Automated Reasoning 2018

Lecture 16: Difference and Octagonal logic

Instructor: Ashutosh Gupta

IITB, India

Compile date: 2018-09-25

Where are we and where are we going?

We have seen

- EUF, LRA, and LIA solvers

We will see solvers for

- Difference logic
- Octagonal logic

Lecture is based on:

The octagon abstract domain.Antoine Miné. In Higher-Order and Symbolic Computation (HOSC), 19(1), 31-100, 2006. Springer.

Topic 16.1

Difference logic

Logic vs. theory

- theory = FOL + axioms
- logic = theory+syntactic restrictions

Example 16.1

LRA is a theory
QF_LRA is a logic that has only quantifier free LRA formulas

Difference Logic

Difference Logic over reals(QF_RDL): Boolean combinations of atoms of the form $x - y \leq b$, where x and y are real variables and b is a real constant.

Difference Logic over integers(QF_IDL): Boolean combinations of atoms of the form $x - y \leq b$, where x and y are integer variables and b is an integer constant.

Widely used in analysis of timed systems for comparing clocks.

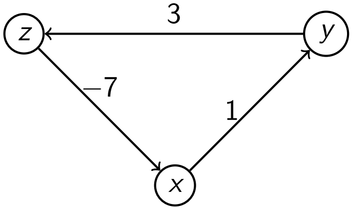
Difference Graph

We may view $x - y \leq b$ as a weighted directed edge between nodes x and y with weight b in a directed graph, which is called difference graph.

Theorem 16.1
A conjunction of difference inequalities is unsatisfiable iff the corresponding difference graph has negative cycles.

Example 16.2

$$x - y \leq 1 \wedge y - z \leq 3 \wedge z - x \leq -7$$



Difference bound matrix

Another view of difference graph.

Definition 16.1

Let F be conjunction of difference inequalities over rational variables $\{x_1, \dots, x_n\}$. The difference bound matrix(DBM) A is defined as follows.

$$A_{ij} = \begin{cases} 0 & i = j \\ b & x_i - x_j \leq b \in F \\ \infty & \textit{otherwise} \end{cases}$$

Let $F[A] \triangleq \bigwedge_{i,j \in 1..n} x_i - x_j \leq A_{ij}$.

Let $A_{i_0 \dots i_m} \triangleq \sum_{k=1}^{m} A_{i_{k-1} i_k}$.

Example: DBM

Example 16.3

Consider:

$x_2 - x_1 \leq 4 \wedge x_1 - x_2 \leq -1 \wedge x_3 - x_1 \leq 3 \wedge x_1 - x_3 \leq -1 \wedge x_2 - x_3 \leq 1$

Constraints has three variables x_1, x_2, and x_3.

The corresponding DBM is

$$\begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & \_\_\_ \\ 3 & \_\_\_ & 0 \end{bmatrix}$$

Exercise 16.1

Fill the blanks

Shortest path closure and satisfiability

Definition 16.2

The *shortest path closure* $A^\bullet$ *of* A *is defined as follows.*

$$(A^\bullet)_{ij} = \min_{i=i_0,i_1,\dots,i_m=j \text{ and } m\leq n} A_{i_0\dots i_m}$$

Theorem 16.2

F *is unsatisfiable iff* $\exists i \in 1..n.\ A^\bullet_{ii} < 0$

Proof.

($\Leftarrow$) If RHS holds, trivially unsat.(why?)

($\Rightarrow$) if LHS holds,
due to farkas lemma 10.8, there is a positive integral linear combination of difference inequalities that is $0 \leq -k$. ...

Shortest path closure: there is a negative loop

Proof(contd.)

claim: there is $A_{i_0,....,i_m} < 0$ and $i_0 = i_m$.

Let $G = (V, E)$ be a weighted directed graph s.t.

- $G = \{x_1, ..., x_n\}$
- $\{\underbrace{(x_i, b, x_j), .., (x_i, b, x_j)}_{\lambda \text{ times}}\} \subseteq E$ if $x_i - x_j \leq b$ has λ coefficient in the proof

Since each x_i cancels out in the proof, x_i has equal in and out degree in G.

Therefore, each SCC of G has a Eulerian cycle (full traversal without repeating an edge). (why?)

The sum along one of the cycles must be negative.(why?) ...

