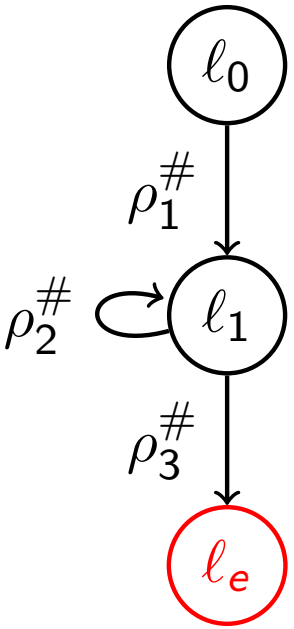
Example 5.1

Consider the following program and sign abstraction $D = \{\top, -, 0, +, \bot\}$.

Program:



Abstract program:



$$\rho_1^\# = \{(-,+),(0,+),(+,+),(\top,+)\}$$
$$\rho_2^\# = \{(-,+),(-,0),(-,+),(0,+),(+,+)\ (\top,\top)\}$$
$$\rho_3^\# = \{(-,-),(\top,-)\}$$

We have only listed pairs that do not have $\bot$ as second component.

Abstract reachability graph

Since $D = (\sqsubseteq, \top, \bot)$ is finite, symbolic execution of $P^{\#} = \textsc{Abstract}(P, D)$ will only produce finitely many symbolic states, which are called abstract states.

Definition 5.2

Abstract reachability graph(ARG) $(reach, R)$ *is the smallest directed graph such that*

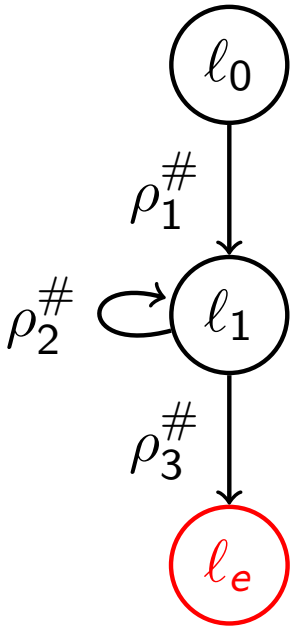
- $reach \subseteq L \times D$
- $(\ell_0, \top) \in reach$
- *If* $((\ell, d), (\ell', d')) \in R$ *iff* $\exists (\ell, \rho^{\#}, \ell') \in E^{\#}.\ \ d' = sp(d, \rho^{\#})$

Theorem 5.3

If $\forall d.\ d \neq bot \wedge (l_e, d) \notin reach$ *then* $P^{\#}$ *is safe.*

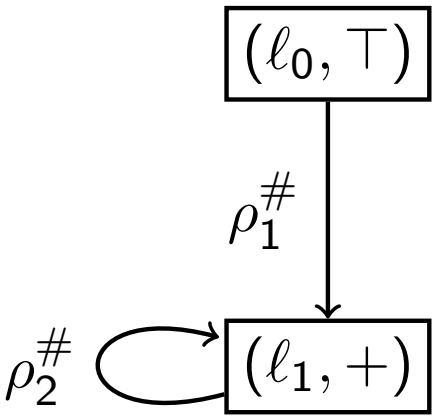
Example: abstract reachability graph

Abstract program:



$\rho_1^\# = \{(-,+),(0,+),(+,+),(\top,+)\}$
$\rho_2^\# = \{(-,+),(-,0),(-,+),(0,+),(+,+)$
$(\top,\top)\}$
$\rho_3^\# = \{(-,-),(\top,-)\}$

Abstract reachability graph:



We are not showing abstract states with $\bot$.

Model checking

The word model checking originated from the area of modal logic, where finding a model that satisfies a formula is called model checking.

In our situation, we have a logical statement $P^{\#}$ is not safe

We search for a model of the statement, i.e., a path in the abstract reachability graph that reaches to error location.

If no model found, then $P^{\#}$ is safe.

Abstract reachability graph may be large.

In contrast, abstract interpretation does not construct large objects.

Abstract model checking

Algorithm 5.1: $\textsc{AbstractMC}(P^{\#} = (V, L, \ell_0, \ell_e, E^{\#}),\ D = (\sqsubseteq, \top, \bot))$

Output: $\textsc{Correct}$ if $P^{\#}$ is safe, abstract counterexample otherwise

$worklist := \{(\ell_0, \top)\}$; $reach := \emptyset$; $covered := \emptyset$;

$parent : reach \cup worklist \to reach \cup worklist := \{((\ell_0, \top), (\ell_0, \top))\}$;

$path : reach \cup worklist \to (\text{sequences of } E^{\#}) := \{((\ell_0, \top), \epsilon)\}$;

while $worklist \neq \emptyset$ **do**

  choose $(\ell, d) \in worklist$; $worklist := worklist \setminus \{(\ell, d)\}$;

  if $d = \bot$ *or* $\exists s \in parent^*((l, d)).\ (\_, s) \in covered$ **then continue**;

  if $\ell = \ell_e$ **then return** $\textsc{Counterexample}(path(\ell, d))$;

  $reach := reach \cup \{(\ell, d)\}$;

  if $\exists (\ell, d') \in reach - range(covered).\ d \sqsubseteq d'$ **then**

    $covered := covered \cup \{((\ell, d'), (\ell, d))\}$

  else

    if $\exists (\ell, d') \in reach - range(covered).\ d' \sqsubseteq d$ **then**

      $covered := covered \cup \{((\ell, d), (\ell, d'))\}$

    foreach $(l, \rho^{\#}, l') \in E^{\#}$ **do**

      $d' := sp(d, \rho^{\#})$; $worklist := worklist \cup \{(l', d')\}$;

      $parent((l', d')) = (l, d)$; $path((l', d')) = path((l, d)).(l, \rho^{\#}, l')$;

$P^{\#}$ accessed only once

return $\textsc{Correct}$

On the fly abstraction

In AbstractMC, we only access $P^{\#}$ to compute post operator over d.This suggests, AbstractMC can be implemented in the following two ways.

