Automated reasoning 2016

Lecture 3: Difference and Octagonal logic

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Where are we and where are we going?

We have seen

- EUF, LRA, and LIA solvers

We will see solvers for

- Difference logic
- Octagonal logic

Lecture is based on:

The octagon abstract domain.Antoine Miné. In Higher-Order and Symbolic Computation (HOSC), 19(1), 31-100, 2006. Springer.

Topic 3.1

Difference logic

Logic vs. theory

In the world of SMT solving, words logic and theory are used differently than rest of formal methods.

- theory = FOL + axioms
- logic = theory+syntactic restrictions

Example 3.1

LRA is a theory
QF_LRA is logic, which has only quantifier free LRA formulas

Difference Logic

Difference Logic over the integers(QF_IDL):
Boolean combinations of inequalities of the form $x - y \leq b$ where x and y are integer variables and b is an integer constant.

Difference Logic over the rationals(QF_RDL):
Boolean combinations of inequalities of the form $x - y \leq b$ where x and y are rational variables and b is an rational constant.

Widely used in analysis of timed systems for comparing clocks.

We will present an $O(n^3)$ method to decide conjunction of literals in QF_RDL and QF_IDL.

Difference Graph

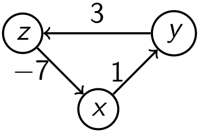
We may view an atom $x - y \leq b$ as a weighted directed edge between two nodes x and y with weight b in graph over variables. This graph is called difference graph.

Theorem 3.1

A conjunction of difference inequalities is unsatisfiable iff the corresponding difference graph has negative cycles.

Example 3.2

$$x - y \leq 1 \wedge y - z \leq 3 \wedge z - x \leq -7$$



Difference bound matrix

Another view of difference graph.

Definition 3.1

Let F be conjunction of difference inequalities over rational variables $\{x_1, \dots, x_n\}$. The difference bound matrix(DBM) A is defined as follows.

$$A_{ij} = \begin{cases} 0 & i = j \\ b & x_i - x_j \leq b \in F \\ \infty & otherwise \end{cases}$$

Let $F[A] \triangleq \bigwedge_{i,j \in 1..n} x_i - x_j \leq A_{ij}$.

Let $A_{i_0 \dots i_m} \triangleq \sum_{k=1}^{m} A_{i_{k-1} i_k}$.

Example: DBM

Example 3.3

Consider:

$x_2 - x_1 \leq 4 \wedge x_1 - x_2 \leq -1 \wedge x_3 - x_1 \leq 3 \wedge x_1 - x_3 \leq -1 \wedge x_2 - x_3 \leq 1$

Constraints has three variables x_1, x_2, *and* x_3.

The corresponding DBM is

$$\begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & \_\_\_ \\ 3 & \_\_\_ & 0 \end{bmatrix}$$

Exercise 3.1

Fill the blanks

Shortest path closure

Definition 3.2

The shortest path closure $A^\bullet$ *of* A *is defined as follows.*

$$(A^\bullet)_{ij} = \min_{i=i_0,i_1,\dots,i_m=j \text{ and } m\leq n} A_{i_0\dots i_m}$$

Theorem 3.2

F is unsatisfiable iff $\exists i \in 1..n.\ A^\bullet_{ii} < 0$

Proof.

If RHS holds, then trivially unsat.(why?)

If LHS holds, then there must be a proof of unsatisfiability, i.e., there is a positive linear combination of difference inequalities that results in $0 \leq -k$.

Wlog, we assume the combination has only integer coefficients. ...

Shortest path closure: there is a negative loop

Proof(contd.)

claim: there is $A_{i_0,.....,i_m} < 0$ and $i_0 = i_m$.

Let $G = (V, E)$ be a graph s.t.