Exercise 16.2

Prove that if a directed graph is a strongly connected component(scc), and each node has equal in and out degree, there is a Eulerian cycle in the graph.

Shortest path closure(contd.)

Proof.

claim: Shortest loop with negative sum has no repeated node

For $0 < p < q < m$, lets suppose $i_0 = i_m$ and $i_p = i_q$.

Since $A_{i_0..i_m} = \underbrace{A_{i_p..i_q}}_{\text{loop}} + \underbrace{(A_{i_q..i_m} + A_{i_m..i_p})}_{\text{loop}}$, one of the two sub-loops is negative.

Therefore, shorter loops exists with negative sum.

Therefore, there is a negative simple loop and RHS holds. □

Exercise 16.3

If F is sat, $A^\bullet_{ij} \leq A^\bullet_{ikj}$.

Floyd-Warshall Algorithm for shortest closure

We can compute $A^\bullet$ using the following iterations generating $A^0, \dots, A^n$.

$$A^0 = A$$
$$A^k_{ij} = \min(A^{k-1}_{ij}, A^{k-1}_{ikj})$$

Theorem 16.3
$A^\bullet = A^n$

Exercise 16.4
a. Prove Theorem 16.3. *Hint: Inductively show each loop-free path is considered*

b. Extend the above algorithm to support strict inequalities

c. Does the above algorithm also works for $\mathbb{Z}$?

Example: DBM

Example 16.4

Consider DBM:

$$A^0 = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & \infty & 0 \end{bmatrix}$$

Apply first iteration:

$$A^1 = min(A^0, \begin{bmatrix} A^0_{111} & A^0_{112} & A^0_{113} \\ A^0_{211} & A^0_{212} & A^0_{213} \\ A^0_{311} & A^0_{312} & A^0_{313} \end{bmatrix}) = min(A^0, \begin{bmatrix} 0 & -1 & -1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Apply second iteration:

$$A^2 = min(A^1, \begin{bmatrix} A^1_{121} & A^1_{122} & A^1_{123} \\ A^1_{221} & A^1_{222} & A^1_{223} \\ A^1_{321} & A^1_{322} & A^1_{323} \end{bmatrix}) = min(A^1, \begin{bmatrix} 3 & -1 & 0 \\ 4 & 0 & 1 \\ 6 & 2 & 2 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Apply third iteration:

$$A^3 = min(A^2, \begin{bmatrix} A^1_{131} & A^1_{132} & A^1_{133} \\ A^1_{231} & A^1_{232} & A^1_{233} \\ A^1_{331} & A^1_{332} & A^1_{333} \end{bmatrix}) = min(A^2, \begin{bmatrix} 2 & 1 & -1 \\ 4 & 3 & 1 \\ 3 & 2 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

Incremental difference logic for SMT solvers

DBMs are not good for SMT solvers, where we need pop and unsat core.

SMT solvers implements difference logic constraints using difference graph. Maintains a current assignment.

- push($x_1 - x_2 \leq b$):
  1. Adds corresponding edge from the graph
  2. If current assignment is feasible with new atom, exit
  3. If not, adjust assignments until it saturates z3:src/smt/diff_logic.h:make_feasible

- Pop($x_1 - x_2 \leq b$):
  - Remove the corresponding edge without worry

- Unsat core
  - If assignment fails to adjust, we can find the set of edges that required the adjustment
  - the edges form negative cycle, and reported as unsat core

Canonical representation

Sometimes a class for formulas have canonical representation.

Definition 16.3

A set of objects R canonically represents a class of formulas Σ if for each $F, F' \in \Sigma$ if $F \equiv F'$ and $o \in R$ represents F then o represents F'.

Tightness

Definition 16.4

A is tight if for all i and j

- *if $A_{ij} < \infty$, $\exists v \models F[A].\ v_i - v_j = A_{ij}$*
- *if $A_{ij} = \infty$, $\forall m < \infty.\ \exists v \models F[A].\ v_i - v_j > m$*

Theorem 16.4

If F is sat, $A^\bullet$ is tight.

Proof.

Suppose there is a better bound $b < A^\bullet_{ij}$ exists s.t. $F[A^\bullet] \Rightarrow x_i - x_j \leq b$.

Like the last proof, there is a path $i_0..i_m$ s.t. $A_{i_0..i_m} \leq b$, $i_0 = i$ and $i_m = j$(why?)