- Precompute $P^{\#}$ and run AbstractMC as presented.
- On the fly construction of $P^{\#}$. We construct transitions of $P^{\#}$ as we need them

Exercise 5.1

Discuss benefits of both the approaches

Finite abstraction

The following abstractions are widely used in modelcheckers

- Cartesian predicate abstraction
- Boolean predicate abstraction

Finite abstraction example : Cartesian predicate abstraction

Cartesian predicate abstraction is defined by a set of predicates

$Preds = \{p_1, \ldots, p_n\}$

$C = \mathfrak{p}(\mathbb{Q}^{|V|})$

$D = \bot \cup \mathfrak{p} Preds$ // $\emptyset$ represents $\top$

$\bot \sqsubseteq S_1 \sqsubseteq S_2$ if $S_2 \subseteq S_1$

$\alpha(c) = \{p \in P | c \Rightarrow p\}$

$\gamma(S) = \bigwedge S$

Example 5.2

$V = \{\mathrm{x}, \mathrm{y}\}$

$P = \{\mathrm{x} \leq 1, -\mathrm{x} - \mathrm{y} \leq -1, \mathrm{y} \leq 5\}$

$\alpha(\{(0,0)\}) = \{\mathrm{x} \leq 1, \mathrm{y} \leq 5\}$

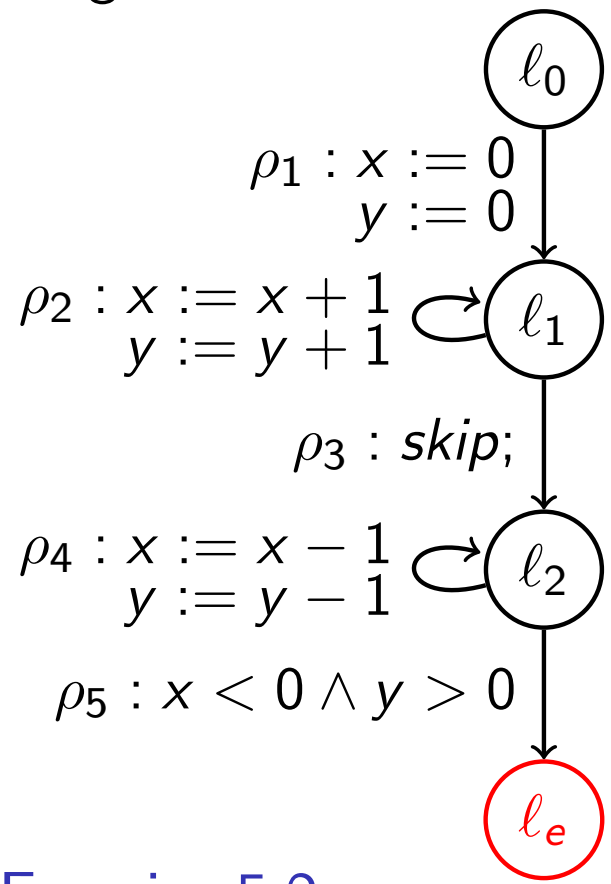
$\alpha((\mathrm{x}-1)^2 + (\mathrm{y}-3)^2 = 1) = \{-\mathrm{x} - \mathrm{y} \leq -1, \mathrm{y} \leq 5\}$

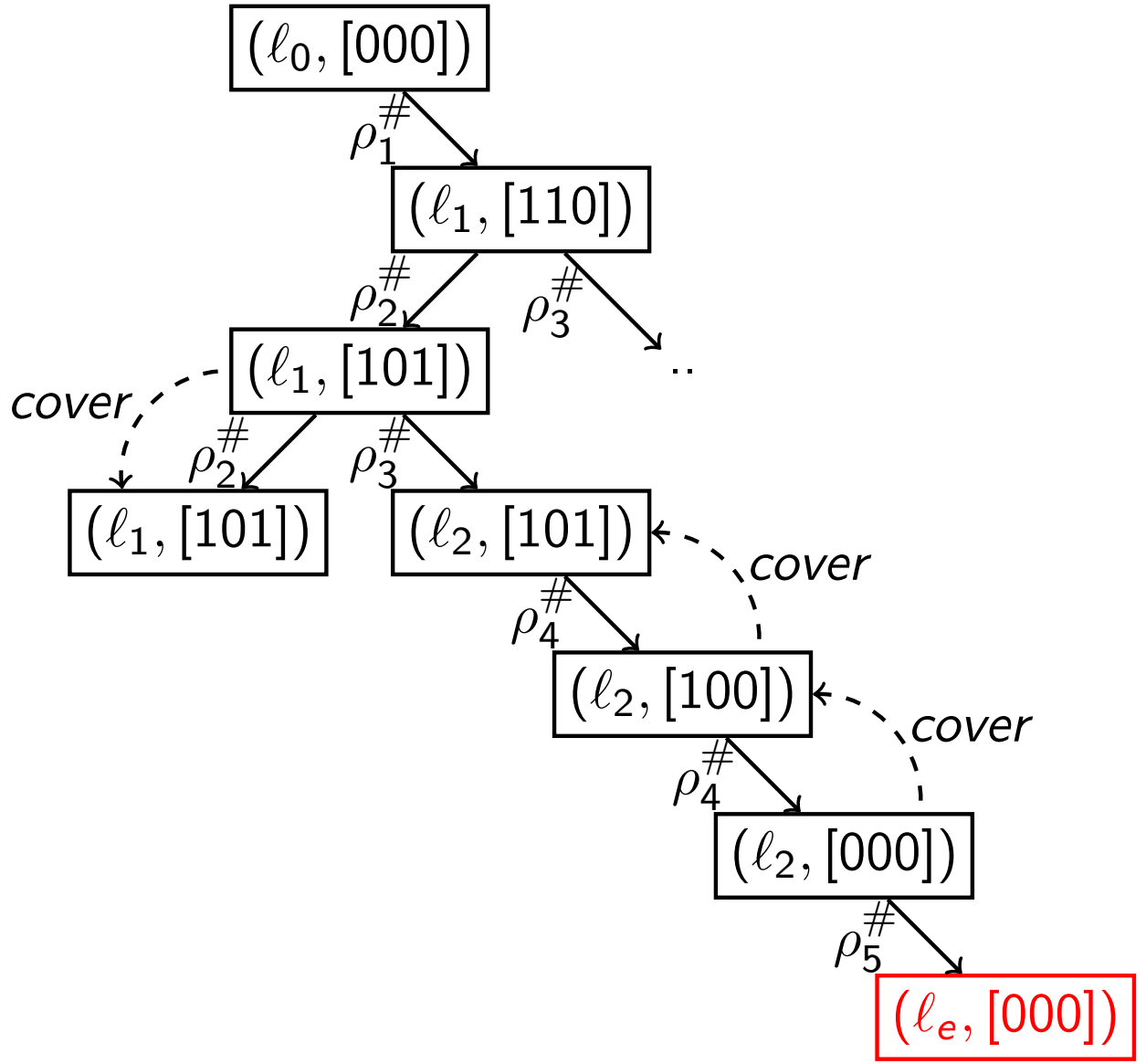
We represent abstract state as bit vectors, e.g., [101] represents $\mathrm{x} \leq 1 \wedge \mathrm{y} \leq 5$

