Automated Reasoning 2018

Lecture 12: Other methods for LRA

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Ideas that do not work!!

We have been seeing the success of SMT solvers and their algorithms.

Today, we will look at the methods that may (or may not) look efficient on paper but do not perform well!

This lecture is a word of caution, we may have stuck in a local maxima of ideas!!

Where are going to look a few less popular methods

- Fourier-Motzkin
- Interior point methods
- Ellipsoid method
- Kermaker's method

Topic 12.1

Fourier-Motzkin

Fourier-Motzkin

The algorithm proceeds by eliminating variables one by one. After eliminating all the variables, if the input reduces to $\top$ then only the input is satisfiable.

For variable x, a conjunction of linear inequalities can be transformed into the following form, where x does not occur in the linear terms s_j, t_k, and u_k.

$$\bigwedge_{j=1}^{m} s_j \leq x \wedge \bigwedge_{i=1}^{\ell} x \leq t_i \wedge \bigwedge_{k=1}^{n} u_k \leq 0$$

$$\Updownarrow$$

$$\bigwedge_{j=1}^{m} \bigwedge_{i=1}^{\ell} s_j \leq t_i \wedge \bigwedge_{k=1}^{n} u_k \leq 0$$

equisatisfiable but without x

Exercise 12.1

a. Add support for equality, dis-equality, and strict inequalities

b. What is the complexity?

Example: Fourier-Motzkin

Example 12.1

Consider: $-x_1 + x_2 + 2x_3 \leq 0 \wedge x_1 - x_2 \leq 0 \wedge x_1 - x_3 \leq 0 \wedge 1 - x_3 \leq 0$

Suppose we eliminate x_1 *first. We transform the constraints into our format.*

$$\underbrace{x_2 + 2x_3 \leq x_1}_{x_1 \textit{ lower bounded}} \wedge \underbrace{(x_1 \leq x_2 \wedge x_1 \leq x_3)}_{x_1 \textit{ upper bounded}} \wedge \underbrace{-x_3 + 1 \leq 0}_{x_1 \textit{ does not occur}}$$

Eliminated constraints: $x_2 + 2x_3 \leq x_2 \wedge x_2 + 2x_3 \leq x_3 \wedge -x_3 + 1 \leq 0$

After simplification: $x_3 \leq 0 \wedge x_2 + x_3 \leq 0 \wedge -x_3 + 1 \leq 0$

Since x_2 *has no lower bound, we can drop* x_2 *atoms:* $x_3 \leq 0 \wedge -x_3 + 1 \leq 0$

Eliminating x_3*:* $1 \leq 0 \leftarrow$ *false formula therefore unsat*

Exercise 12.2

How to generate the proof for unsatisfiable conjunctions

Another example

Example 12.2

Consider: $-x_1 + x_2 + 2x_3 \leq 0 \wedge x_1 - x_2 \leq 0 \wedge x_1 - x_3 \leq 0 \wedge -x_1 \leq 2$

Let us transform to eliminate x_1*.*

$$\underbrace{x_2 + 2x_3 \leq x_1 \wedge -2 \leq x_1}_{x_1 \textit{ lower bounded}} \wedge \underbrace{(x_1 \leq x_2 \wedge x_1 \leq x_3)}_{x_1 \textit{ upper bounded}}$$

Eliminated constraints: $x_2 + 2x_3 \leq x_2 \wedge x_2 + 2x_3 \leq x_3 \wedge -2 \leq x_2 \wedge -2 \leq x_3$

After simplification: $x_3 \leq 0 \wedge x_2 + x_3 \leq 0 \wedge -2 \leq x_2 \wedge -2 \leq x_3$

Eliminating x_2*:* $x_3 \leq 0 \wedge -2 \leq -x_3 \wedge -2 \leq x_3$

After simplification: $x_3 \leq 0 \wedge -2 \leq x_3$

Eliminating x_3*:* $-2 \leq 0 \leftarrow$ *true formula therefore sat*

Exercise 12.3

How to generate the model for satisfiable conjunctions?