If $i_0..i_m$ has a loop then the sum along the loop must be positive.
Therefore, there must be a shorter path from i to j with smaller sum.(why?)
Therefore, a loopfree path from i to j exists with sum less than b.

Therefore, $A^\bullet$ is tight □

Implication checking and canonical form

Theorem 16.5

The set of shortest path closed DBMs canonically represents difference logic formulas.

Exercise 16.5

Give an efficient method of checking equisatisfiablity and implication using DBMs.

Topic 16.2

Octagonal constraints

Octagonal constraints

Definition 16.5

Octagonal constraints are boolean combinations of inequalities of the form $\pm x \pm y \leq b$ *or* $\pm x \leq b$ *where* x *and* y *are* $\mathbb{Z}/\mathbb{Q}$ *variables and* b *is an* $\mathbb{Z}/\mathbb{Q}$ *constant.*

We can always translate octagonal constraints into equisatisfiable difference constraints.

Octagon to difference logic encoding (contd.)

Consider conjunction of octagonal atoms F over variables $V = \{x_1, \ldots, x_n\}$.

We construct a difference logic formula F' over variables $V' = \{x'_1, \ldots, x'_{2n}\}$.

In the encoding, x'_{2i-1} represents x_i and x'_{2i} represents $-x_i$.

Octagon to difference logic encoding

F' is constructed as follows

$$
\begin{array}{lrcll}
F \ni & x_i \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2i} \leq 2b & \in F' \\
F \ni & -x_i \leq b & \rightsquigarrow & x'_{2i} - x'_{2i-1} \leq 2b & \in F' \\
F \ni & x_i - x_j \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2j-1} \leq b,\ x'_{2j} - x'_{2i} \leq b & \in F' \\
F \ni & x_i + x_j \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2j} \leq b, \quad x'_{2j-1} - x'_{2i} \leq b & \in F' \\
F \ni & -x_i - x_j \leq b & \rightsquigarrow & x'_{2i} - x'_{2j-1} \leq b, \quad x'_{2j} - x'_{2i-1} \leq b & \in F'
\end{array}
$$

Theorem 16.6

If F is over $\mathbb{Q}$ then

- *If $(v_1, \dots, v_n) \models F$ then $(v_1, -v_1, \dots, v_n, -v_n) \models F'$*
- *If $(v_1, v_2, \dots, v_{2n-1}, v_{2n}) \models F'$ then $(\frac{(v_1 - v_2)}{2}, \dots, \frac{(v_{2n-1} - v_{2n})}{2}) \models F$*

Exercise 16.6

a. Prove the above. *b. Give an example over $\mathbb{Z}$ when Theorem 16.6 fails*

Example: octagonal DBM

Definition 16.6

The DBM corresponding to F' are called octagonal DBMs(ODBMs).

Exercise 16.7

Consider:

$x_1 + x_2 \leq 4 \wedge x_2 - x_1 \leq 5 \wedge x_1 - x_2 \leq 3 \wedge -x_1 - x_2 \leq 1 \wedge x_2 \leq 2 \wedge -x_2 \leq 7$

Corresponding ODBM

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

$x_1 + x_2 \leq 4 \rightsquigarrow x_1 - x_4 \leq 4, x_3 - x_2 \leq 4$

$x_2 - x_1 \leq 5 \rightsquigarrow x_3 - x_1 \leq 5, x_2 - x_4 \leq 5$

$x_1 - x_2 \leq 3 \rightsquigarrow x_1 - x_3 \leq 3, x_4 - x_2 \leq 3$

$-x_1 - x_2 \leq 1 \rightsquigarrow x_1 - x_4 \leq 1, x_3 - x_2 \leq 1$

$x_2 \leq 2 \rightsquigarrow x_3 - x_4 \leq 4$

$-x_2 \leq 7 \rightsquigarrow x_3 - x_4 \leq 14$

Relating indices and coherence

Let $\overline{2k} \triangleq 2k-1$ and $\overline{2k-1} \triangleq 2k$

Example 16.5

$\bar{1}\bar{1} = 22 \quad \bar{2}\bar{1} = 12 \quad \bar{2}\bar{2} = 11$

Exercise 16.8

- $\bar{3}\bar{1} =$
- $\bar{3}\bar{2} =$
- $\bar{4}\bar{2} =$
- $\bar{1}\bar{1} =$

Relating indices and coherence II

Consider the following DBM due to 2 variable octagonal constraints.

