Automated Reasoning 2018

Lecture 5: Interpolants

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Topic 5.1

Interpolants

Interpolants

Definition 5.1

For mutually unsat formulas A and B, a formula I is interpolant between A and B if

1. $A \Rightarrow \mathrm{I}$,
2. $B \wedge \mathrm{I} \Rightarrow \bot$, *and*
3. $vars(\mathrm{I}) \subseteq (vars(A) \cap vars(B))$

Example 5.1

Consider:

$A = x_1 + x_2 \leq 2 \wedge x_3 - x_2 \leq 0$
$B = 6x_4 - 2x_1 \leq -8 \wedge -3x_4 - x_3 \leq 0$

$vars(A) = \{x_1, x_2, x_3\}$ $\quad vars(B) = \{x_1, x_3, x_4\}$ $\quad vars(\mathrm{I}) \subseteq \{x_1, x_3\}$

$\mathrm{I} = x_1 + x_3 \leq 2$

A-local, B-local, global symbols

Definition 5.2

A symbol is A-local if it only occurs in A.

Definition 5.3

A symbol is B-local if it only occurs in B.

Definition 5.4

A symbol is global if it occurs in both A and B.
The set of global symbols is written $Globals = vars(A) \cap vars(B)$

cc by nc sa

Interpolants - equivalent definition

Definition 5.5

Let $A \Rightarrow B$. A formula I is interpolant between A and B if

1. $A \Rightarrow \mathrm{I}$,
2. $\mathrm{I} \Rightarrow B$, *and*
3. $vars(\mathrm{I}) \subseteq Globals$

Exercise 5.1

Show both the definitions are equivalent.

In this form, interpolant captures a general idea behind every human inquiry.

Example 5.2

$$\textit{Human activity} \Rightarrow \underbrace{\textit{Increase in CO2 level}}_{\textit{interpolant}} \Rightarrow \textit{Climate change}$$

A concise explanation of cause and effect

Interpolants from proofs

Back to the original definition of interpolants.

We seek interpolants for mutually unsat A and B.

Interpolant captures the interactions of A and B that makes them mutually unsat.

If we have unsat proof for $A \wedge B$, we may use the reasoning that proves $A \wedge B$ to find a interpolant.

As proofs have intermediate results, we should have equivalent concept of intermediate interpolants.

Partial interpolants

Definition 5.6

Let A,B and C be formulas such that $A \wedge B \Rightarrow C$.

A partial interpolant I_C between A and B for C is a formula such that

- $A \Rightarrow \mathrm{I}_C$
- $B \wedge \mathrm{I}_C \Rightarrow C$
- $vars(\mathrm{I}_C) \subseteq Globals \cup vars(C)$

Interpolation via proofs

We present here a proof based method for computing interpolants.

Consider A and B are mutually unsat formulas in some given theory.

1. produce proof of unsatisfiablity of $A \wedge B$
2. annotate each intermediate derived formulas with partial interpolants inductively
3. Since $\perp$ is the final node of the proof, the annotation of $\perp$ is the interpolant between A and B.

We will write annotations in proof rules within square brackets after derived formulas.

$$\dots \frac{\dots}{C[\mathrm{I}_C]} \dots$$

Topic 5.2

Interpolation in $\mathcal{T}_{LRA}$

Annotation rules for conjunction of linear arithmetic formulas

Linear arithmetic proof system

$$hyp\,\frac{}{aX \leq c}\,aX \leq c \in A, B \qquad comb\,\frac{a_1X \leq c_1 \quad a_2X \leq c_2}{(\lambda_1 a_1 + \lambda_2 a_2)X \leq (\lambda_1 c_1 + \lambda_2 c_2)}\,\lambda_1, \lambda_2 \geq 0$$

Proof rules with partial interpolant annotations

$$\text{HYP-A}\,\frac{}{aX \leq c[aX \leq c]}\,aX \leq c \in A \qquad \text{HYP-B}\,\frac{}{aX \leq c[0 \leq 0]}\,aX \leq c \in B$$

$$\text{COMB}\,\frac{a_1X \leq c_1[a'_1X \leq c'_1] \quad a_2X \leq c_2[a'_2X \leq c'_2]}{(\lambda_1 a_1 + \lambda_2 a_2)X \leq (\lambda_1 c_1 + \lambda_2 c_2)[(\lambda_1 a'_1 + \lambda_2 a'_2)X \leq (\lambda_1 c'_1 + \lambda_2 c'_2)]}\,\begin{array}{l}\lambda_1 \geq 0\\ \lambda_2 \geq 0\end{array}$$