Example: ARG with Cartesian predicate abstraction

$Preds = \{x \geq 0, y \leq 0, x \geq 1\}$.

Program:





Exercise 5.2

Complete the ARG

Spurious counterexample

$\textsc{AbstractMC}(P^{\#}, D)$ may fail to prove $P^{\#}$ correct and return a path $e_1^{\#} \dots e_m^{\#}$, which is called abstract counterexample.

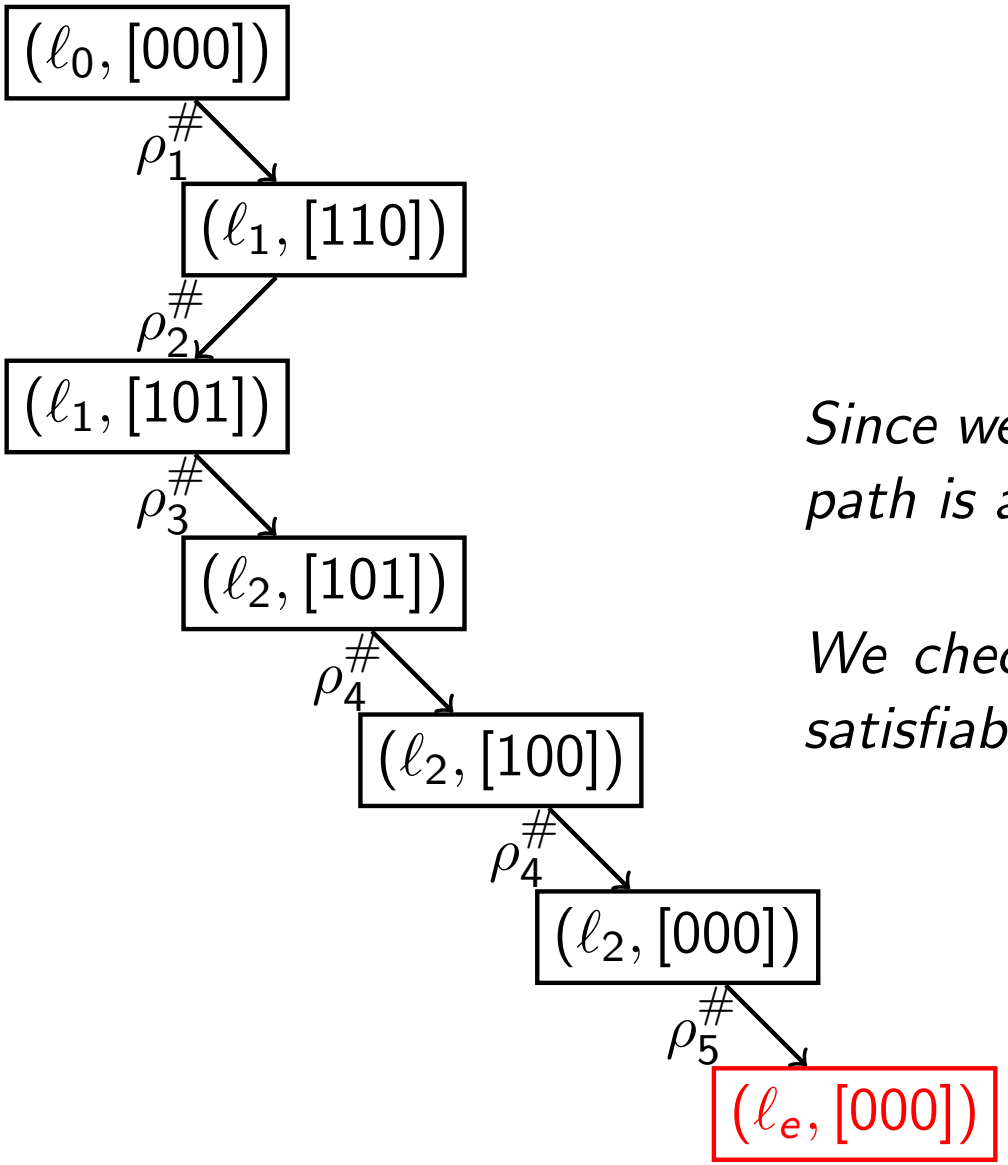
Let $e_1 \dots e_m$ be the corresponding path in P. Now we have two possibilities.

- $e_1 \dots e_m$ is feasible. Then, we have found a bug
- $e_1 \dots e_m$ is not feasible. Then, we call $e_1 \dots e_m$ as spurious counterexample.

We need to fix our abstraction such that we do not get the spurious counter example.

Example : spurious counterexample

Example 5.3



Since we cannot execute $\rho_1\rho_2\rho_3\rho_4\rho_4\rho_5$, the path is a spurious counterexample.

We check the feasibility of the path using satisfiability of path constraints.

