Automated Reasoning 2018

Lecture 14: Integers and Simplex+Gomery cut

Instructor: Ashutosh Gupta

IITB, India

Compile date: 2018-09-04

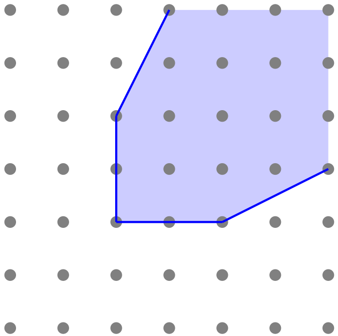
Topic 14.1

Total duality integrality

Integral

Definition 14.1

A polyhedron P is integral if all faces of P have integral vectors.



Faces include any thing that is facing exterior

- Vertices (minimal face)
- Edges
- Many dimensional surfaces

Some properties of faces

- Faces are obtained by converting one or more inequalities to equality.
- Faces are polyhedron themselves.
- Faces have subfaces
- There are minimal dimensional faces.
- All minimal dimensional faces must has same dimension, are subspaces and are translation of each other.

Condition for being integral

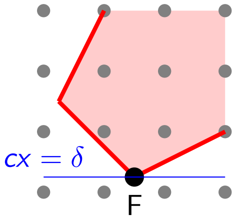
The hyperplanes that are "touching" P

Theorem 14.1

A rational polyhedron P is integral, iff each *supporting hyperplane* *of P has integral vectors.*

Proof.

($\Rightarrow$) trivial.



($\Leftarrow$) Assume $\neg$LHS. We prove $\neg$RHS.
Let $P = \{x|Ax \leq b\}$ for integral A and b, and
$F = \{x|A'x = b'\}$ be a minimal face of P, where $A'x \leq b'$ is a subsystem of $Ax \leq b$, without integral vectors.

Due to theorem 13.2, there is a y such that yA' is integral and yb' is not.
We add positive integers to components of y to make it positive.
Still yA' is integral and yb' is not. Let $c = yA'$ and $\delta = yb'$.
Clearly, $cx = \delta$ has no integral vectors.
Since $F \subseteq cx = \delta$ and $P \subseteq cx \leq \delta_{\text{(why?)}}$, $cx = \delta$ is a supporting hyperplane. $\square$

Commentary: Theorem 22.1 in Schrijver

Total duality integrality(TDI)

Definition 14.2

A rational system $Ax \leq b$ is totally dual integral if the minimum in the LP-duality equation

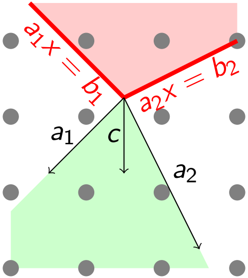
$$max\{cx|Ax \leq b\} = min\{yb|y \geq 0 \wedge yA = c\}$$

has an integral optimum y for each integral c for which the minimum is finite.

Example 14.1

max reaches optima at the corner of the red polyhedron, if c is in the green cone.
TDI says that integral c is nonnegative integral combination of a_1 and a_2.

Therefore, a_1 and a_2 form an Hilbert basis.



Exercise 14.1

Prove: If $Ax \leq b$ is TDI, and $Ax \leq b \Rightarrow ax \leq \beta$, $Ax \leq b \wedge ax \leq \beta$ is a TDI.

TDI has integral optimum solutions

Theorem 14.2

If $Ax \leq b$ is TDI and b is integral, $\{x|Ax \leq b\}$ is integral.

Proof.

Let c be an integral row vector such that $max\{cx|Ax \leq b\}$ is finite.
Since $Ax \leq b$ is TDI and b is integral, $min\{yb|y \geq 0 \wedge yA = c\}$ is integer.(why?)
$\delta = max\{cx|Ax \leq b\}$ is integer.
Let $H = \{x|cx = \delta\}$. H is a supporting hyperplane.
Wlog, we assume $gcd(c) = 1$. Therefore, $cx = \delta$ has integer solutions.

Due to theorem 14.1, $\{x|Ax \leq b\}$ is integral. □

Exercise 14.2

Let $Ax \leq b$ be TDI. If b and c are integral, and $max\{cx|Ax \leq b\}$ is finite, the max achieves optima at integral x.

Commentary: Theorem 22.1a-c in Schrijver

A face of TDI-system is TDI-system

Theorem 14.3

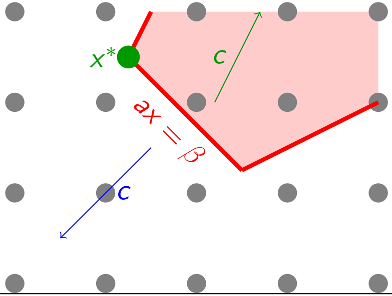
Let $Ax \leq b \wedge ax \leq \beta$ be TDI. Then, $Ax \leq b \wedge ax = \beta$ is also TDI.

