# Automated reasoning 2016

## Lecture 2: Theory of linear rational arithmetic (LRA)

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# Where are we and where are we going?

We have

- 

We will

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Topic 2.1

# Theory of rational linear arithmetic

# Rational linear arithmetic

**Rational linear arithmetic:** quantifier-free first order formulas with structure $\Sigma = (\{+/2, 0, 1, \dots\}, \{< /2\})$ with a set of axioms

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We will discuss a number of methods to find satisfiability of conjunction of linear inequalities.

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- Simplex

We will not cover following methods

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We present the above methods using non-strict linear inequalities. However, they are extendable to strict inequalities, equalities, dis-equalities.

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## Exercise 2.1

*a. Add support for equality, dis-equality, and strict inequalities*

*b. What is the complexity?*

# Example: Fourier-Motzkin

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*Consider:* $-x_1 + x_2 + 2x_3 \leq 0 \wedge x_1 - x_2 \leq 0 \wedge x_1 - x_3 \leq 0 \wedge 1 - x_3 \leq -1$

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Both complexity and practical performance of the algorithm are bad.

Topic 2.2

## Simplex

# Simplex

Simplex was originally for linear optimization

A variation of simplex is used to check satisfiablity, which is called incremental simplex

- incremental simplex takes atoms one by one and finds satisfying assignment of so for taken atoms before processing the next atom.

# Incremental simplex as theory solver

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The above constraints will be denoted by

$$Ax = 0 \text{ and } \bigwedge_{i=1}^{m+n} l_i \leq x_i \leq u_i.$$

$l_i$ and $u_i$ are $+\infty$ and $-\infty$ if there is no lower and upper bound, respectively.

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We have slack variables $s_1$ and $s_2$. In matrix form,

$$\left[\begin{array}{cc:cc} -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \left[\begin{array}{c} s_1 \\ s_2 \\ \hdashline x \\ y \end{array}\right] = 0 \qquad \begin{array}{c} s_1 \leq -2 \\ s_2 \leq 3 \end{array}$$

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*Currently,* $s_1$ *and* $s_2$ *are basic and* $x$ *and* $y$ *are nonbasic.*
$B = \{1, 2\}$, $NB = \{3, 4\}$, $k_1 = 1$, *and* $k_2 = 2$.

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*Simplex maintains* current assignment $v : x \to \mathbb{Q}$ *s.t.*

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Initially, $v = \{\underbrace{x \mapsto 0, y \mapsto 0}_{\text{Choose values for nonbasic variables, others follow!}}, s_1 \mapsto 0, s_2 \mapsto 0\}$

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*We will mark the active bounds by $^*$.*

### Definition 2.5

*The NB set and bound activity defines the current state of simplex.*

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*Since all nonbasic variables have no bounds, no bound is marked active.*

# Simplex - pivot operation

If $v$ violates a bound constraint of some basic variable, then simplex correct it by applying pivot operation

## Definition 2.6

*Let us suppose $x_j$ is basic, column $j$ has $-1$ at row $k$, and $x_i$ is nonbasic.*

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & & j & & i & & \\ & & \downarrow & & \downarrow & & \end{array} \\
k \longrightarrow & \begin{bmatrix} & \vdots & 0 & \vdots & a & \vdots & \\ & \dots & -1 & \dots & b & \dots & \\ & \vdots & 0 & \vdots & c & \vdots & \end{bmatrix}
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$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & j & & i & \\ & \downarrow & & \downarrow & \end{array} \\
k \longrightarrow & \begin{bmatrix} \vdots & 0 & \vdots & a & \vdots \\ \dots & -1 & \dots & b & \dots \\ \vdots & 0 & \vdots & c & \vdots \end{bmatrix}
\end{array}
$$

# Simplex - pivot operation

If $v$ violates a bound constraint of some basic variable, then simplex correct it by applying pivot operation

## Definition 2.6

*Let us suppose $x_j$ is basic, column $j$ has $-1$ at row $k$, and $x_i$ is nonbasic. A pivot operation between $i$ and $j$ exchanges the roll between $x_i$ and $x_j$. The pivot operation applies row operations until column $i$ has $-1$ at row $k$ and all other entries in the column are zero.*