Fourier-Motzkin in practice

Both complexity and practical performance of the algorithm are bad.

Almost never used in practice, except for some bounded simplifications.

Topic 12.2

Ellipsoid method

Ellipsoid method : a Soviet scare

Khachian found the first polynomial time method for linear programming.

ARCHIVES | 1979

A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE NOV. 7, 1979


The New York Times

Similar Transportation Bond Issues Are Passed in New York and Jersey

CARTER SAID TO ASK FOR LOYALTY CHECK OF AIDES ON POLITICS

IRAN'S CIVIL GOVERNMENT OUT; HOSTAGES FACE DEATH THREAT; OIL EXPORTS BELIEVED HALTED

STUDENTS WARN U.S.

Ayatollah Instructs Secret Revolutionary Council to Form a Cabinet

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring its practical applications.

http://www.nytimes.com/1979/11/07/archives/a-soviet-discovery-rocks-world-of-mathematics-russians-surprise.html

Elliposid method

We want to check satisfiability of

$$Ax \leq b.$$

We assume that space $Ax \leq b$ is bounded and full dimensional.

Coeffcient bounded solution existence

Size are define as follows

- For a number r, $size(r)$ is $\log_2(r)$
- For a row/vector c, $c = n + \sum_i size(c_i)$
- For a matrix A, $size(A) = mn + \sum_{ij} size(A_{ij})$

Theorem 12.1

If solution exists of $Ax \leq b$, then there is a solution with size less than $4n^2\phi$, where ϕ is the maximum row size.

Proof.

Every vertex is a solution of n equalities $A'x = b'$ from $Ax \leq b$.

The vertex is $x = A'^{-1}b'$, which depends on the size of determinant of A'.

Since determinant sums multiples of n numbers, its size is bounded by $2n\phi$.

Therefore, $size(x_i) \leq 4n\phi$. Therefore, $size(x) \leq 4n^2\phi$. □

Minimum volume condition

Theorem 12.2
If $Ax \leq b$ is satisfiable, volume of $Ax \leq b$ is bigger than $2^{-2n(4n^2\phi)}$

Proof sketch.
Since

- all numbers have finite precision,
- the $Ax \leq b$ is sat, and
- the $Ax \leq b$ is full-rank,

there is a smallest volume convex hull that $Ax \leq b$ can represent.

Therefore, there is a lower bound. □

Idea!

- We know that if there is a solution, it is in the finite space.
- The finite space can be divided in finite granularity!!
- If we can iteratively divide the space, we may have an efficient algorithm.

We use ellipses to describe and split the finite space.

Positive definite matrix

Definition 12.1
A symmetric matrix is positive definite if all its eigenvalues are positive.

Theorem 12.3
The following statements are equivalent

1. D *is positive definite*
2. $D = B^T B$ *for some non-singular* B
3. *For each* x, $x^T Dx > 0$

Proof.
(1)⇒(2)

- D can be diagonalized, i.e., $D = P^T D' P$ where D' is a diagonal matrix
- Since all eigenvalues are positive we can split $D' = D'' D''$
- Therefore, $B = P^T D''$ □

Exercise 12.4
a. Prove (2)⇒(3) *b. Prove (3)⇒(1)*

Represnting ellipses

Definition 12.2

The following defines interior of a ellipse

$$ell(z, D) := \{x | (x - z)^T D^{-1} (x - z) \leq 1\}$$

where is D a $n \times n$ positive definite matrix.

- z is the center of the ellipse
- D defines the direction and length of axes

Example 12.3

2-D unit ball is

$$ell(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathtt{I}) = \{x | x^T \mathtt{I} x \leq 1\} = \{x| \ \ |x|^2 \leq 1\}$$

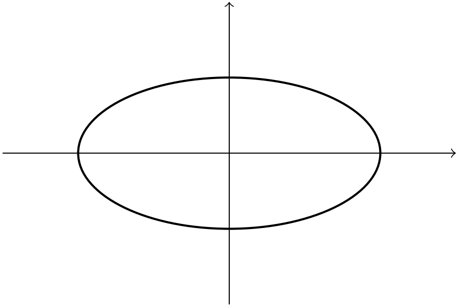
Shorthand for ball $ball(z, r) := ell(z, r^2\mathtt{I})$