- $G = \{x_1, ..., x_n\}$
- $\{\underbrace{(x_i, x_j), .., (x_i, x_j)}_{\lambda}\} \subseteq E$ if $x_i - x_j \leq b$ has λ coefficient in the combination

Since each x_i has to cancel out in the combination, x_i has equal in and out degree in G

Therefore, G has a Eularian walk (full traversal without repeating an edge).

The sum along the walk must be negative. ...

Shortest path closure(contd.)

Proof.

claim: Shortest loop with negative sum has no sub-loops

For $0 < p < q < m$, lets suppose $i_p = i_q$.

Since $A_{i_0,..,i_m} = \underbrace{A_{i_p..i_q}}_{\text{loop}} + \underbrace{(A_{i_q..i_m} + A_{i_m..i_p})}_{\text{loop}}$,
one of the two loops must be a negative loop.

Therefore, shorter path exists with negative sum.

Therefore, RHS holds. □

Exercise 3.2 *If F is sat,* $A^{\bullet}_{ij} \leq A^{\bullet}_{ikj}$.

Tightness

Definition 3.3

A is tight if for all i and j

- *if* $A_{ij} < \infty$, $\exists v \models F[A].\ v_i - v_j = A_{ij}$
- *if* $A_{ij} = \infty$, $\forall m < \infty.\ \exists v \models F[A].\ v_i - v_j > m$

Theorem 3.3

If F is sat, $A^\bullet$ *is tight.*

Proof.

Suppose there is a better bound $b < A^\bullet_{ij}$ exists s.t. $F[A^\bullet] \Rightarrow x_i - x_j \leq b$.

Like the last proof, there is a path $i_0..i_m$ s.t. $A_{i_0..i_m} \leq b$, $i_0 = i$ and $i_m = j$ (why?)

If $i_0..i_m$ has a loop then the sum along the loop must be positive.
Therefore, there must be a shorter path from i to j with smaller sum. (why?)
Therefore, a loopfree path from i to j exists with sum less than b.

Therefore, $A^\bullet$ is tight ∎

Implication checking and canonical form

Definition 3.4

A set of objects R represents a class of formulas Σ canonically if for each $F, F' \in \Sigma$ if $F \equiv F'$ and $o \in R$ represents F then o represents F'.

Theorem 3.4

The set of shortest path closed DBMs canonically represents difference logic formulas.

Exercise 3.3

Give an efficient method of checking equisatisfiablity and implication using DBMs.

Floyd-Warshall Algorithm for shortest closure

We can compute $A^\bullet$ using the following iterations generating $A^0, \dots, A^n$.

$$A^0 = A$$
$$A^k_{ij} = \min(A^{k-1}_{ij}, A^{k-1}_{ikj})$$

Theorem 3.5

$A^\bullet = A^n$

Exercise 3.4

Prove Theorem 3.5.

Hint: Inductively show each loop-free path is considered

Exercise 3.5

a. Extend the above algorithm to support strict inequalities

b. Does the above algorithm also works for $\mathbb{Z}$?

Example: DBM

Example 3.4

Consider DBM:

$$A^0 = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & \infty & 0 \end{bmatrix}$$

Apply first iteration:

$$A^1 = min(A^0, \begin{bmatrix} A^0_{111} & A^0_{112} & A^0_{113} \\ A^0_{211} & A^0_{212} & A^0_{213} \\ A^0_{311} & A^0_{312} & A^0_{313} \end{bmatrix}) = min(A^0, \begin{bmatrix} 0 & -1 & -1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Apply second iteration:

$$A^2 = min(A^1, \begin{bmatrix} A^1_{121} & A^1_{122} & A^1_{123} \\ A^1_{221} & A^1_{222} & A^1_{223} \\ A^1_{321} & A^1_{322} & A^1_{323} \end{bmatrix}) = min(A^1, \begin{bmatrix} 3 & -1 & 0 \\ 4 & 0 & 1 \\ 6 & 2 & 2 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Apply third iteration:

$$A^3 = min(A^2, \begin{bmatrix} A^2_{131} & A^2_{132} & A^2_{133} \\ A^2_{231} & A^2_{232} & A^2_{233} \\ A^2_{331} & A^2_{332} & A^2_{333} \end{bmatrix}) = min(A^2, \begin{bmatrix} 2 & 1 & -1 \\ 4 & 3 & 1 \\ 3 & 2 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Topic 3.2