Refinement relation

Definition 5.3

Consider abstractions

$$(C, \subseteq) \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} (D_1, \sqsubseteq_1) \quad \textit{and} \quad (C, \subseteq) \underset{\alpha_1}{\overset{\gamma_2}{\leftrightarrows}} (D_2, \sqsubseteq_2).$$

D_2 *refines* D_1 *if*

$$\forall c \in C.\ \gamma_1(\alpha_1(c)) \subseteq \gamma_2(\alpha_2(c))$$

Exercise 5.3

$\gamma_1 \circ \alpha_2$ *is order embedding.*

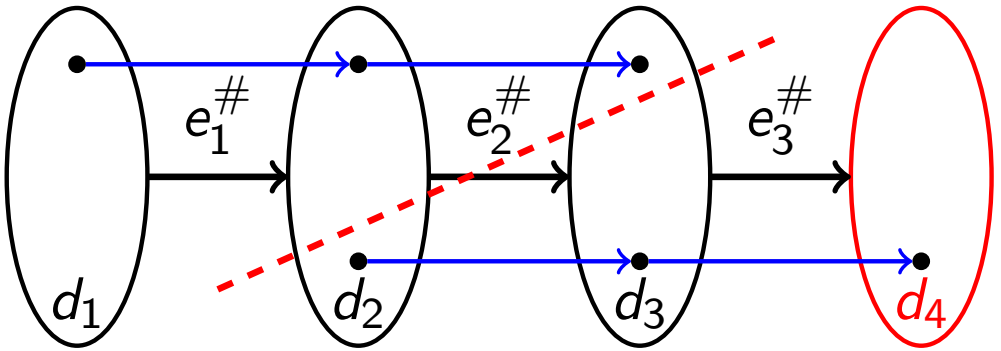
Abstraction refinement

Theorem 5.4

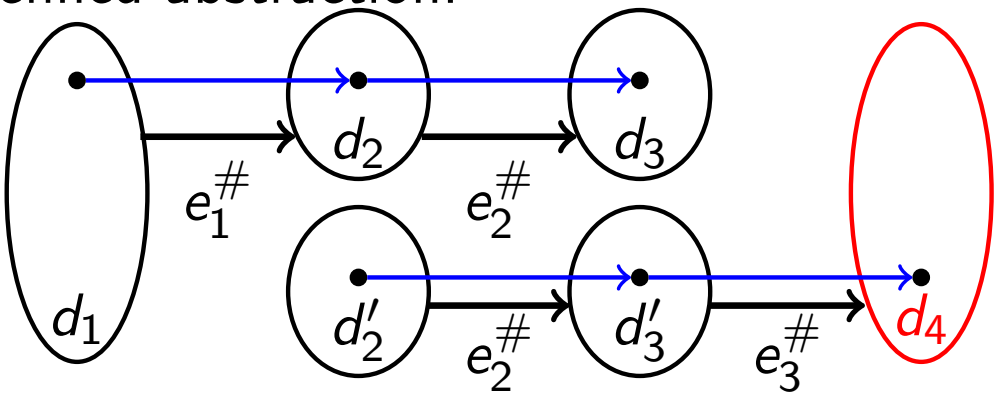
If $\textsc{Abstract}(P, D_1)$ *exhibits a spurious counterexample then there is an abstraction* D_2 *such that* D_2 *refines* D_1 *and* $\textsc{Abstract}(P, D_2)$ *does not exhibit the same counter example.*

Proof sketch.

Spurious counterexample:



Refined abstraction:



We say the refinement to D_2 from D_1 ensures progress, i.e., counterexamples are not repeated if ARG is build again with D_2

Refinment Strategy for predicate abstraction

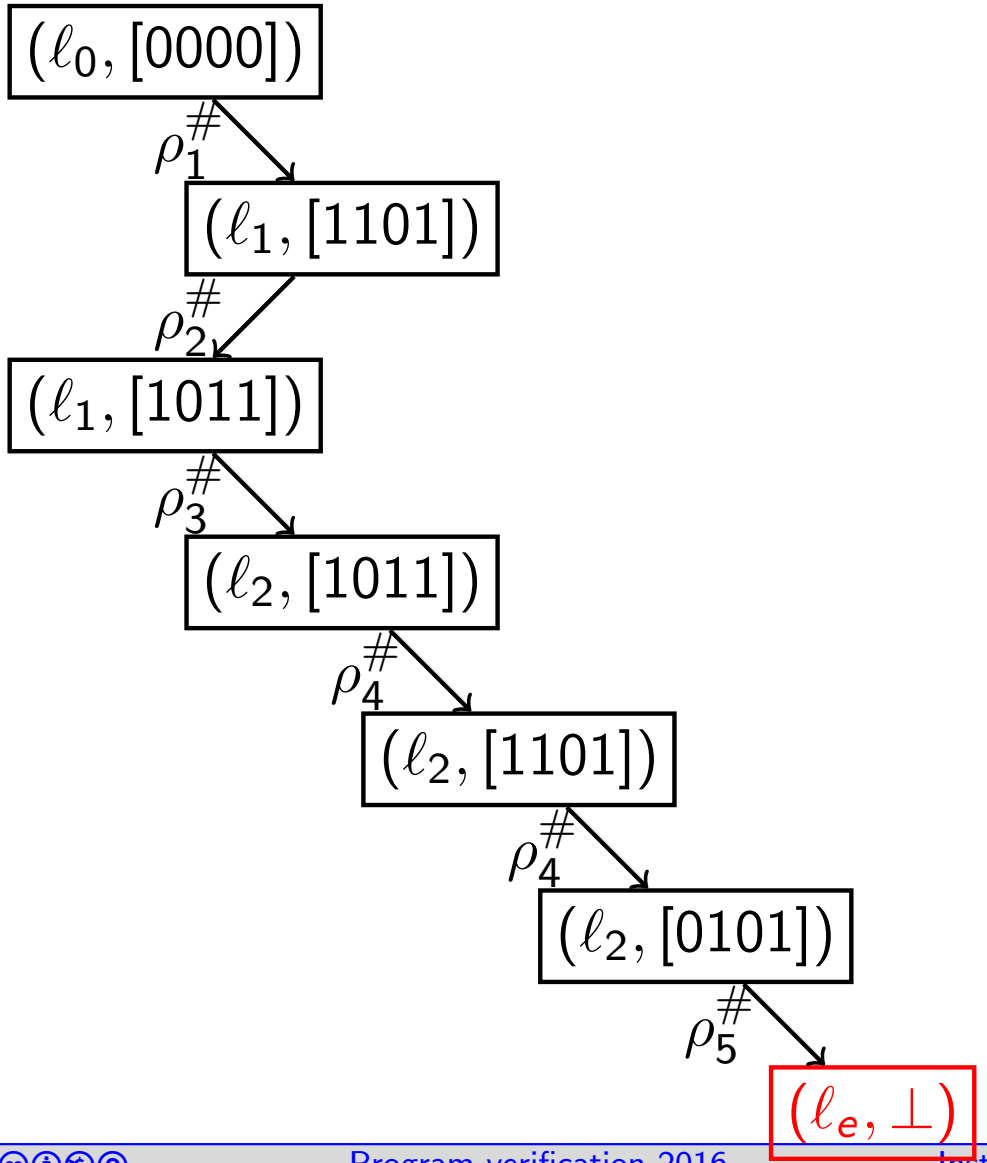
General refinement strategy
Split abstract states such that the spurious counterexample is disconnected.