After pivot operation between $i$ and $j$

$$
\begin{array}{c c}
 & \begin{array}{ccccc} & j & & i & \\ & \downarrow & & \downarrow & \end{array} \\
k \longrightarrow & \begin{bmatrix} \vdots & 0 & \vdots & a & \vdots \\ \dots & -1 & \dots & b & \dots \\ \vdots & 0 & \vdots & c & \vdots \end{bmatrix}
\end{array}
=>
\begin{bmatrix} \vdots & \frac{a}{b} & \vdots & 0 & \vdots \\ \dots & \frac{1}{b} & \dots & -1 & \dots \\ \vdots & \frac{c}{b} & \vdots & 0 & \vdots \end{bmatrix}
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# Simplex - choosing non-basic column for pivot

Wlog, let $1 \in B$ , $k_1 = 1$, and $v(x_1)$ violates $u_1$. We need to decrease $v(x_1)$.
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### Definition 2.7

*A column $i \in NB$ is suitable if*

- *$x_i$ is unbounded,*
- *$x_i = u_i$ and $a_{1i} > 0$, or*
- *$x_i = l_i$ and $a_{1i} < 0$.*

*$i$ is selected suitable column if $i$ is the smallest suitable column.*

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*Column 3 and 4 are suitable.*

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### Exercise 2.2

*Write other cases that are ignored due to "wlog"*

*Column 3 and 4 are suitable.*

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$$ch := min\bigcup_{j\in B}\{\frac{v(x_j)-u_j}{a_{k_ji}}|a_{k_ji}<0\}\cup\{\frac{v(x_j)-l_j}{a_{k_ji}}|a_{k_ji}>0\}$$

*Let $j$ be the smallest index for which the above min is attained.*

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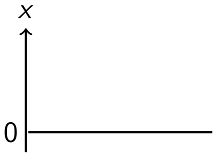

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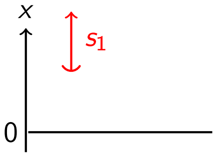

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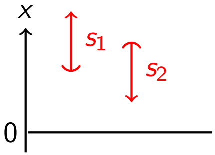

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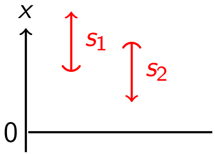

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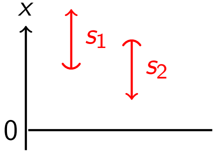
*Clearly, $ch = 3$ and $j = 2$.*



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We carry $ch$ and $j$ from the last slide. Wlog, $ch = \frac{v(x_j) - u_j}{a_{k_j i}}$.

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Now there are three possibilities

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### Example 2.8

*After pivoting between* 3 *and* 2*.*

$$\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ x \\ y \end{bmatrix} = 0 \qquad \begin{matrix} s_1 \leq -2 \\ s_2 \leq 3^* \end{matrix}$$

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*Now $v$ is satisfying.*

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New atom $ax \leq b$ is added in the following steps.

- A fresh slack variable $s$ is introduced
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Therefore, current assignment can only violate the bound of $s$.

# Example: inserting a new atom

## Example 2.9

*Let us add atom* $-2x - y \leq -8$ *in our running example*

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*Let us add atom $-2x - y \leq -8$ in our running example*
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## Exercise 2.3

*Now $s_3$ is violated. Pivot if possible.*

# Simplex - iterations

Simplex is a sequence of pivot operations

- If simplex fails to find a suitable column for some violation then input is unsat.
- If a state is reached without violation then $v$ is a satisfying assignment.

## Example 2.10

*$s_3$ is still in violation.*

$$\begin{bmatrix} -1 & -1 & -3 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s_3 \\ s_1 \\ s_2 \\ x \\ y \end{bmatrix} = 0 \qquad \begin{array}{c} s_3 \leq -8 \\ s_1 \leq -2^* \\ s_2 \leq 3^* \end{array}$$

*Now, we can not find a suitable column.*

*Therefore, the constraints are unsat.*

# Example 2.11

*Run simplex on* $x_1 \leq 5 \wedge 4x_1 + x_2 \leq 25 \wedge -2x_1 - x_2 \leq -25$

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## Exercise 2.4

*Finish the run*

# Popping atoms

If we want to remove some atom from simplex state, we

# Popping atoms

If we want to remove some atom from simplex state, we

- make the corresponding slack variable basic variable and
- remove the corresponding row and bound constraints on the slack variable

# Unsat core extraction

If input is unsat, then there must be a basic variable whose value is violated.