Proof.

Let c be an integral vector, with

$$max\{cx|Ax \leq b \wedge ax = \beta\} = min\{yb + (\lambda - \mu)\beta | y, \lambda, \mu \geq 0 \wedge yA + (\lambda - \mu)a = c\}.$$

Let x^*, y^*, λ^* and μ^* attain the optima. ...



Two possibilities:

1. $\lambda * - \mu * \geq 0$
2. $\lambda * - \mu * < 0$

The second case can be handled by rotating c. No need of cases.

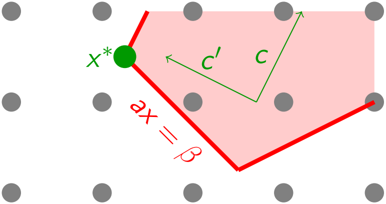
Commentary: Theorem 22.2 in Schrijver

A face of TDI-system is TDI-system II

Proof(contd.)

Let $c' = c + Na$ for some integer N such that $N \geq \mu*$ and Na is integral.

Removes negative a component from c



;

Then optima

$$max\{c'x | Ax \leq b \wedge ax \leq \beta\} = min\{yb + \lambda\beta | y, \lambda \geq 0 \wedge yA + \lambda a = c'\}$$

is finite because

- $x := x^*$ satisfies $Ax \leq b \wedge ax \leq \beta$
- $y := y^*$, and $\lambda := \lambda^* + N - \mu^*$ satisfies $y, \lambda \geq 0 \wedge yA + \lambda a = c'$.

Commentary: N can vary well be $|max(0, \lambda^* - \mu^*)|$

A face of TDI-system is TDI-system III

Proof(contd.)

Since $Ax \leq b \wedge ax \leq \beta$ is TDI, the minimum in the above is attained by integral solution, say y_0, λ_0. Therefore, $y_0 b + \lambda_0 \beta \leq y^* b + (\lambda^* + N - \mu^*)\beta$.

claim: $y = y_0$,$\lambda = \lambda_0$, $\mu = N$ also attains minimum in

$max\{cx | Ax \leq b \wedge ax = \beta\} = min\{yb + (\lambda - \mu)\beta | y, \lambda, \mu \geq 0 \wedge yA + (\lambda - \mu)a = c\}$.

Since $y_0 b + \lambda_0 \beta \leq y^* b + (\lambda^* + N - \mu^*)\beta$, after moving $N\beta$ rhs to lhs

$$y_0 b + (\lambda_0 - N)\beta \leq y^* b + (\lambda^* - \mu^*)\beta$$

Since $y = y^*, \lambda = \lambda^*, \mu = \mu^*$ attains the minimum, therefore $y = y_0$,$\lambda = \lambda_0$, $\mu = N$ attains the minimum. □

Hilbert basis and TDI

An inequality $ax \leq \delta$ of $Ax \leq b$ is active in F if $F \Rightarrow ax = \delta$

Theorem 14.4

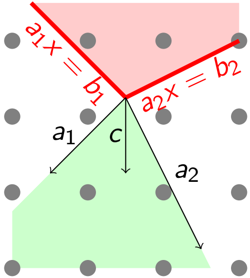
Let $Ax \leq b$ be TDI iff, for each face F of $\{x | Ax \leq b\}$, the inequalities of $Ax \leq b$ that are active in F form a Hilbert basis.

Proof.

$(\Rightarrow)$

Let $a_1 \leq \delta_1, \ldots, a_t \leq \delta_t$ be active on F.
Choose an integral vector c in the cone of $\{a_1, .., a_t\}$.
The maximum attained in the following

$$max\{cx | Ax \leq b\} = min\{yb | y \geq 0 \wedge yA = c\}$$



is achieved by x in F.(why?)
Since $Ax \leq b$ is TDI, the minimum is achieved by integral y.
Due to complementary slackness, the components of y for non-active rows is 0.
Hence c is nonnegative integral combination of $a_1, ...a_t$. ...

Commentary: Theorem 22.5 in Schrijver

Hilbert and TDI

Proof(contd.)