Example: LRA annotations

Example 5.3

Consider:

$A = x_1 + x_2 \le 2 \land x_3 - x_2 \le 0$

$B = 6x_4 - 2x_1 \le -8 \land -3x_4 - x_3 \le 0$

$$\dfrac{\dfrac{x_1 + x_2 \le 2[x_1 + x_2 \le 2] \quad 6x_4 - 2x_1 \le -8[0 \le 0]}{3x_4 + x_2 \le -2[x_1 + x_2 \le 2]} \qquad \dfrac{x_3 - x_2 \le 0[x_3 - x_2 \le 0] \quad -3x_4 - x_3 \le 0[0 \le 0}{-3x_4 - x_2 \le 0[x_3 - x_2 \le 0]}}{0 \le -2[x_1 + x_3 \le 2]}$$

Annotation correctness

Theorem 5.1

The annotations in the proof rules are partial interpolants

Proof.

We show the following facts for annotations that imply the conditions of partial interpolants.

Let $a'X \leq c'$ be the annotation for $aX \leq c$.

1. $A \Rightarrow a'X \leq c'$, //1st cond. of partial interpolants
2. $B \Rightarrow (a - a')X \leq (c - c')$, //implies 2nd cond.(why?)
3. B-locals do not occur in $a'X \leq c'$, and
4. A-locals have same coefficient in $aX \leq c$ and $a'X \leq c'$. //3-4 imply 3rd cond.

Annotation correctness(contd.)

Proof(contd.)

□

base case:

$$\textsc{Hyp-A}\frac{}{aX \leq c[aX \leq c]} aX \leq c \in A$$

1. $A \Rightarrow aX \leq c$
2. $B \Rightarrow (a-a)X \leq (c-c)$
3. B-locals do not occur in $aX \leq c$, and
4. A-locals have same coefficient in $aX \leq c$ and $a'X \leq c'$

$$\textsc{Hyp-B}\frac{}{aX \leq c[0 \leq 0]} aX \leq c \in B$$

1. $A \Rightarrow 0 \leq 0$
2. $B \Rightarrow (a-0)X \leq (c-0)$
3. B-locals do not occur in $0 \leq 0$, and
4. A-locals have same coefficient in $0 \leq 0$ and $aX \leq c$

Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\text{COMB}\frac{a_1 X \leq c_1[a'_1 X \leq c'_1] \quad a_2 X \leq c_2[a'_2 X \leq c'_2]}{(\lambda_1 a_1 + \lambda_2 a_2)X \leq (\lambda_1 c_1 + \lambda_2 c_2)[(\lambda_1 a'_1 + \lambda_2 a'_2)X \leq (\lambda_1 c'_1 + \lambda_2 c'_2)]}\begin{matrix}\lambda_1 \geq 0\\ \lambda_2 \geq 0\end{matrix}$$

Due to ind. hyp., the conditions of the partial interpolants of the antecedents holds.

1. $A \Rightarrow a'_1 X \leq c'_1$
2. $B \Rightarrow (a_1 - a'_1)X \leq (c_1 - c'_1)$
3. No B-locals in $a'_1 X \leq c'_1$
4. A-locals have same coefficient in $a_1 X \leq c_1$ and $a'_1 X \leq c'_1$.

1. $A \Rightarrow a'_2 X \leq c'_2$
2. $B \Rightarrow (a_2 - a'_2)X \leq (c_2 - c'_2)$
3. No B-locals in $a'_2 X \leq c'_2$
4. A-locals have same coefficient in $a_2 X \leq c_2$ and $a'_2 X \leq c'_2$.

Rest is exercise. □

Exercise 5.2

Show the above four properties hold for the annotation of the consequent.

Topic 5.3

Interpolation in Propositional logic

Some notation

For a clause C and a formula B, let

$$C|_B = \{\ell \in C | vars(\ell) \subseteq vars(B)\}$$

and

$$C/_B = \{\ell \in C | vars(\ell) \not\subseteq vars(B)\}.$$

Theorem 5.2

$C = C|_B \vee C/_B$

Example 5.4

Consider formula $B = (p \vee q) \wedge (q \wedge r)$.
Let $C = p \vee \neg r \vee \neg x \vee z$.