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

Cells with matching colors are pairs $(ij, \overline{ji})$.

Definition 16.7

A DBM A is coherent if $\forall i, j.\ A_{ij} = A_{\overline{ji}}$.

Unsatisfiability

For $\mathbb{Q}$, any method of checking unsat of difference constraints will work on ODBMs.

Let A be ODBM of F. $A^{\bullet}$ will let us know in $2n$ steps if F is sat.

For $\mathbb{Z}$, we may need to interpret ODBMs differently.
We will cover this shortly.

Implication checking and canonical form

Floyd-Warshall Algorithm does not obtain canonical form for ODBMs.

$x'_k = -x'_{\bar{k}}$ is not needed for satisfiablity check. Consequently, $A^\bullet$ is not canonical over $\mathbb{Q}$.

We need to tighten the bounds that may be proven due to the above equalities.

Exercise 16.9

Give an example such that $A^\bullet$ is not tight for octagonal constraints.

Canonical closure for octagonal constraints

Let us define closure property for ODBMs.

Definition 16.8

For a ODBM A, let $F[A]$ define the corresponding formula over original variables.

Definition 16.9

For both $\mathbb{Z}$ and $\mathbb{Q}$, an ODBM A is tight if for all i and j

- *if $A_{ij} < \infty$ then $\exists v \models F[A].\ v'_i - v'_j = A_{ij}$ and*
- *if $A_{ij} = \infty$ then $\forall m < \infty.\ \exists v \models F[A].\ v'_i - v'_j > m$,*

where $v'_{2k-1} \triangleq v_k$ and $v'_{2k} \triangleq -v_k$

Theorem 16.7

If A is tight then A is a canonical representation of $F[A]$

$\mathbb{Q}$ tightness condition

Theorem 16.8

Let us suppose $F[A]$ is sat.
If $\forall i, j, k,\ A_{ij} \leq A_{ikj}$ and $A_{ij} \leq (A_{i\bar{i}} + A_{\bar{j}j})/2$ then A is tight

Proof.

Consider cell ij in A s.t. $i \neq j$.(otherwise trivial)
Suppose A_{ij} is finite.
Let $A' = A[ji \mapsto -A_{ij}, \overline{ij} \mapsto -A_{ij}]$

claim: $v \models F[A]$ and $v'_i - v'_j = A_{ij}$ iff $v \models F[A']$
Forward direction easily holds.(why?)
Since A has no negative cycles, $A_{ij} + A_{ji} \geq 0$. So, $A_{ji} \geq -A_{ij}$. So, $A_{ji} \geq A'_{ji}$.
Therefore, A is pointwise greater than A'. Therefore, $F[A'] \Rightarrow F[A]$.
Since $A'_{ij} = -A'_{ji}$, if $v \models F[A']$ then $v'_i - v'_j = A_{ij}$. Backward direction holds.

...

$\mathbb{Q}$ tightness condition(contd.)

Proof(contd.)

Now we are only left to show the following.

claim: $F[A']$ is sat, which is there are no negative cycles in A'

A' can have negative cycles only if ji or $\overline{ij}$ occur in the cycle.(why?)

Wlog, we assume only ji occurs in a negative cycle $i = i_0..i_m = j$

Therefore, $A'_{ji} + \sum_{l \in 1..m} A'_{i_{(l-1)} i_l} < 0$. Therefore, $-A_{ij} + \sum_{l \in 1..m} A_{i_{(l-1)} i_l} < 0$.

Therefore, $\sum_{l \in 1..m} A_{i_{(l-1)} i_l} < A_{ij}$. Contradiction.

Now we assume both ji and $\overline{ij}$ occur in a negative cycle $i = i_0..i_m i'_0..i_{m'} = j$, where $i_m = \bar{i}$ and $\bar{j} = i'_0$.(one case missing)

Therefore, $A'_{ji} + A'_{\overline{ij}} + \sum_{l \in 1..m} A'_{i_{l-1} i_l} + \sum_{l \in 1..m'} A'_{i'_{l-1} i'_l} < 0$.

Therefore, $-2A_{ij} + \sum_{l \in 1..m} A'_{i_{l-1} i_l} + \sum_{l \in 1..m'} A'_{i'_{l-1} i'_l} < 0$.