In predicate abstraction, we only need to add more predicates. The new abstraction will certainly be refinement.

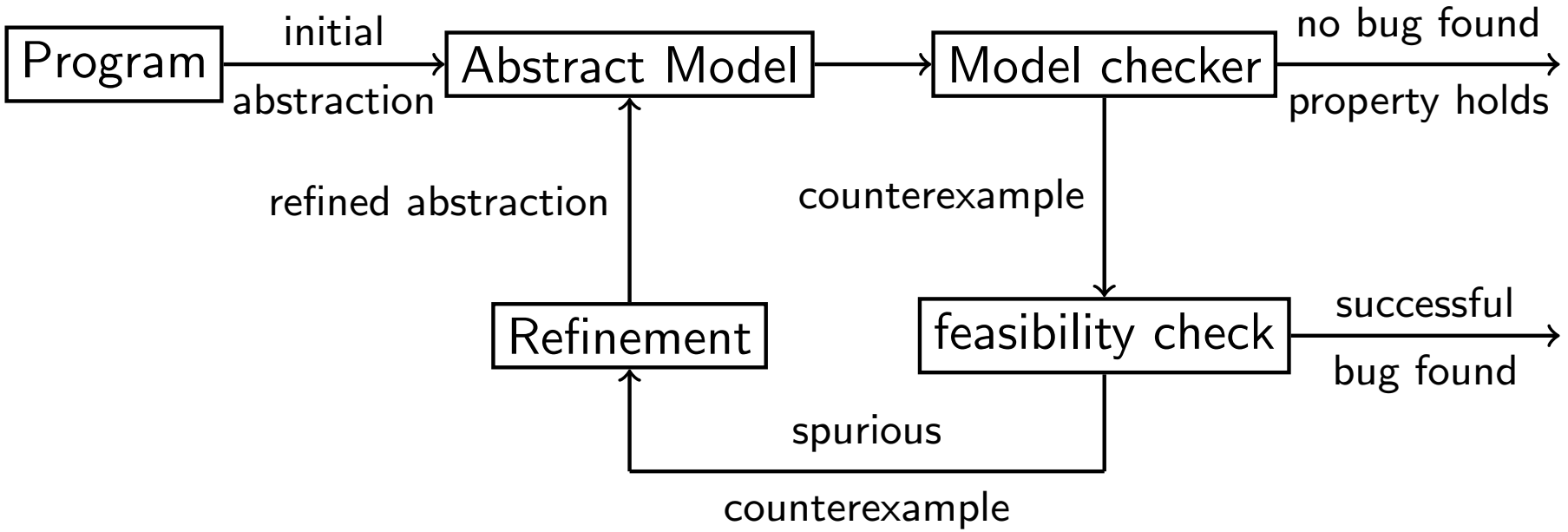
Example: refinement

Adding predicate $y \leq -1$ will remove the spurious counterexample.

$Preds = \{x \geq 0, y \leq 0, x \geq 1, y \leq 1\}$



CEGAR: CounterExample Guided Abstraction Refinement



Computing refinement

In order to automate CEGAR, we need an effective method for computing new predicates that result in the desired refinement.

Here, we discuss the following two methods for refinement of a predicate abstraction.

- Syntax based refinement
- Interpolation based refinement

Let us first define(remind) some notations.

Path constraints for spurious counterexample (reminder)

V_i be the vector of variables obtained by adding subscript i after each variable in V.

Definition 5.4

For a spurious counterexample $e_1 \dots e_n$, path constraints $pathCons(e_1 \dots e_n)$ is

$$\bigwedge_{i \in 1..n} e_i(V_{i-1}, V_i)$$

A path is feasible if corresponding path constraints is satisfiable.

Note: Path constraints are also known as "SSA formulas".

Syntax based refinement

$core$ = unsatCore($pathCons(e_1 \dots e_n)$).

$preds$ = atoms of $core$ after erasing subscripts in its variables

Add $preds$ in the predicate domain to obtain the refined abstract domain

Interpolation

Definition 5.5

Let A and B be formulas such that $A \wedge B$ is unsat. An interpolant I between A and B is a formula such that

- $A \Rightarrow \mathrm{I}$
- $B \wedge \mathrm{I} \Rightarrow \bot$
- $vars(\mathrm{I}) \subseteq vars(A) \cap vars(B)$

Theorem 5.5 (Craig interpolation theorem)

Interpolant always exists.

Example 5.4

Consider:

$A = x_1 + x_2 \leq 2 \wedge x_3 - x_2 \leq 0$

$B = 6x_4 - 2x_1 \leq -8 \wedge -3x_4 - x_3 \leq 0$

$vars(A) = \{x_1, x_2, x_3\} \quad vars(B) = \{x_1, x_3, x_4\} \quad vars(\mathrm{I}) \subseteq \{x_1, x_3\}$

$\mathrm{I} = x_1 + x_3 \leq 2$

Interpolation chain

We can extend the definition of interpolant to our setting

Definition 5.6

Consider unsat formula $\bigwedge_{i \in 1..m} e_i(V_{i-1}, V_i)$. *An interpolant chain is a sequence of formulas such that* $\mathrm{I}_0 \ldots \mathrm{I}_m$ *such that*

- $\mathrm{I}_0 = \top$
- $\forall i \in 1..m\ \mathrm{I}_{i-1} \wedge e_i(V_{i-1}, V_i) \Rightarrow \mathrm{I}_i$
- $\mathrm{I}_m = \bot$
- $vars(\mathrm{I}_i) \subseteq V_i$