Octagonal constraints

Octagonal constraints

Definition 3.5

Octagonal constraints are boolean combinations of inequalities of the form $\pm x \pm y \leq b$ or $\pm x \leq b$ where x and y are $\mathbb{Z}/\mathbb{Q}$ variables and b is an $\mathbb{Z}/\mathbb{Q}$ constant.

We can always translate octagonal constraints into equisatisfiable difference constraints.

Octagon to difference logic encoding (contd.)

Consider conjunction of octagonal atoms F over variables $V = \{x_1, \dots, x_n\}$.

We construct a difference logic formula F' over variables $V' = \{x'_1, \dots, x'_{2n}\}$.

In the encoding, x'_{2i-1} represents x_i and x'_{2i} represents $-x_i$.

Octagon to difference logic encoding

F' is constructed as follows

$$
\begin{array}{llcll}
F \ni & x_i \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2i} \leq 2b & \in F' \\
F \ni & -x_i \leq b & \rightsquigarrow & x'_{2i} - x'_{2i-1} \leq 2b & \in F' \\
F \ni & x_i - x_j \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2j-1} \leq b, x'_{2j} - x'_{2i} \leq b & \in F' \\
F \ni & x_i + x_j \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2j} \leq b, \quad x'_{2j-1} - x'_{2i} \leq b & \in F' \\
F \ni & -x_i - x_j \leq b & \rightsquigarrow & x'_{2i} - x'_{2j-1} \leq b, \quad x'_{2j} - x'_{2i-1} \leq b & \in F'
\end{array}
$$

Definition 3.6

The DBM corresponding to F' are called octagonal DBMs(ODBMs).

Theorem 3.6

If F is over $\mathbb{Q}$ then

- *If $(v_1, \dots, v_n) \models F$ then $(v_1, -v_1, \dots, v_n, -v_n) \models F'$*
- *If $(v_1, v_2, \dots, v_{2n-1}, v_{2n}) \models F'$ then $(\frac{(v_1 - v_2)}{2}, \dots, \frac{(v_{2n-1} - v_{2n})}{2}) \models F$*

Exercise 3.6

a. Prove Theorem 3.6. b. Give an example over $\mathbb{Z}$ when Theorem 3.6 fails

Example: octagonal DBM

Exercise 3.7

Consider:

$x_1 + x_2 \leq 4 \wedge x_2 - x_1 \leq 5 \wedge x_1 - x_2 \leq 3 \wedge -x_1 - x_2 \leq 1 \wedge x_2 \leq 2 \wedge -x_2 \leq 7$

Corresponding ODBM

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

$x_1 + x_2 \leq 4 \rightsquigarrow x_1 - x_4 \leq 4, x_3 - x_2 \leq 4$

$x_2 - x_1 \leq 5 \rightsquigarrow x_3 - x_1 \leq 5, x_2 - x_4 \leq 5$

$x_1 - x_2 \leq 3 \rightsquigarrow x_1 - x_3 \leq 3, x_4 - x_2 \leq 3$

$-x_1 - x_2 \leq 1 \rightsquigarrow x_1 - x_4 \leq 1, x_3 - x_2 \leq 1$

$x_2 \leq 2 \rightsquigarrow x_3 - x_4 \leq 4$

$-x_2 \leq 7 \rightsquigarrow x_3 - x_4 \leq 14$

Relating indices and coherence

Let $\overline{2k} \triangleq 2k - 1$ and $\overline{2k-1} \triangleq 2k$

Example 3.5

$\bar{1}\bar{1} = 22 \quad \bar{2}\bar{1} = 12 \quad \bar{2}\bar{2} = 11 \quad \bar{3}\bar{1} = 42 \quad \bar{4}\bar{2} = 31 \quad \bar{3}\bar{2} = 41$

Consider the following DBM due to 2 variable octagonal constraints.