Therefore, $-2A_{ij} + A_{i\bar{i}} + A_{j\bar{j}} < 0$. Contradiction. $\square$

Exercise 16.10

a. Prove the $A_{ij} = \infty$ case. *b. Does converse of the theorem holds?*

Computing canonical closure for octagonal constraints

Due to the previous theorem and desire of efficient computation, let us redefine $A^\bullet$ for ODBMs.

Definition 16.10

We compute $A^\bullet$ using the following iterations generating $A^0, \ldots, A^{2n} = A^\bullet$. Let $o = 2k-1$ for some $k \in 1..n$.

$$\begin{aligned} A^0 &= A \\ (A^{o+1})_{ij} &= \min(A^o_{ij}, \tfrac{A^o_{i\bar{i}} + A^o_{\bar{j}j}}{2}) && \textit{(odd rule)} \\ (A^o)_{ij} &= \min(A^{o-1}_{ij}, A^{o-1}_{ioj}, A^{o-1}_{i\bar{o}j}, A^{o-1}_{io\bar{o}j}, A^{o-1}_{i\bar{o}oj}) && \textit{(even rule)} \end{aligned}$$

In the even rule, three new paths are considered to exploit the structure of ODBMs.

We will prove that $A^\bullet$ is tight in post lecture slides.

Even rule intuition

In octagon formulas, x_k variable may insert itself between variables $x_{\lceil i/2 \rceil}$ and $x_{\lceil j/2 \rceil}$ in several ways.

Consider the following scenarios.

1. $\pm x_{\lceil i/2 \rceil} - x_k \leq A_{io}$ and $x_k \pm x_{\lceil j/2 \rceil} \leq A_{oj}$
   - Update using $A_{io} + A_{oj}$
2. $\pm x_{\lceil i/2 \rceil} + x_k \leq A_{i\bar{o}}$ and $-x_k \pm x_{\lceil j/2 \rceil} \leq A_{\bar{o}j}$
   - Update using $A_{i\bar{o}} + A_{\bar{o}j}$
3. $\pm x_{\lceil i/2 \rceil} + x_k \leq A_{i\bar{o}}$, $x_k \pm x_{\lceil j/2 \rceil} \leq A_{oj}$, and $-x_k \leq A_{\bar{o}o}/2$
   - Update using $A_{i\bar{o}} + A_{\bar{o}o} + A_{oj}$
4. $\pm x_{\lceil i/2 \rceil} - x_k \leq A_{io}$, $-x_k \pm x_{\lceil j/2 \rceil} \leq A_{\bar{o}j}$, and $x_k \leq A_{o\bar{o}}/2$
   - Update using $A_{io} + A_{o\bar{o}} + A_{\bar{o}j}$

Each of the above case is the considered four paths in the definition 16.10.

Example: canonical closure of ODBM

Example 16.6

Consider:

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

First we apply the even rule $o = 1$:

$A^1_{ij} = A^1_{\bar{j}\bar{i}} = \min(A^0_{ij}, A^0_{i1j}, A^0_{i2j}, A^0_{i12j}, A^0_{i21j})$

$A^1_{12} = A^1_{21} = \min(A^0_{12}, A^0_{112}, A^0_{122}, A^0_{1122}, A^0_{1212}) = min(\infty, \infty, \infty, \infty, \infty) = \infty$

$A^1_{24} = A^1_{13} = min(A^0_{24}, A^0_{214}, A^0_{224}, A^0_{2124}, A^0_{2214}) = min(5, \infty, 5, \infty, \infty) = 5$

$A^1_{34} = A^1_{34} = \min(A^0_{34}, A^0_{314}, A^0_{324}, A^0_{3124}, A^0_{3214}) = min(4, 9, 9, \infty, \infty) = 4$

$A^1_{43} = A^1_{43} = \min(A^0_{43}, A^0_{413}, A^0_{423}, A^0_{4123}, A^0_{4213}) = min(14, 4, 4, \infty, \infty) = 4$

Exercise 16.11

Find the tight ODBM for the following octagonal constraints:

$2 \leq x + y \leq 7 \wedge x \leq 9 \wedge y - x \leq 1 \wedge -y \leq 1$

Octagonal constraints over $\mathbb{Z}$

For $\mathbb{Z}$, we need a stronger property to ensure tightness.