Interpolation for refinement

We compute interpolation chain $\mathrm{I}_0 \ldots \mathrm{I}_m$ for $\mathit{pathCons}(\mathit{cons})$

preds = atoms in $\mathrm{I}_0 \ldots \mathrm{I}_m$ after erasing subscripts in its variables

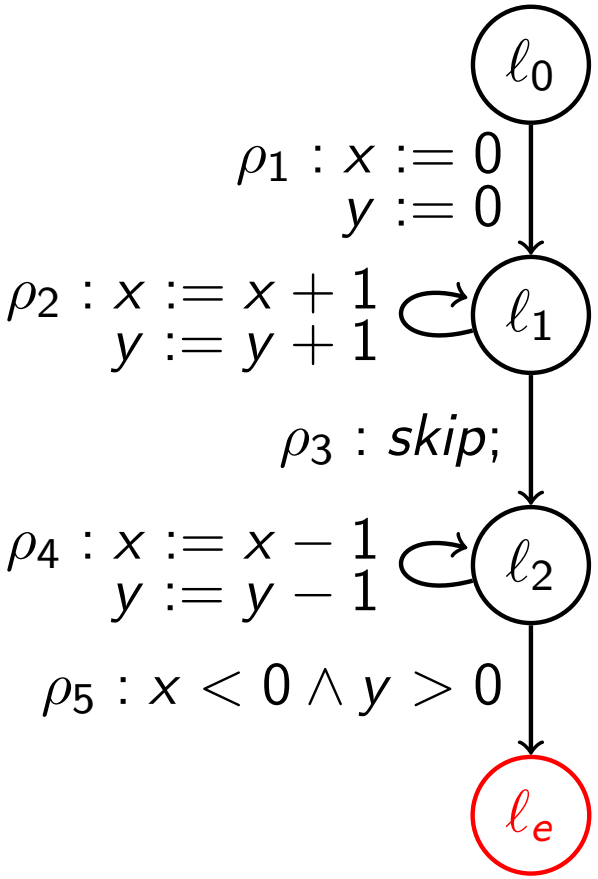
Add preds in the predicate domain to obtain the refined abstract domain

Theorem 5.6

The new abstract domain eliminates spurious counterexample $e_1 \ldots e_n$

Example: interpolation for refinement

Program:



Spurious counterexample: $\rho_1\rho_2\rho_3\rho_4\rho_4\rho_5$.

$\top$

$\rho_1(x_0, y_0, x_1, y_1) = (x_1 = 0 \wedge y_1 = 0)$

$I_1 = y_1 \leq 0$

$\rho_2(x_1, y_1, x_2, y_2) = (x_2 = x_1 + 1 \wedge y_2 = y_1 + 1)$

$I_2 = y_2 \leq 1$ $\leftarrow$ New predicate

$\rho_3(x_2, y_2, x_3, y_3) = (x_3 = x_2 \wedge y_3 = y_2)$

$I_3 = y_3 \leq 0$

$\rho_4(x_3, y_3, x_4, y_4) = (x_4 = x_3 - 1 \wedge y_4 = y_3 - 1)$

$I_4 = y_4 \leq 0$

$\rho_4(x_4, y_4, x_5, y_5) = (x_5 = x_4 - 1 \wedge y_5 = y_4 - 1)$

$I_5 = y_5 \leq 0$

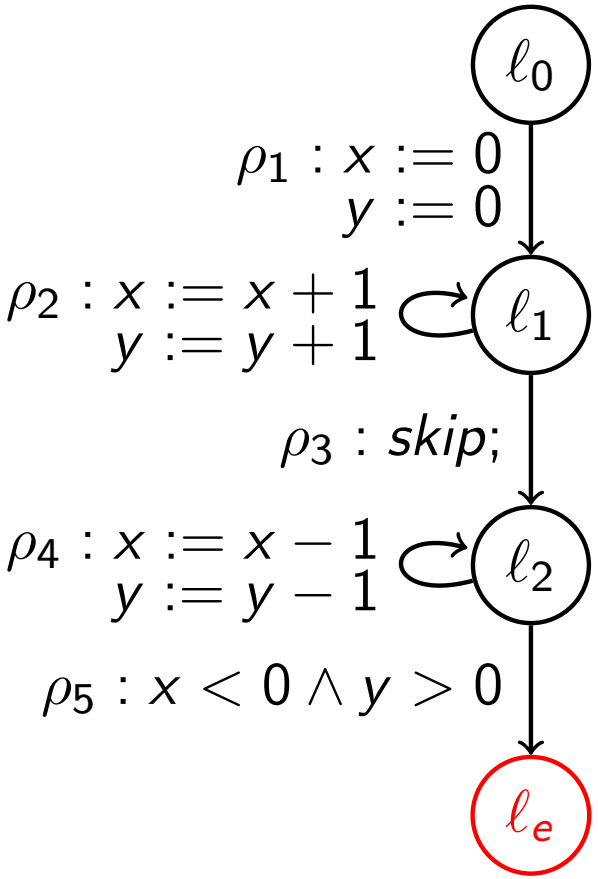
$\rho_5(x_5, y_5, x_6, y_6) = (x_5 < 0 \wedge y_5 > 0)$