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

Cells with matching colors are pairs $(ij, \overline{ji})$.

Definition 3.7

A DBM A is coherent *if* $\forall i, j.\ A_{ij} = A_{\overline{ji}}$.

Unsatisfiability

For $\mathbb{Q}$, any method of checking unsat of difference constraints will work on ODBMs.

Let A be ODBM of F. $A^{\bullet}$ will let us know in $2n$ steps if F is sat.

For $\mathbb{Z}$, we may need to interpret ODBMs differently.
We will cover this shortly.

Implication checking and canonical form

Floyd-Warshall Algorithm does not obtain canonical form for ODBMs.

$x'_k = -x'_{\overline{k}}$ is not needed for satisfiablity check. Consequently, $A^\bullet$ is not canonical over $\mathbb{Q}$.

We need to tighten the bounds that may be proven due to the above equalities.

Exercise 3.8

Give an example such that $A^\bullet$ is not tight for octagonal constraints.

Canonical closure for octagonal constraints

Let us define closure property for ODBMs.

Definition 3.8

For a ODBM A, let $F[A]$ define the corresponding formula over original variables.

Definition 3.9

For both $\mathbb{Z}$ and $\mathbb{Q}$, an ODBM A is tight if for all i and j

- *if $A_{ij} < \infty$ then $\exists v \models F[A].\ v'_i - v'_j = A_{ij}$ and*
- *if $A_{ij} = \infty$ then $\forall m < \infty.\ \exists v \models F[A].\ v'_i - v'_j > m$,*

where $v'_{2k-1} \triangleq v_k$ and $v'_{2k} \triangleq -v_k$

Theorem 3.7

If A is tight then A is a canonical representation of $F[A]$

$\mathbb{Q}$ tightness condition

Theorem 3.8

Let us suppose $F[A]$ *is sat.*
If $\forall i, j, k,\ A_{ij} \leq A_{ikj}$ *and* $A_{ij} \leq (A_{i\bar{i}} + A_{\bar{j}j})/2$ *then* A *is tight*

Proof.

Consider cell ij in A s.t. $i \neq j$.(otherwise trivial)
Suppose A_{ij} is finite.
Let $A' = A[ji \mapsto -A_{ij}, \overline{ij} \mapsto -A_{ij}]$

claim: $v \models F[A]$ and $v'_i - v'_j = A_{ij}$ iff $v \models F[A']$
Forward direction easily holds.(why?)
Since A has no negative cycles, $A_{ij} + A_{ji} \geq 0$. So, $A_{ji} \geq -A_{ij}$. So, $A_{ji} \geq A'_{ji}$.
Therefore, A is pointwise greater than A'. Therefore, $F[A'] \Rightarrow F[A]$.
Since $A'_{ij} = -A'_{ji}$, if $v \models F[A']$ then $v'_i - v'_j = A_{ij}$. Backward direction holds.

...

$\mathbb{Q}$ tightness condition(contd.)

Proof(contd.)

Now we are only left to show the following.

claim: $F[A']$ is sat, which is there are no negative cycles in A'

A' can have negative cycles only if ji or $\overline{ij}$ occur in the cycle.(why?)

Wlog, we assume only ji occurs in a negative cycle $i = i_0..i_m = j$

Therefore, $A'_{ji} + \sum_{l \in 1..m} A'_{i_{(l-1)} i_l} < 0$. Therefore, $-A_{ij} + \sum_{l \in 1..m} A_{i_{(l-1)} i_l} < 0$.

Therefore, $\sum_{l \in 1..m} A_{i_{(l-1)} i_l} < A_{ij}$. Contradiction.