Theorem 16.9

Let A be ODBM interpreted over $\mathbb{Z}$.
if $\forall i, j, k,\ A_{ij} \leq A_{ikj}$, $A_{ij} \leq (A_{i\bar{i}} + A_{j\bar{j}})/2$, and $A_{i\bar{i}}$ is even then A is tight.

Exercise 16.12

Prove the above theorem.

Computing canonical closure for octgonal DBMs over $\mathbb{Q}$

In this case, let us present an incremental version of the closure iterations.

Lets suppose A is tight and we add another octagonal atom in A that updates $A_{i_0 j_0}$ and $A_{\bar{j}_0 \bar{i}_0}$. (Observe: always updated together)

Let A^0 be the updated DBM.

$$(A^1)_{ij} = \min(A^0_{ij}, A^0_{ii_0j_0j}, A^0_{i\bar{j}_0\bar{i}_0j}) \qquad \text{if } i \neq \bar{j}$$

$$(A^1)_{i\bar{i}} = \min(A^0_{i\bar{i}}, A^0_{i\bar{j}_0\bar{i}_0i_0j_0\bar{i}}, A^0_{ii_0j_0\bar{j}_0\bar{i}_0\bar{i}}, 2\lfloor\frac{A^0_{ii_0j_0\bar{i}}}{2}\rfloor)$$

$$(A^2)_{ij} = \min(A^1_{ij}, \frac{A^1_{i\bar{i}} + A^1_{j\bar{j}}}{2})$$

Theorem 16.10

A^2 is tight

Topic 16.3

Problem

Difference logic for integers

Exercise 16.13

Consider a difference logic formula with integer bounds. Show that it has an integer solution if it has a rational solution.

End of Lecture 16

Topic 16.4

Post lecture proofs

Tightness of $A^\bullet$

Theorem 16.11

$A^\bullet$ (defined in 16.10) is tight.

Proof.

For each i, j, and k, we need to show $A^\bullet_{ij} \leq (A^\bullet_{i\bar{i}} + A^\bullet_{j\bar{j}})/2$ and $A^\bullet_{ij} \leq A^\bullet_{ikj}$.

claim: For $k > 0$, $A^{2k}_{ij} \leq (A^{2k}_{i\bar{i}} + A^{2k}_{j\bar{j}})/2$

Note $A^{2k}_{i\bar{i}} = A^{2k-1}_{i\bar{i}}$.(why?)

By def,

$$(A^{2k})_{ij} \leq \frac{A^{2k-1}_{i\bar{i}} + A^{2k-1}_{j\bar{j}}}{2}.$$

Therefore,

$$(A^{2k})_{ij} \leq \frac{A^{2k}_{i\bar{i}} + A^{2k}_{j\bar{j}}}{2}.$$

...

Tightness of $A^{\bullet}$(contd.)

Proof(contd.)

We are yet to prove $\forall i, j.\ A^{\bullet}_{ij} \leq A^{\bullet}_{ikj}$.

Let $Fact(k, o) \triangleq \forall i, j.\ A^{o}_{ij} \leq A^{o}_{ikj} \land A^{o}_{ij} \leq A^{o}_{i\bar{k}j}$

So we need to prove $\forall k \in 1..n.\ Fact(2k, 2n)$.

the following three will prove the above by induction:(why?)

1. In odd rules ($o = 2k' - 1$), $Fact(k, o) \Rightarrow Fact(k, o + 1)$ (preserve)
2. In even rules ($o = 2k'$), $Fact(k, o) \Rightarrow Fact(k, o + 1)$ (preserve)
3. After even rules ($o = 2k'$), $Fact(o, o)$ (establish)

...

Tightness of $A^\bullet$: odd rules preserve the facts

Proof(contd.)

claim: odd rule, if $\forall i,j.\ A^o_{ij} \leq A^o_{ikj} \wedge A^o_{ij} \leq A^o_{i\bar{k}j}$ then $\forall i,j.\ A^{o+1}_{ij} \leq A^{o+1}_{ikj}$.

We have four cases(why?) and denoted them by pairs.