Ellipse example: stretched

Example 12.4

Ellipse $x_1^2 + 4x_2^2 \leq 4$ *will be encoded as follows*

$$ell(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}) = \{x | x^T \begin{bmatrix}1/4 & 0\\0 & 1\end{bmatrix} x \leq 1\}$$

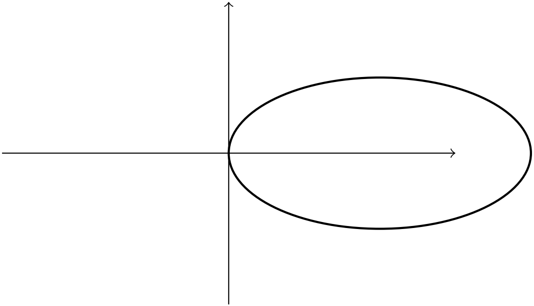


Ellipse example: shifted

Example 12.5

Ellipse $x_1^2 - 4x_1 + 4 + 4x_2^2 \leq 4$ *will be encoded as follows*

$$ell(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}) = \{x|(x - \begin{bmatrix}2\\0\end{bmatrix})^T \begin{bmatrix}1/4 & 0\\0 & 1\end{bmatrix}(x - \begin{bmatrix}2\\0\end{bmatrix}) \leq 1\}$$

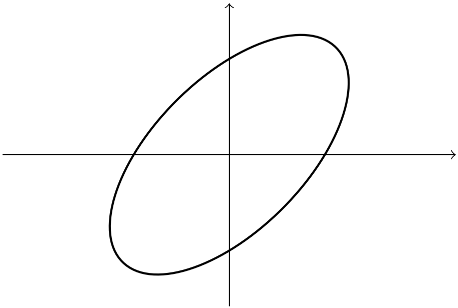


Ellipse example: rotated

Example 12.6

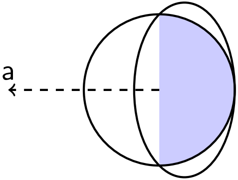
Ellipse $5x_1^2 + 5x_2^2 - 6x_1x_2 \leq 8$ *will be encoded as follows*

$$ell([0,0], \begin{bmatrix} \frac{5}{8} & \frac{-3}{8} \\ \frac{-3}{8} & \frac{5}{8} \end{bmatrix}^{-1}) = \{x | x^T \begin{bmatrix} \frac{5}{8} & \frac{-3}{8} \\ \frac{-3}{8} & \frac{5}{8} \end{bmatrix} x \leq 1\}$$



Smallest covering ellipse

What is the smallest ellipse in area that covers a half circle?



Let a be the unit row vector that defines the half circle, i.e., $Ball(0,1) \cap ax \leq 0$.

The smallest ellipse is

$$ell\left(-\frac{a^T}{n+1} \quad , \quad \frac{n^2}{n^2-1}(\mathrm{I} - \frac{2a^T a}{n+1}) \right)$$

Volume decays exponentially

One axis of the ball is shrunk by $\frac{n}{n+1}$.

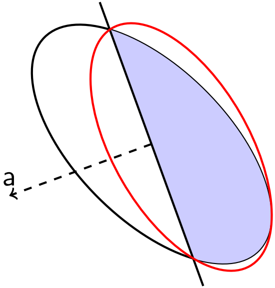
And the other $n-1$ axes are expanded by $\frac{n^2}{n^2-1}^{1/2}$.(why?)

Therefore the volume is changed by the factor of $\frac{n}{n+1}\frac{n^2}{n^2-1}^{(n-1)/2}$

$$\frac{n}{n+1}\frac{n^2}{n^2-1}^{(n-1)/2} < e^{-1/(2n+2)}.\text{(why?)}$$

In more general form

Consider the following ellipse $ell(z, D)$ and direction a



The smaller $\textsc{CoveringEllipse}(z, D, a) :=$

$$ell\left(\quad z - \frac{Da^T}{(n+1)\sqrt{aDa^T}} \quad , \quad \frac{n^2}{n^2-1}\left(D - \frac{2Da^T aD}{(n+1)\sqrt{aDa^T}}\right) \quad \right)$$

Since all ellipse are linear transformations of unit ball, the exponential volume reduction still holds.