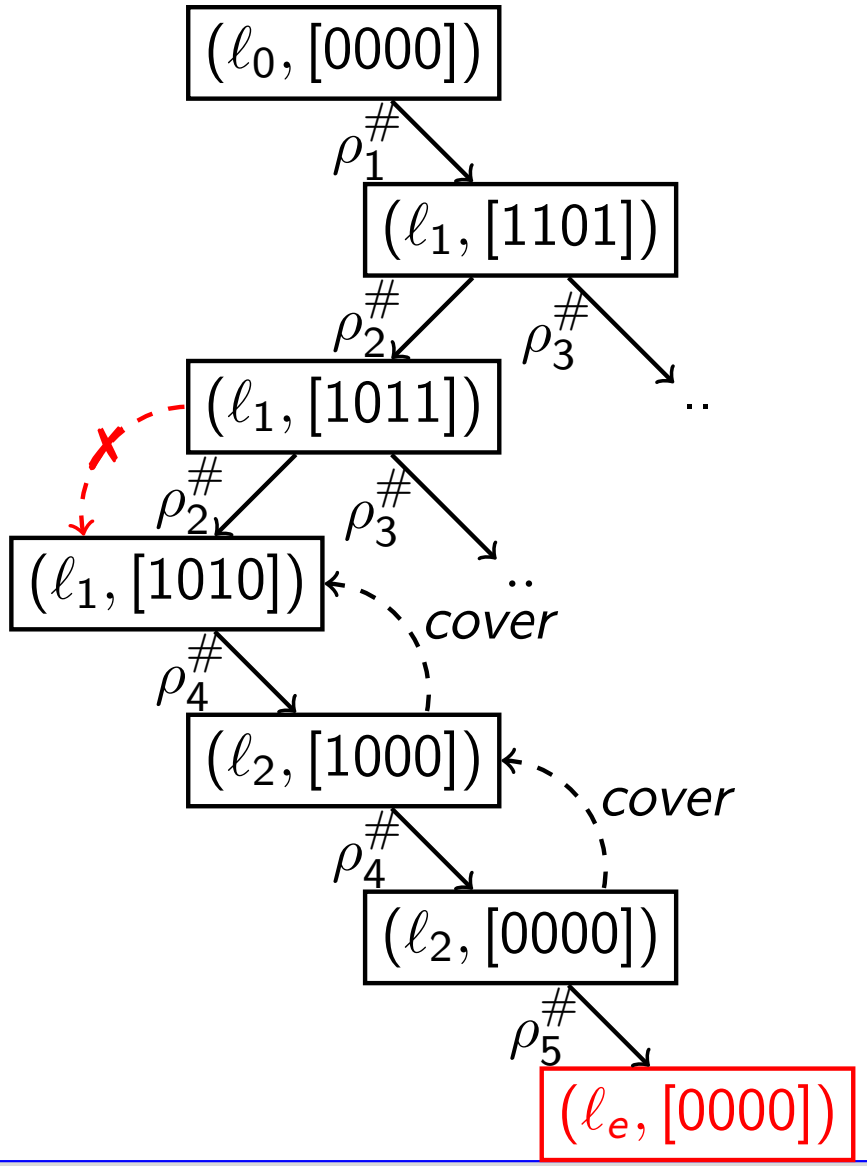
$\bot$

Example: refined reachability graph

$$Preds = \{x \geq 0, y \leq 0, x \geq 1, y \leq 1\}$$

Program:



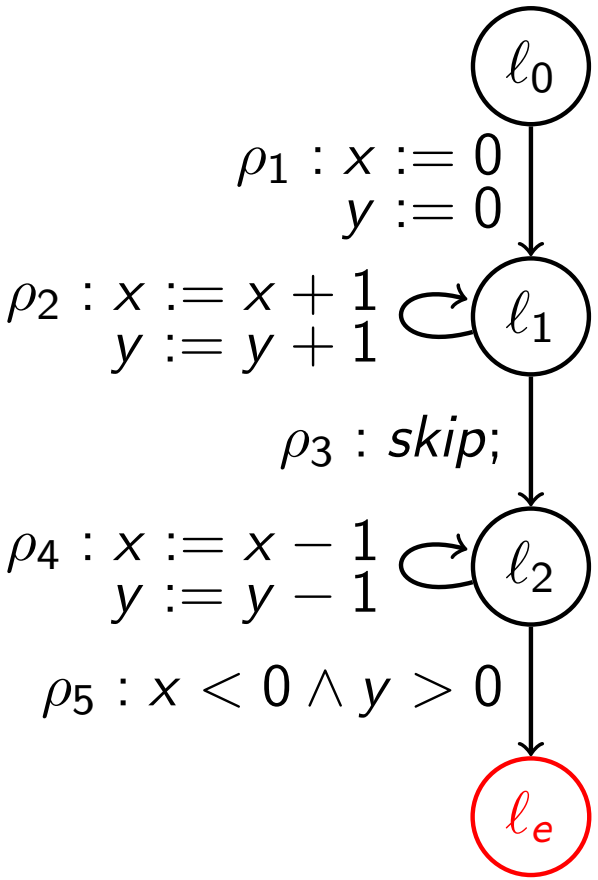


Exercise 5.4

Complete the ARG

Example: good refinement

Program:



Consider the earlier spurious counterexample again: $\rho_1\rho_2\rho_3\rho_4\rho_4\rho_5$.