- $C|_B = p \vee \neg r$
- $C/_B = \neg x \vee z$

Annotation rules for propositional formulas

Resolution proof system

$$\textsc{hyp}\frac{}{C}\, C \in A, B \qquad \textsc{Res}\frac{C \vee x \quad D \vee \neg x}{C \vee D}$$

Proof rules with partial interpolant annotations

$$\textsc{hyp-A}\frac{}{C[C|_B \vee C/_B]}\, C \in A \quad \textsc{hyp-B}\frac{}{C[\top \vee C/_B]}\, C \in B$$

$$\textsc{Res-B}\frac{C \vee x[\mathrm{I}_1 \vee C/_B] \quad D \vee \neg x[\mathrm{I}_2 \vee D/_B]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B]}\, x \in vars(B)$$

$$\textsc{Res-A}\frac{C \vee x[\mathrm{I}_1 \vee (C/_B \vee x)] \quad D \vee \neg x[\mathrm{I}_2 \vee (D/_B \vee \neg x)]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B]}\, x \notin vars(B)$$

Simplified annotations

Sometimes $C/_B$ part of a partial interpolant is not mentioned and implicitly assumed to be around. The shorthanded annotation rules are as follows.

$$\text{HYP-A}\frac{}{C[C|_B]}C \in A \quad \text{HYP-B}\frac{}{C[\top]}C \in B$$

$$\text{RES-B}\frac{C \vee x[\mathrm{I}_1] \quad D \vee \neg x[\mathrm{I}_2]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2)]}x \in vars(B)$$

$$\text{RES-A}\frac{C \vee x[\mathrm{I}_1] \quad D \vee \neg x[\mathrm{I}_2]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2)]}x \notin vars(B)$$

Example: interpolation for propositional formulas

Example 5.5

$A = (p \lor \neg q) \land (\neg p \lor \neg r) \qquad B = (\neg q \lor r) \land (q \lor s) \land \neg s$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg p \lor \neg r[\neg r] \quad \neg q \lor r[\top]}{\neg p \lor \neg q[\neg r]} \quad p \lor \neg q[\neg q]}{\neg q[\neg r \lor \neg q]} \quad q \lor s[\top]}{s[\neg r \lor \neg q]} \quad \neg s[\top]}{\bot[\neg r \lor \neg q]}$$

The interpolant between A and B is $\neg r \lor \neg q$

Exercise 5.3

Compute interpolants between the following pairs of formulas

1. $(q \lor p \lor t) \land \neg q \land (q \lor p \lor \neg u)$ *and* $(u \lor \neg t \lor \neg s) \land s \land \neg p$
2. $(q \lor p \lor t) \land \neg q \land (q \lor p \lor \neg u) \land \neg p$ *and* $(u \lor \neg t \lor \neg s) \land s$
3. $(q \lor p \lor t) \land \neg q \land (q \lor p \lor \neg u) \land \neg p \land (u \lor \neg t \lor \neg s)$ *and* s

Resolution Annotation correctness

Theorem 5.3

The annotations in the above proof rules are partial interpolants

Proof.

We will prove stronger conditions than the partial interpolation conditions.
Assume $(\mathrm{I} \vee C/_B)$ is an annotation of C (non-shorthanded annotations).

1. $A \Rightarrow (\mathrm{I} \vee C/_B)$
2. $B \wedge \mathrm{I} \Rightarrow C|_B$, which implies the second condition $B \wedge (\mathrm{I} \vee C/_B) \Rightarrow C$ (why?)
3. $\mathrm{I} \subseteq \mathit{Globals}$, which implies the 3rd condition

base case:

$$\textsc{hyp-A}\ \frac{}{C[C|_B \vee C/_B]}\ C \in A$$

1. $A \Rightarrow C|_B \vee C/_B$
2. $B \wedge C|_B \Rightarrow C|_B$
3. $\mathit{vars}(C|_B) \subseteq \mathit{Globals}$ (why?)

$$\textsc{hyp-B}\ \frac{}{C[\top \vee C/_B]}\ C \in B$$

1. $A \Rightarrow \top \vee C/_B$
2. $B \wedge \top \Rightarrow C|_B$ (why?)
3. $\mathit{vars}(\top) \subseteq \mathit{Globals}$

Resolution Annotation correctness(contd.)