$(\Leftarrow)$
Let c be an integral row vector for which the following is finite.

$$max\{cx|Ax \leq b\} = min\{yb|y \geq 0 \wedge yA = c\}$$

Consider the largest F such that all x in F attain the maximum.(why?)
Let $a_1 \leq \delta_1, \dots, a_t \leq \delta_t$ be active on F.
c must be in the cone of $a_1, ..., a_t$.
Since they form an Hilbert basis $c = \lambda_1 a_1 + \dots + \lambda_t a_t$ for $\lambda_1, ..., \lambda_t \geq 0$.
By zero padding, we can construct integral y such that $yA = c$ and $yb = yAx = cx$ for each x in F.
Therefore, y achives the minimum. Therefore, $Ax \leq b$ is TDI. □

Exercise 14.3

Why we need largest face F?

There is a TDI-system for each polyhedron

Theorem 14.5

For each rational polyhedron P, there is a TDI-system $Ax \leq b$with A integral matrix and rational vector b such that $P = \{x|Ax \leq b\}$.

Proof.

Consider a minimal face F of P.

Let C_F be the cone of vectors c such that $max\{cx|x \in P\}$ is attained by $x \in F$

Let $a_1, \dots, a_t$ be integral Hilbert basis of C_F.

Let $x_0 \in F$. Therefore, for $1 \leq i \leq t$, $P \Rightarrow a_i x \leq a_i x_0$.

Let $A_F = \{a_1 x \leq a_1 x_0, ..., a_t x \leq a_t x_0\}$.

Let $Ax \leq b$ be union of inequalities A_F for each minimal F.

$Ax \leq b$ defines $P_{\text{(why?)}}$ and is TDI due to theorem 14.4. $\square$

Exercise 14.4

a. Why we need minimal face F?

b. Give algorithm for transforming $Ax \leq b$ into a TDI-system?

Commentary: Theorem 22.6 in Schrijver

Topic 14.2

Cutting planes

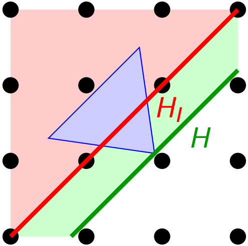
Cutting half spaces

Let $H = \{x | cx \leq \beta\}$ be half space, where $gcd(c) = 1$.

Definition 14.3

For a polyhedron P. Let

$$P' = \bigcap_{P \implies H} H_I$$



Clearly, $P \supseteq P' \supseteq P'' ... \supseteq P^t \supseteq ... \supseteq P_I$.

We will show that the chain will saturate in finite steps.

Exercise 14.5

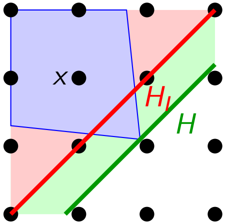
Give a P such that the saturation takes take multiple steps.

TDI-systems quickly finds P'

Theorem 14.6

Let $Ax \leq b$ be TDI and A is integral. Let $P = \{x | Ax \leq b\}$.

$$P' = \{x | Ax \leq \lfloor b \rfloor\}$$



Proof.

If $P = \emptyset$, trivial.(why?)

Let us assume $P \neq \emptyset$.

Clearly, $P' \subseteq \{x | Ax \leq \lfloor b \rfloor\}$.(why?)

claim: $P' \supseteq \{x | Ax \leq \lfloor b \rfloor\}$

Let $H = \{x | cx \leq \delta\}$ be a rational half-space such that $P \subseteq H$.

Wlog we assume $gcd(c) = 1$. Then, $H_I = \{x | cx \leq \lfloor \delta \rfloor\}$.

We have $\delta \geq max\{cx | Ax \leq b\} = min\{yb | y \geq 0 \wedge yA = c\}$.

Since $Ax \leq b$ is TDI, the above min is attained by an integral y_0.

Chose x such that $Ax \leq \lfloor b \rfloor$. Therefore, $cx = y_0 Ax \leq y_0 \lfloor b \rfloor \leq \lfloor y_0 b \rfloor \leq \lfloor \delta \rfloor$.

So $\{x | Ax \leq \lfloor b \rfloor\} \subseteq H_I$.

As this is true for each rational half-space, the claim holds. □

Commentary: Theorem 23.1 in Schrijver

P' carries over to faces

Theorem 14.7

Let F be face of a rational polyhedron P. Then $F' = P' \cap F$

Proof.

Let $P = \{x | Ax \leq b\}$, with A integral and $Ax \leq b$ TDI.
Let $F = \{x | Ax \leq b \wedge ax = \beta\}$ for integral a and β and $P \Rightarrow ax \leq \beta$.(why?)

Since $Ax \leq b \wedge ax \leq \beta$ is TDI(why?), $Ax \leq b \wedge ax = \beta$ is TDI.