Now we assume both ji and $\overline{ij}$ occur in a negative cycle $i = i_0..i_m i'_0..i_{m'} = j$, where $i_m = \bar{i}$ and $\bar{j} = i'_0$.(one case missing)

Therefore, $A'_{ji} + A'_{\overline{ij}} + \sum_{l \in 1..m} A'_{i_{l-1} i_l} + \sum_{l \in 1..m'} A'_{i'_{l-1} i'_l} < 0$.

Therefore, $-2A_{ij} + \sum_{l \in 1..m} A'_{i_{l-1} i_l} + \sum_{l \in 1..m'} A'_{i'_{l-1} i'_l} < 0$.

Therefore, $-2A_{ij} + A_{i\bar{i}} + A_{j\bar{j}} < 0$. Contradiction. □

Exercise 3.9

a. Prove the $A_{ij} = \infty$ case. *b. Does converse of the theorem holds?*

Computing canonical closure for octagonal constraints

Due to the previous theorem and desire of efficient computation, let us redefine $A^\bullet$ for ODBMs.

Definition 3.10

We compute $A^\bullet$ using the following iterations generating $A^0, \dots, A^{2n} = A^\bullet$. Let $o = 2k-1$ for some $k \in 1..n$.

$$
\begin{aligned}
A^0 &= A \\
(A^{o+1})_{ij} &= \min(A^o_{ij}, \frac{A^o_{i\bar{i}} + A^o_{\bar{j}j}}{2}) && \text{(odd rule)} \\
(A^o)_{ij} &= \min(A^{o-1}_{ij}, A^{o-1}_{ioj}, A^{o-1}_{i\bar{o}j}, A^{o-1}_{io\bar{o}j}, A^{o-1}_{i\bar{o}oj}) && \text{(even rule)}
\end{aligned}
$$

Why so complicated update rules?

- In the even rule, three new paths are analyzed to exploit the implicit structure of ODBMs

We need to prove that $A^\bullet$ is tight.

Example: canonical closure of ODBM

Example 3.6

Consider:

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

First we apply the even rule $o = 1$:

$A^1_{ij} = A^1_{\bar{j}\bar{i}} = \min(A^0_{ij}, A^0_{i1j}, A^0_{i2j}, A^0_{i12j}, A^0_{i21j})$

$A^1_{12} = A^1_{21} = \min(A^0_{12}, A^0_{112}, A^0_{122}, A^0_{1122}, A^0_{1212}) = min(\infty, \infty, \infty, \infty, \infty) = \infty$

$A^1_{24} = A^1_{13} = min(A^0_{24}, A^0_{214}, A^0_{224}, A^0_{2124}, A^0_{2214}) = min(5, \infty, 5, \infty, \infty) = 5$

$A^1_{34} = A^1_{34} = \min(A^0_{34}, A^0_{314}, A^0_{324}, A^0_{3124}, A^0_{3214}) = min(4, 9, 9, \infty, \infty) = 4$

$A^1_{43} = A^1_{43} = \min(A^0_{43}, A^0_{413}, A^0_{423}, A^0_{4123}, A^0_{4213}) = min(14, 4, 4, \infty, \infty) = 4$

Exercise 3.10

Find the tight ODBM for the following octagonal constraints:

$2 \leq x + y \leq 7 \wedge x \leq 9 \wedge y - x \leq 1 \wedge -y \leq 1$

Tightness of $A^\bullet$

Theorem 3.9

$A^\bullet$ is tight.

Proof.

For each i, j, and k, we need to show $A^\bullet_{ij} \leq (A^\bullet_{i\bar{i}} + A^\bullet_{j\bar{j}})/2$ and $A^\bullet_{ij} \leq A^\bullet_{ikj}$.

claim: For $k > 0$, $A^{2k}_{ij} \leq (A^{2k}_{i\bar{i}} + A^{2k}_{j\bar{j}})/2$

Note $A^{2k}_{i\bar{i}} = A^{2k-1}_{i\bar{i}}$.(why?)