Ellipsoid method

Input $Ax \leq 0$.

We know if $(Ax \leq 0)$ is sat, $(Ax \leq 0) \subseteq ball(0, 2^{4n^2\phi})$.

Let us suppose we have initial ellipse $ell(z, D) := ball(0, 2^{4n^2\phi})$.

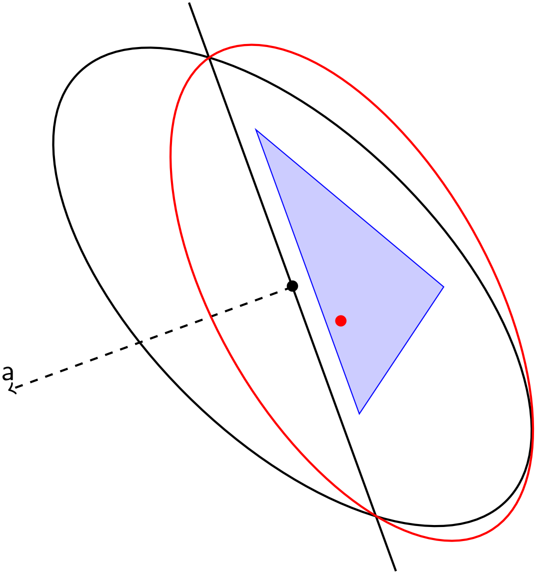
1. if z satisfies $Ax \leq b$, **return** z
2. Otherwise, find inequality $ax \leq \delta$ in $Ax \leq b$ such that $az > \delta$
3. $z, D := \textsc{CoveringEllipse}(z, D, a)$
4. If volume of $ell(z, D)$ is too small, **return** unsatisfiable
5. goto 1

Exercise 12.5

a. Why $ax \leq \delta$ exists at 2?

b. Why smaller ellipses will continue to contain $Ax \leq b$?

Ellipsoid method illustration



Ellipsoid method is polynomial

Theorem 12.4

Ellipsoid method runs less than $16n^2(4n^2\phi)$ *iterations.*

Proof.

1. Initial ellipse has volume less that $(2 \times 2^{4n^2\phi})^n$
2. Volume threshold is $2^{-2n(4n^2\phi)}$
3. Ellipse sizes decreases by the factor of $e^{-1/2(n+1)}$

$\square$

Exercise 12.6

Prove that within the $16n^2(4n^2\phi)$ *iterations the ellipse volume will reduce below the threshold*

Ineffcient method

- Number of iterations depends on the size of the numbers
- Square root needs to be computed, i.e., high precision computation (can be avoided!)
- Experiments show that it can not compete with simplex.

Topic 12.3

Karmakar's method

Karmakar's method: west strikes back

Karmakar fixed some of the problems in ellipsoid method

ARCHIVES | 1984

BREAKTHROUGH IN PROBLEM SOLVING

By **JAMES GLEICK**

The New York Times

AMERICANS IN POLL VIEW GOVERNMENT MORE CONFIDENTLY

TREASURY MAY ASK INCREASES IN TAXES OF SOME CONCERNS

Albany Leaders Predicting a Cut In Income Taxes

Cocaine Traffickers Kill 17 in Peru Raid On Antidrug Team

Vote Comes to a 'Homeland,' But African Problems Linger

Breakthrough in Problem Solving

Homeless Spend Nights in City Welfare Office

High Cuban Defector Speaks Out, Denouncing Castro as 'Impulsive'

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

http://www.nytimes.com/1984/11/19/us/breakthrough-in-problem-solving.html

Efficacy of Karmakar's method

- There are claims that it is far more efficient
- No large scale study to demonstrate (as far as I know!!)
- All SMT solvers still implement simplex

End of Lecture 12