Proof(contd.)

induction step:

$$\textsc{Res-B}\,\frac{C \vee x[\mathrm{I}_1 \vee C/_B] \quad D \vee \neg x[\mathrm{I}_2 \vee D/_B]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B]}\, x \in \mathit{vars}(B)$$

1. Since $A \Rightarrow \mathrm{I}_1 \vee C/_B$ and $A \Rightarrow \mathrm{I}_2 \vee D/_B$, $A \Rightarrow (\mathrm{I}_1 \wedge \mathrm{I}_2) \vee (C \vee D)/_B$.(why?)
2. Since $B \wedge \mathrm{I}_1 \Rightarrow (C \vee x)|_B$ and $B \wedge \mathrm{I}_2 \Rightarrow (D \vee \neg x)|_B$,

$$B \wedge \mathrm{I}_1 \wedge \mathrm{I}_2 \Rightarrow (C|_B \vee x) \wedge (D|_B \vee \neg x) \Rightarrow (C \vee D)|_B.$$

3. $\mathit{vars}(\mathrm{I}_1 \wedge \mathrm{I}_2) \subseteq \mathit{Globals}$

Resolution Annotation correctness(contd.)

Proof(contd.)

$$\text{Res-A}\ \frac{C \vee x[\mathrm{I}_1 \vee (C/_B \vee x)] \quad D \vee \neg x[\mathrm{I}_2 \vee (D/_B \vee \neg x)]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B]} x \notin vars(B)$$

1. Since $A \Rightarrow \mathrm{I}_1 \vee C/_B \vee x$ and $A \Rightarrow \mathrm{I}_2 \vee D/_B \vee \neg x$,

   $$A \Rightarrow (\mathrm{I}_1 \vee \mathrm{I}_2) \vee (C \vee D)/_B.\text{(why?)}$$

2. Since $B \wedge \mathrm{I}_1 \Rightarrow C|_B$ and $B \wedge \mathrm{I}_2 \Rightarrow D|_{B\text{(why?)}}$,

   $$B \wedge (\mathrm{I}_1 \vee \mathrm{I}_2) \Rightarrow C|_B \vee D|_B.$$

3. $vars(\mathrm{I}_1 \vee \mathrm{I}_2) \subseteq Globals$.

Topic 5.4

Interpolation in $\mathcal{T}_{EUF}$

Proof rules for $\mathcal{T}_{EUF}$

Proof rules of $\mathcal{T}_{EUF}$

$$\textsc{Hyp}\frac{}{x \bowtie y}\, x \bowtie y \in A, B$$

$$\textsc{Sym}\frac{x \approx y}{y \approx x} \qquad \textsc{Trans}\frac{x \approx y \quad y \approx z}{x \approx z}$$

We will skip annotating congruence.

$$\textsc{Cong}\frac{x_1 \approx y_1 \quad .. \quad x_n \approx y_n}{f(x_1, .., x_n) \approx f(y_1, .., y_n)}$$

$$\textsc{Contra}\frac{x \approx y \quad x \not\approx y}{\bot}$$

Structure of interpolants for equality only

In LRA, interpolants are inequalities. In EUF, interpolants are complicated.

Since we are not considering uninterpreted functions, we can get away with something simple.

The following is the potential structure of interpolants in EUF.

$$x \approx y[F, (g_1, g_2)]$$

where F is a formula over global symbols, and g_1 and g_2 are

- global terms or
- equal to either x or y.

Interpretation of the interpolants

We interpret the annotation as follows.

1. $A \Rightarrow F \land x \approx g_1 \land y \approx g_2$
2. $B \land F \land x \approx g_1 \land y \approx g_2 \Rightarrow x \approx y$
3. $F \land x \approx g_1 \land y \approx g_2 \subseteq \mathit{Globals} \cup \mathit{vars}(x \approx y)$

F is contributed by A.

$x \approx g_1 \land y \approx g_2$ are also contributed by A.