By def,

$$(A^{2k})_{ij} \leq \frac{A^{2k-1}_{i\bar{i}} + A^{2k-1}_{j\bar{j}}}{2}.$$

Therefore,

$$(A^{2k})_{ij} \leq \frac{A^{2k}_{i\bar{i}} + A^{2k}_{j\bar{j}}}{2}.$$

...

Tightness of $A^\bullet$(contd.)

Proof(contd.)

We are yet to prove $\forall i, j.\ A^\bullet_{ij} \leq A^\bullet_{ikj}$.

Let $Fact(k, o) \triangleq \forall i, j.\ A^o_{ij} \leq A^o_{ikj} \land A^o_{ij} \leq A^o_{i\bar{k}j}$

So we need to prove $\forall k \in 1..n.\ Fact(2k, 2n)$.

the following three will prove the above by induction:(why?)

1. In odd rules ($o = 2k' - 1$), $Fact(k, o) \Rightarrow Fact(k, o + 1)$ (preserve)
2. In even rules ($o = 2k'$), $Fact(k, o) \Rightarrow Fact(k, o + 1)$ (preserve)
3. After even rules ($o = 2k'$), $Fact(o, o)$ (establish)

...

Tightness of $A^\bullet$: odd rules preserve the facts

Proof(contd.)

claim: odd rule, if $\forall i,j.\ A^o_{ij} \leq A^o_{ikj} \land A^o_{ij} \leq A^o_{i\bar{k}j}$ then $\forall i,j.\ A^{o+1}_{ij} \leq A^{o+1}_{ikj}$.

We have four cases(why?) and denoted them by pairs.

(1,1) $A^{o+1}_{ik} = A^o_{ik}$, $A^{o+1}_{kj} = A^o_{kj}$: $\underbrace{A^{o+1}_{ij} \leq A^o_{ij}}_{\text{odd rule}} \underbrace{\leq A^o_{ikj}}_{\text{lhs}} \underbrace{= A^{o+1}_{ikj}}_{\text{case cond.}}$

(2,1) $A^{o+1}_{ik} = (A^o_{i\bar{i}} + A^o_{k\bar{k}})/2$, $A^{o+1}_{kj} = A^o_{kj}$:

$$\underbrace{A^o_{ij} \leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}j}}{2}}_{\text{odd rule}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}kj}}{2}}_{\text{lhs}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}\bar{k}kj}}{2}}_{\text{lhs}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{k}k} + A^o_{\bar{j}\bar{k}} + A^o_{kj}}{2}}_{\text{rewrite}}$$

$$\underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{k\bar{k}}}{2} + A^o_{kj}}_{\text{coherence}} \underbrace{= A^{o+1}_{ikj}}_{\text{case cond.}}$$

(2,1) $A^{2k}_{ik} = A^o_{ik}$, $A^{o+1}_{kj} = (A^o_{k\bar{k}} + A^o_{\bar{j}j})/2$ (Symmetric to the last case)

(2,2) $A^{o+1}_{ik} = (A^o_{i\bar{i}} + A^o_{k\bar{k}})/2$ and $A^{o+1}_{kj} = (A^o_{k\bar{k}} + A^o_{\bar{j}j})/2$ ...

Exercise 3.11 *Prove the last case.*

Tightness of $A^\bullet$: even rules preserve the facts

Proof(contd.)

claim: even rule, if $\forall i,j.\ A^{o-1}_{ij} \leq A^{o-1}_{ikj} \wedge A^{o-1}_{ij} \leq A^{o-1}_{i\bar{k}j}$ then $\forall i,j.\ A^{o}_{ij} \leq A^{o}_{ikj}$.

Here, we have 25 cases(why?) and denoted them by pairs:

(1,1) $A^o_{ik} = A^{o-1}_{ik}, A^o_{kj} = A^{o-1}_{kj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ij}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{ikj}}_{\text{lhs}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$

(2,1) $A^o_{ik} = A^{o-1}_{iok}, A^o_{kj} = A^{o-1}_{kj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{iokj}}_{\text{lhs}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$.