Therefore,

$$P' \cap F = \{x | Ax \leq \lfloor b \rfloor \wedge ax = \beta\} = \{x | Ax \leq \lfloor b \rfloor \wedge ax \leq \lfloor \beta \rfloor \wedge ax \geq \lfloor \beta \rfloor\} = F'$$

□

Commentary: Lemma of Theorem 23.1 in Schrijver

$P^t = P_I$

Theorem 14.8

For each rational polyhedron P, there exists a number t such that $P^t = P_I$.

Proof.

We apply induction over dimension d of P.

The case $P = \emptyset$ and $d = 0$ are trivial.

case: Let us suppose $affine.Hull(P)$ has no integers.

Therefore, there is integral vector c and non-integer δ such that $affine.Hull(P) \subseteq \{x|cx = \delta\}$. Hence,

$$P' \subseteq \{x|cx \leq \lfloor\delta\rfloor \land cx \geq \lceil\delta\rceil\} = \emptyset.$$

Therefore, $P' = P_I$. ...

Commentary: Theorem 23.2 in Schrijver

$P^t = P_I$ II

Proof(contd).

case: Let us suppose $affine.Hull(P)$ has integers.

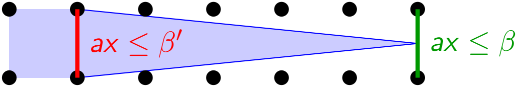
If $affine.Hull(P)$ is not full dimensional, we project it to lower dimensions using Hermite Normal form and apply induction hypothesis. (how?)

Therefore, we may assume $affine.Hull(P)$ is full dimensional.

Due to theorem 13.5, we know $P_I = \{cx | x \leq b'\}$ and $P = \{Ax \leq b\}$.

Let $ax \leq \beta'$ in $Ax \leq b'$, and there is a corresponding $ax \leq \beta$ in $Ax \leq b$.

Let $H = \{x | ax \leq \beta'\}$. ...



$P^t = P_I$ III

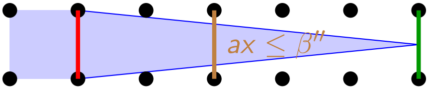
Proof(contd.)

claim: $P^s \subseteq H$ for some s

Let us suppose for each s, we have $P^s \not\subseteq H$.

Therefore, there is an integer β'' and an integer r such that $\beta' < \beta'' \leq \lfloor \beta \rfloor$.

$$\{x|ax \leq \beta'' - 1\} \not\supseteq P^s \subseteq \{x|ax \leq \beta''\} \qquad \text{for each } s \geq r$$



Let $F = P^r \cap \{x|ax = \beta''\}$.

Due to $dim(F) < dim(P)$, F does not contain any integer(why?), and induction hypothesis, $F^u = \emptyset$ for some u.

Therefore,

$$\emptyset = F^u = P^{(r+u)} \cap F = P^{(r+u)} \cap \{x|ax = \beta''\}$$

Therefore, $P^{(r+u)} \subseteq \{x|ax < \beta''\}$.

Therefore, $P^{(r+u+1)} \subseteq \{x|ax \leq \beta'' - 1\}$. Contradiction. $\square$

Cutting plane proofs

Let $Ax \leq b$ be a system of inequalities, and let $cx \leq \delta$ be an inequality.

Definition 14.4

A sequence of inequalities $c_1x \leq \delta_1, \ldots, c_mx \leq \delta_m$ is a cutting plane proof of $cx \leq \delta$ from $Ax \leq b$ if

- $c_m = c$, $\delta_m = \delta$,
- c_1,....c_m *are integral,*
- $c_i = \Lambda A + \lambda_1 c_1 + \cdots + \lambda_{i-1} c_{i-1}$, *and*
- $\delta_i \geq \lfloor \Lambda\delta + \lambda_1\delta_1 + \cdots + \lambda_{i-1}\delta_{i-1} \rfloor$, *where* $\Lambda, \lambda_1, \ldots, \lambda_{i-1} \geq 0$.

m is the length of the proof.

Cutting plane proofs always exist

Theorem 14.9

Let $P = \{x | Ax \leq b\}$ be a nonempty rational polyhedron.

- *If $P_I \neq \emptyset$ and $P_I \Rightarrow cx \leq \delta$, then there is a cutting plane proof of $cx \leq \delta$ from $Ax \leq b$.*
- *If $P_I = \emptyset$, then there is a cutting plane proof of $0 \leq -1$ from $Ax \leq b$.*

Proof.

Let t be such that $P^t = P_I$.