Annotations for hypothesis

$$\text{HYP-A}\frac{}{x \approx y[\top, (y, x)]} x \approx y \in A \quad x \notin vars(B), y \notin vars(B)$$

$$\text{HYP-A}\frac{}{x \approx y[\top, (y, y)]} x \approx y \in A \quad x \notin vars(B), y \in vars(B)$$

$$\text{HYP-A}\frac{}{x \approx y[\top, (x, x)]} x \approx y \in A \quad x \in vars(B), y \notin vars(B)$$

$$\text{HYP-A}\frac{}{x \approx y[x \approx y, (x, y)]} x \approx y \in A \quad x \in vars(B), y \in vars(B)$$

$$\text{HYP-B}\frac{}{x \approx y[\top, (x, y)]} x \approx y \in B$$

Annotating for transitivity and symmetry

$$\text{SYM}\frac{x \approx y[F, (g_1, g_2)]}{y \approx x[F, (g_2, g_1)]}$$

$$\text{TRANS}\frac{x \approx y[F_1, (g_1, g_2)] \quad y \approx z[F_2, (g_3, g_4)]}{x \approx z[F_1 \land F_2 \land g_2 \approx g_3, (g_1, g_4)]} vars(g_2, g_3) \subseteq Globals$$

$$\text{TRANS}\frac{x \approx y[F_1, (y, x)] \quad y \approx z[F_2, (g_3, g_4)]}{x \approx z[F_1 \land F_2, (g_3, g_4)]} vars(g_3) \subseteq Globals$$

$$\text{TRANS}\frac{x \approx y[F_1, (g_1, g_2)] \quad y \approx z[F_2, (z, y)]}{x \approx z[F_1 \land F_2, (g_1, g_2)]} vars(g_2) \subseteq Globals$$

$$\text{TRANS}\frac{x \approx y[F_1, (y, x)] \quad y \approx z[F_2, (z, y)]}{x \approx z[F_1 \land F_2, (z, x)]}$$

Topic 5.5

Problems

Asymmetric annotation rules

Exercise 5.4

Consider the following equivalent definition of interpolants

1. $A \Rightarrow \mathrm{I}$,
2. $B \Rightarrow \neg\mathrm{I}$, *and*
3. $vars(\mathrm{I}) \subseteq (vars(A) \cap vars(B))$

The above definition is symmetric upto negation, but the following annotation rules are not.

$$\textsc{hyp-A}\frac{}{C[C|_B]}\, C \in A \quad \textsc{hyp-B}\frac{}{C[\top]}\, C \in B$$

$$\textsc{Res-B}\frac{C \vee x[\mathrm{I}_1] \quad D \vee \neg x[\mathrm{I}_2]}{C \vee D[(\mathrm{I}_1 \wedge \mathrm{I}_2)]}\, x \in vars(B)$$

$$\textsc{Res-A}\frac{C \vee x[\mathrm{I}_1] \quad D \vee \neg x[\mathrm{I}_2]}{C \vee D[(\mathrm{I}_1 \vee \mathrm{I}_2)]}\, x \notin vars(B)$$

Find annotation rules that are mirror to the above rules and symmetry is broken in a similar way. Prove that your rules are also correct.

Symmetric annotation rules

Exercise 5.5

Prove that the following annotation rules will always annotate $\perp$ with the interpolant between A and B.

$$\text{HYP-A}\frac{}{C[\perp]}C \in A \quad \text{HYP-B}\frac{}{C[\top]}C \in B$$

$$\text{RES-B-LOCAL}\frac{C \vee x[\mathrm{I}_1] \quad D \vee \neg x[\mathrm{I}_2]}{C \vee D[\mathrm{I}_1 \wedge \mathrm{I}_2]}x \in vars(B) - Globals$$

$$\text{RES-A-LOCAL}\frac{C \vee x[\mathrm{I}_1] \quad D \vee \neg x[\mathrm{I}_2]}{C \vee D[\mathrm{I}_1 \vee \mathrm{I}_2]}x \in vars(A) - Globals$$

$$\text{RES-GLOBAL}\frac{C \vee x[\mathrm{I}_1] \quad D \vee \neg x[\mathrm{I}_2]}{C \vee D[(\mathrm{I}_1 \vee x) \wedge (\neg x \vee \mathrm{I}_2)]}x \in Globals$$

End of Lecture 5