(4,5) $A^o_{ik} = A^{o-1}_{io\bar{o}k}, A^o_{kj} = A^{o-1}_{k\bar{o}oj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{ioj} + A^{o-1}_{o\bar{o}o} + A^{o-1}_{\bar{o}k\bar{o}}}_{\text{no negative loops}}$

$\underbrace{\leq A^{o-1}_{io\bar{o}k} + A^{o-1}_{k\bar{o}oj}}_{\text{rewrite}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$

...

Exercise 3.12

Prove cases (1,4), (2,3) and (3,3).

Hint: key proof technique: introduce cycles, introduce k

Tightness of $A^\bullet$: even rule establishes the fact

Proof(contd.)

claim: even rule, $\forall i,j.\ A^o_{ij} \leq A^o_{ioj} \wedge A^o_{ij} \leq A^o_{i\bar{o}j}$

We only prove $A^o_{ij} \leq A^o_{ioj}$, the other inequality is symmetric.

Again, we have 25 cases.(why?)

Since there are no negative cycles and $A^o_{oo} = 0$,

$A_{io} = A_{ioo} \leq A_{io\bar{o}o}$ and $i\bar{o}o \leq i\bar{o}oo$.

Therefore, only four cases left to consider.(why?)

(1,1) $A^o_{io} = A^{o-1}_{io}, A^o_{oj} = A^{o-1}_{oj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{= A^o_{ioj}}_{\text{case cond.}}$

(2,2) $A^o_{io} = A^{o-1}_{i\bar{o}o}, A^o_{oj} = A^{o-1}_{o\bar{o}j}$:

$$\underbrace{A^o_{ij} \leq A^{o-1}_{i\bar{o}j}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{i\bar{o}j} + A^{o-1}_{o\bar{o}o}}_{\text{no negative cycles}} \underbrace{\leq A^{o-1}_{i\bar{o}o} + A^{o-1}_{o\bar{o}j}}_{\text{rewrite}} \underbrace{= A^o_{ioj}}_{\text{case cond.}}$$

□

Exercise 3.13

Prove case (1,2).

Octagonal constraints over $\mathbb{Z}$

For $\mathbb{Z}$, we need a stronger property to ensure tightness.

Theorem 3.10

Let A be ODBM interpreted over $\mathbb{Z}$.
if $\forall i, j, k,\ A_{ij} \leq A_{ikj}$, $A_{ij} \leq (A_{i\bar{i}} + A_{\bar{j}j})/2$, *and* A_{ii} *is even then A is tight.*

Computing canonical closure for octgonal DBMs over $\mathbb{Q}$

In this case, we present an incremental version of the closure iterations.
Lets suppose A is tight and we add another octagonal atom in A that updates $A_{i_0 j_0}$ and $A_{\bar{j}_0 \bar{i}_0}$ (Observe: always updated together).
Let A^0 be the updated DBM.

$$(A^1)_{ij} = \min(A^0_{ij}, A^0_{i i_0 j_0 j}, A^0_{i \bar{j}_0 \bar{i}_0 j}) \qquad \text{if } i \neq \bar{j}$$

$$(A^1)_{i\bar{i}} = \min(A^0_{i\bar{i}}, A^0_{i\bar{j}_0\bar{i}_0 i_0 j_0 \bar{i}}, A^0_{i i_0 j_0 \bar{j}_0 \bar{i}_0 \bar{i}}, 2\lfloor \frac{A^0_{i i_0 j_0 \bar{i}}}{2} \rfloor)$$

$$(A^2)_{ij} = \min(A^1_{ij}, \frac{A^1_{i\bar{i}} + A^1_{j\bar{j}}}{2})$$

Theorem 3.11
A^2 is tight

End of Lecture 3