For each $i \geq 1$, there is a system $A_i x \leq b_i$ that defines P^i such that

- For each $\alpha x \leq \beta$ in $A_i x \leq b_i$, there is $yA_{i-1} = \alpha$ and $\beta = \lfloor yb_{i-1} \rfloor$.
- $A_0 = A$ and $b_0 = b$.

...

Commentary: Theorem 23.2b in Schrijver

Cutting plane proofs always exist

Proof(contd.)

If $P_I \neq \emptyset$ and $P_I \Rightarrow cx \leq \delta$, due to the Farkas lemma (affine form) $yA_t = c$ and $\delta \geq yb_t$.
Therefore, the following is the cutting proof of $cx \leq b$ from $Ax \leq b$,

$$A_1 x \leq b_1, ..., A_t x \leq b_t, cx \leq b.$$

If $P_I = \emptyset$, then $yA_t = 0$ and $yb_t = -1$ for some $y \geq 0$.
Therefore, the following is the cutting proof of $0 \leq -1$ from $Ax \leq b$.

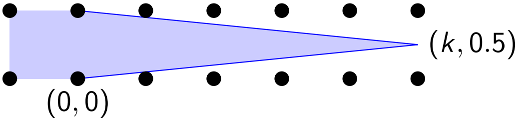
$$A_1 x \leq b_1, ..., A_t x \leq b_t, 0x \leq -1.$$

$\square$

Length of cutting plane proofs

The number of cutting planes depends on the size of numbers!

The following will trigger at least k cuts.



Topic 14.3

Theory of integer linear arithmetic

Integer linear arithmetic(QF_LIA)

Syntax is same as rational integer linear arithmetic with a different axiom set.

We will discuss a number of methods to find satisfiability of conjunction of linear inequalities.

- Cooper's method
- Branch and Bound
- Gomery Cut
- Omega test method

Topic 14.4

Gomery cut

Simplex for integers

Recall our normal form for the input problem

$$Ax = 0 \text{ and } \bigwedge_{i=1}^{m+n} l_i \leq x_i \leq u_i.$$

l_i and u_i are $+\infty$ and $-\infty$ if there is no lower and upper bound, respectively.

In the following presentation of Gomery cut, we assume that

- at least one bound is finite for each variable and
- all finite bounds are integral.

Simplex+Gomery cut

Gomery cut chips away non-integer parts of the solution space.

The algorithm proceeds as follows

1. Run simplex as if all variables are rationals and find an assignment v
2. if v is integral, **return** v
3. if for some $i \in B$, $v(x_i)$ is not integer then add a constraint to eliminate the neighbouring non-integer space.

Consider the row k_i of A, $x_i = \sum_{j \in NB} a_{k_i j} x_j$.

An integer solution must satisfy the equality

Wlog, we assume all upper bounds are active for the nonbasic variables.

$$v(x_i) := \sum_{j \in NB} a_{k_i j} u_j$$

After a rewrite,

$$v(x_i) = x_i + \sum_{j \in NB} a_{k_i j}(u_j - x_j).$$

Simplex+Gomery cut (II)

Consider the following inequality

$\{\delta\} = \delta - \lfloor \delta \rfloor$

$$\{v(x_i)\} \leq \sum_{j \in NB} \{a_{k_i j}\}(u_j - x_j)$$

claim: v does not satisfy the above inequality.

- Since $v(x_i)$ is not an integer, $\{v(x_i)\}$ is positive.
- Under v the rhs is 0.(why?)

claim: Any integer solution of the input satisfies the above inequality.

An integer solution x must satisfy $v(x_i) = x_i + \sum_{j \in NB} a_{k_i j}(u_j - x_j)$.

Therefore,

- $\sum_{j \in NB}\{a_{k_i j}\}(u_j - x_j) \geq 0$(why?)
- $\{v(x_i)\} = \{\sum_{j \in NB}\{a_{k_i j}\}(u_j - x_j)\}$
- $\{v(x_i)\} \leq \sum_{j \in NB}\{a_{k_i j}\}(u_j - x_j)$

Therefore, the inequality separates v from the integer solutions.

We push the above inequality in simplex and run it again.

Branch and bound: Unbounded cases

Let us suppose there is a nonbasic variable that has no bounds.

We can not apply Gomery cut. We may need to case split.

We generate two simplex problems with the following two inequalities respectively.

- $x_i \leq \lfloor v(x_i) \rfloor$
- $x_i \geq \lceil v(x_i) \rceil$

We have removed space $\lceil v(x_i) \rceil > x_i > \lfloor v(x_i) \rfloor$

Solve the two problems separately.

The splits are called branch and bound method.

End of Lecture 14