$\top$

$\rho_1(x_0, y_0, x_1, y_1) = (x_1 = 0 \wedge y_1 = 0)$

$I_1 = y_1 \leq x_1$

$\rho_2(x_1, y_1, x_2, y_2) = (x_2 = x_1 + 1 \wedge y_2 = y_1 + 1)$

$I_2 = y_2 \leq x_2$ $\leftarrow$ New predicate

$\rho_3(x_2, y_2, x_3, y_3) = (x_3 = x_2 \wedge y_3 = y_2)$

$I_3 = y_3 \leq x_3$

$\rho_4(x_3, y_3, x_4, y_4) = (x_4 = x_3 - 1 \wedge y_4 = y_3 - 1)$

$I_4 = y_4 \leq x_4$

$\rho_4(x_4, y_4, x_5, y_5) = (x_5 = x_4 - 1 \wedge y_5 = y_4 - 1)$

$I_5 = y_5 \leq x_5$

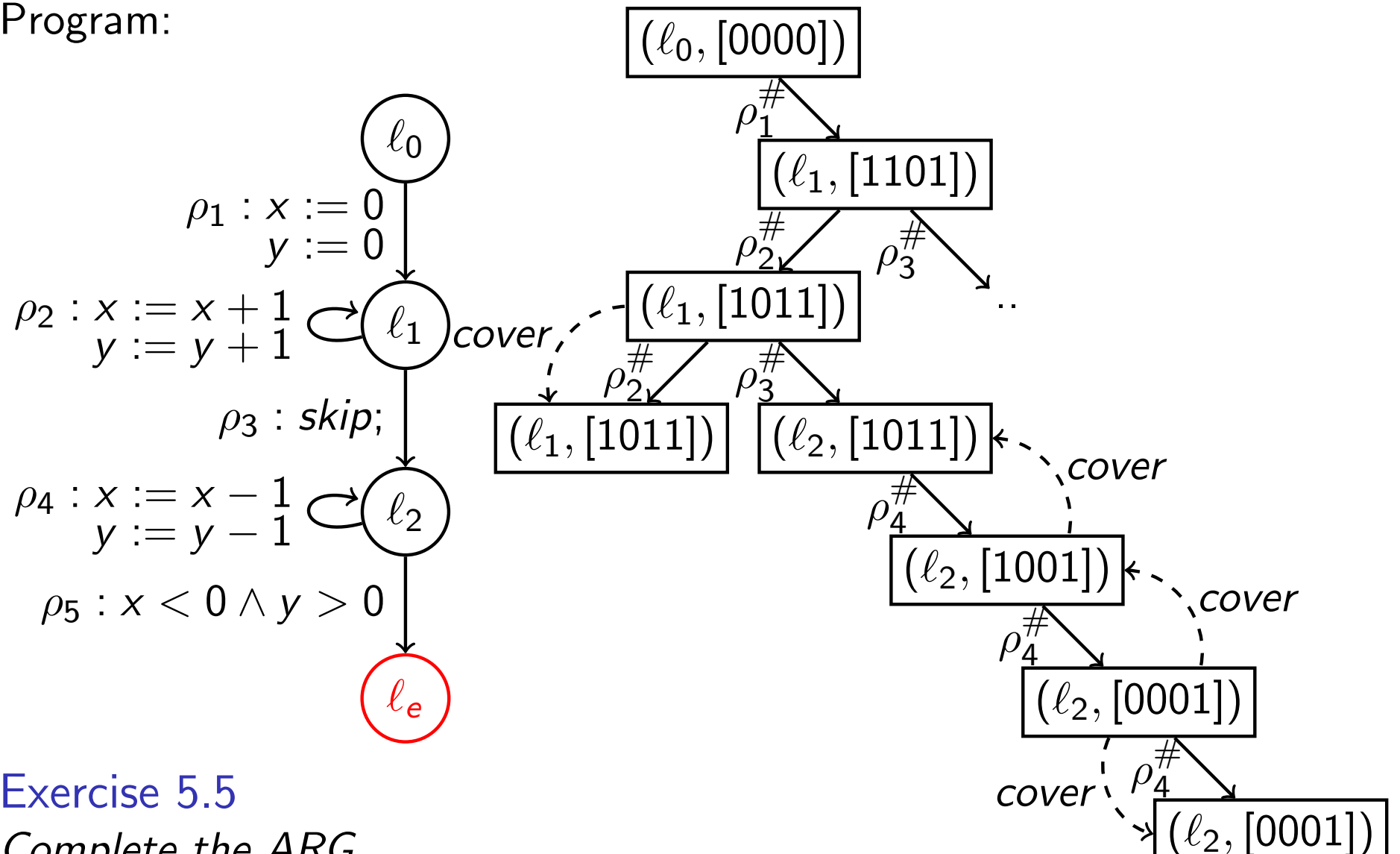
$\rho_5(x_5, y_5, x_6, y_6) = (x_5 < 0 \wedge y_5 > 0)$

$\bot$

Example: ARG without spurious counterexample

$Preds = \{x \geq 0, y \leq 0, x \geq 1, y \leq x\}$

Program:



Exercise 5.5

Complete the ARG

Observation on CEGAR

- Bad predicates waste our time
- We may reuse the ARG computed in last iteration
- Potential exponential blowup

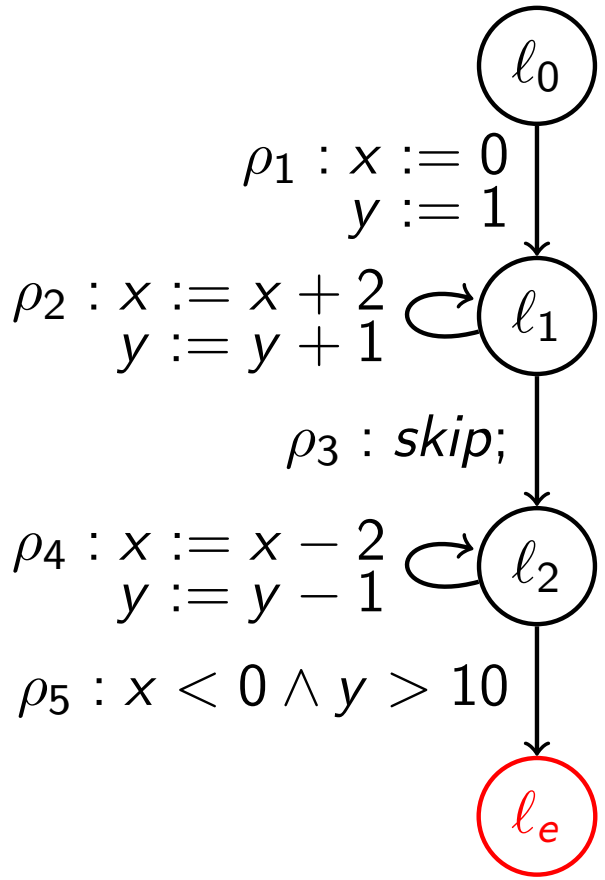
Topic 5.2

Problems

Abstract reachability graph

Exercise 5.6

Choose a set of predicates that will prove the following program correct and show the ARG of the program using the predicates.



CPAchecker

Exercise 5.7

Download CPAchecker: https://cpachecker.sosy-lab.org/
Apply the tool on the following example and report the generated ARG.

```
int x=0; y=0; z=0; w=0;
while( * )) {
  if( * ) {
     x = x+1;
     y = y+100;
  }else if ( * ) {
    if (x >= 4) {
      x = x+1;
      y = y+1;
    }
  }else if (y > 10*w && z >= 100*x) {
     y = -y;
  }
  w = w+1;
  z = z+10;
}
if (x >= 4 && y <= 2)
  error();
```

LTL to Bübhi

Exercise 5.8

Convert the following LTL formula into a Büchi automatom

$$\Box\Diamond a \wedge \Diamond\Box b$$

End of Lecture 5