- we collect the slack variables that appear in the row that defines the basic variable
- the atoms corresponding to the slack variables are part of unsat core

# Simplex- geometric intuition

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For example, $A^p$ is the value of $A$ at $p$th iteration.

# Simplex - geometric intuition: meaning of suitable column

Let us introduce the following object in each iterations

- Let $\mu^p$ be a row vector of length $2(m+n)$ s.t.

$$\begin{bmatrix} 1 & \underbrace{0}_{m-1} & \mu^p \end{bmatrix} \begin{bmatrix} A^p \\ \mathrm{I} \\ -\mathrm{I} \end{bmatrix} = \begin{bmatrix} -1 & \underbrace{0}_{m+n-1} \end{bmatrix}$$

$$\mu^p_k = \begin{cases} -A^p_{1k} & k \in NB^p \text{ and } u_k \text{ is active at } p\text{th iteration} \\ A^p_{1(k-(m+m))} & (k-(m+n)) \in NB^p \text{ and } l_{k-(m+n)} \text{ is active at } p\text{th} \\ 0 & \text{otherwise} \end{cases}$$

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## Theorem 2.2

*Let $i'$ be the smallest index for which $\mu^p$ has a negative number and $i$ be the selected suitable column for the next pivoting. Then,*

$$i = \begin{cases} i' & i' \leq m+n \\ i'-(m+n) & \textit{otherwise.} \end{cases}$$

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## Exercise 2.5

*Prove the above.*

# Simplex - geometric intuition: update direction

Selection of suitable column induces the idea of update direction

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Exercise 2.6

*Show* $[-1\ 0]y^p > 0$

# Simplex - geometric intuition: limit on update

- The change in direction $y$ only violate bounds on basic variables

$$ch := min \bigcup_{j \in 1..m} \{ \frac{u_j - v(x_j)}{y_j^p} | y_j^p > 0 \} \cup \{ \frac{l_j - v(x_j)}{-y_j^p} | y_j^p > 0 \}$$

Let $j$ be the smallest index for which the above min is attained, which is used for pivoting.

## Exercise 2.7

*Check the basis column $j$ selected above is same as the pivot basis column selected earlier*

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## Lemma 2.1

*Simplex terminates.*

## Proof.

In every step the violation difference $(v(x_1) - u_1)$ reduces or stays same.

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Wlog, let us suppose the states of $s$th and $t$th iterations of simplex is same and there is no change in $v(x_1) - u_1$ from $p$ to $q$.

Let $r$ be the largest index column which left and reentered $NB$ at iteration $p$ and $q$ respectively, where $s \leq p < q \leq t$.

# Simplex - termination(contd.)

Now Consider,

$$\begin{bmatrix} 1 & \underbrace{0}_{m-1} & \mu^p \end{bmatrix} \begin{bmatrix} A^p \\ \mathtt{I} \\ -\mathtt{I} \end{bmatrix} y^q = [-1 \;\; 0] y^q > 0$$

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Let us apply a different calculation on the above term.

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$$= \begin{bmatrix} 1 & \underbrace{0}_{m-1} & \mu^p \end{bmatrix} \begin{bmatrix} 0 \\ \mathtt{I} y^q \\ -\mathtt{I} y^q \end{bmatrix} = \mu^p \begin{bmatrix} y^q \\ -y^q \end{bmatrix}$$

# Termination

Let $\hat{y}^p \triangleq \begin{bmatrix} y^q \\ -y^q \end{bmatrix}$

Now we show every $\mu_j \hat{y}_j^p$ is non-positive.

- $j \in B^p$ or $j - n \in B^p$ or $j$th bound is inactive, $\mu_j^p = 0$
- $j \in NB^p$ or $j - n \in NB^p$, and $j$th bound is active
  - $j > r$, $y_i^q = 0$
  - $j = r$, $\underbrace{u_r^p < 0}$ and $y_r^q > 0(why?)$

    because $r$ is selected to leave $NB^p$

  - $j > r$, $\overbrace{u_j^p \geq 0}$ and $y_j^q \leq 0$ (why?)

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Ellipsoid method is a polynomial time algorithm for linear constraints. In practice, simplex performs better in many classes of problems.

# End of Lecture 2