(1,1) $A^{o+1}_{ik} = A^o_{ik},\ A^{o+1}_{kj} = A^o_{kj}$: $\underbrace{A^{o+1}_{ij} \leq A^o_{ij}}_{\text{odd rule}} \underbrace{\leq A^o_{ikj}}_{\text{lhs}} \underbrace{= A^{o+1}_{ikj}}_{\text{case cond.}}$

(2,1) $A^{o+1}_{ik} = (A^o_{i\bar{i}} + A^o_{k\bar{k}})/2,\ A^{o+1}_{kj} = A^o_{kj}$:

$$\underbrace{A^o_{ij} \leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}j}}{2}}_{\text{odd rule}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}kj}}{2}}_{\text{lhs}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}\bar{k}kj}}{2}}_{\text{lhs}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{k}k} + A^o_{\bar{j}\bar{k}} + A^o_{kj}}{2}}_{\text{rewrite}}$$

$$\underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{k\bar{k}}}{2} + A^o_{kj}}_{\text{coherence}} \underbrace{= A^{o+1}_{ikj}}_{\text{case cond.}}$$

(2,1) $A^{2k}_{ik} = A^o_{ik},\ A^{o+1}_{kj} = (A^o_{k\bar{k}} + A^o_{\bar{j}j})/2$ (Symmetric to the last case)

(2,2) $A^{o+1}_{ik} = (A^o_{i\bar{i}} + A^o_{k\bar{k}})/2$ and $A^{o+1}_{kj} = (A^o_{k\bar{k}} + A^o_{\bar{j}j})/2$ …

Exercise 16.14 *Prove the last case.*

Tightness of $A^\bullet$: even rules preserve the facts

Proof(contd.)

claim: even rule, if $\forall i,j.\ A^{o-1}_{ij} \leq A^{o-1}_{ikj} \wedge A^{o-1}_{ij} \leq A^{o-1}_{i\bar{k}j}$ then $\forall i,j.\ A^{o}_{ij} \leq A^{o}_{ikj}$.

Here, we have 25 cases(why?) and denoted them by pairs:

(1,1) $A^o_{ik} = A^{o-1}_{ik}, A^o_{kj} = A^{o-1}_{kj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ij}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{ikj}}_{\text{lhs}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$

(2,1) $A^o_{ik} = A^{o-1}_{iok}, A^o_{kj} = A^{o-1}_{kj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{iokj}}_{\text{lhs}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$.

(4,5) $A^o_{ik} = A^{o-1}_{io\bar{o}k}, A^o_{kj} = A^{o-1}_{k\bar{o}oj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{ioj} + A^{o-1}_{o\bar{o}o} + A^{o-1}_{\bar{o}k\bar{o}}}_{\text{no negative loops}}$

$\underbrace{\leq A^{o-1}_{io\bar{o}k} + A^{o-1}_{k\bar{o}oj}}_{\text{rewrite}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$

...

Exercise 16.15

Prove cases (1,4), (2,3) and (3,3).

Hint: key proof technique: introduce cycles, introduce k

Tightness of $A^\bullet$: even rule establishes the fact

Proof(contd.)

claim: even rule, $\forall i,j.\ A^o_{ij} \leq A^o_{ioj} \wedge A^o_{ij} \leq A^o_{i\bar{o}j}$

We only prove $A^o_{ij} \leq A^o_{ioj}$, the other inequality is symmetric.

Again, we have 25 cases.(why?)

Since there are no negative cycles and $A^o_{oo} = 0$,

$A_{io} = A_{ioo} \leq A_{io\bar{o}o}$ and $i\bar{o}o \leq i\bar{o}oo$.

Therefore, only four cases left to consider.(why?)

(1,1) $A^o_{io} = A^{o-1}_{io}, A^o_{oj} = A^{o-1}_{oj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{= A^o_{ioj}}_{\text{case cond.}}$

(2,2) $A^o_{io} = A^{o-1}_{i\bar{o}o}, A^o_{oj} = A^{o-1}_{o\bar{o}j}$:

$$\underbrace{A^o_{ij} \leq A^{o-1}_{i\bar{o}j}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{i\bar{o}j} + A^{o-1}_{o\bar{o}o}}_{\text{no negative cycles}} \underbrace{\leq A^{o-1}_{i\bar{o}o} + A^{o-1}_{o\bar{o}j}}_{\text{rewrite}} \underbrace{= A^o_{ioj}}_{\text{case cond.}}$$

□

Exercise 16.16

Prove case (1,2).