Lecture 5: $\epsilon$-transitions

Instructor: Ashutosh Gupta

IITB, India

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Topic 5.1

$\epsilon$-transitions
**ε-transitions**

Let us add another feature to our automaton.

Let us allow it to jump states without reading inputs. We will call such moves **ε-transitions**.

![Diagram of ε-transitions](image)

**Example 5.1**

*Consider the following automaton with ε-transitions that recognizes integers without leading zeros.*

![Diagram of Example 5.1](image)
Notation alert: specifying range of symbols

‘-’ is used to specify range of symbols.

Example 5.2

0-9 means \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}

k-z means \{k, \ell, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}
\( \epsilon \)-transitions

Does the new power brings new languages?

No!
Definition 5.1

An \(\epsilon\)-nondeterministic finite automaton (\(\epsilon\)-NFA) \(A\) is a five-tuple

\[(Q, \Sigma, \delta, q_0, F)\]

where

- \(Q\) is a finite set of states,
- \(\Sigma\) is a finite set of input symbols,
- \(\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow \mathcal{P}(Q)\) is the transition function,
- \(q_0 \in Q\) is the start/initial state, and
- \(F \subseteq Q\) is a set of accepting states.
\( \epsilon \) closure

**Definition 5.2**
Let \((Q, \Sigma, \delta, q_0, F)\) be an \( \epsilon \)-NFA. For each set \( S \subseteq Q \), \( ECLOSE(S) \) is the set of states reachable via \( \epsilon \)-transitions from \( S \).

**Example 5.3**

\[
\begin{align*}
ECLOSE(\{q_0\}) &= \{q_0, q_1\} \\
ECLOSE(\{q_1\}) &= \{q_1\} \\
ECLOSE(\{q_2\}) &= \{q_2, q_3\} \\
ECLOSE(\{q_0, q_2\}) &= \{q_0, q_1, q_2, q_3\}
\end{align*}
\]
Extending the transition function to words

Definition 5.3
Let $A = (Q, \Sigma, \delta, q_0, F)$ be an $\epsilon$-NFA. Let $\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$ be defined as follows.

$$\hat{\delta}(q, \epsilon) \triangleq \text{ECLOSE}(\{q\})$$
$$\hat{\delta}(q, wa) \triangleq \bigcup_{q' \in \hat{\delta}(q, w)} \text{ECLOSE}(\delta(q', a))$$

Example 5.4

$$\hat{\delta}(q_0, \epsilon) = \{q_0, q_1\} \quad \hat{\delta}(q_0, 1) = \{q_2, q_3\}$$
$$\hat{\delta}(q_0, 10) = \bigcup_{q' \in \hat{\delta}(q_0, 1)} \delta(q', 0) = \text{ECLOSE}(\delta(q_2, 1)) \cup \text{ECLOSE}(\delta(q_3, 1))$$
$$= \text{ECLOSE}(\{q_2\}) \cup \text{ECLOSE}(\emptyset) = \{q_2, q_3\} \cup \emptyset = \{q_2, q_3\}$$
Exercise: extended transitions

Exercise 5.1

Give value of the following function applications

\( \hat{\delta}(q_2, ba) = \)
\( \hat{\delta}(q_1, bb) = \)
\( \hat{\delta}(q_0, a) = \)
\( \hat{\delta}(q_0, bbb) = \)

\[ q_0 \xrightarrow{a} q_1 \xrightarrow{b} q_2 \]
\[ q_0 \xrightarrow{\epsilon} q_1 \xrightarrow{b} q_2 \]
\[ q_0 \xrightarrow{a, b} q_2 \]
Accepted word

Definition 5.4
A word $w$ is accepted by an $\epsilon$-NFA $A = (Q, \Sigma, \delta, q_0, F)$ if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$.

Example 5.5
Consider the following NFA

Since $\hat{\delta}(q_0, 10) = \{q_2, q_3\}$, 10 is accepted by the above $\epsilon$-NFA.
Eliminating $\epsilon$ transitions

Theorem 5.1
Let $A = (Q, \Sigma, \delta, q_0, F)$ be an $\epsilon$-NFA. There is an NFA $A'$ such that $L(A) = L(A')$.

Proof.
We construct $A' = (Q, \Sigma, \delta', q_0, F)$, where for each $a \in \Sigma$ and $q \in Q$

$$\delta'(q, a) \triangleq \text{ECLOSE}(\delta(q, a)).$$

Intuitively, the above construction works.

Commentary: The above construction adds only new transitions. No new state.
Example: eliminating $\epsilon$ transitions

Example 5.6

Consider automaton

After removing $\epsilon$ transition, we obtain the black transitions.

The above red transitions are added after applying $\epsilon$ closure.
Example II: eliminating $\epsilon$ transitions

Example 5.7

Consider automaton

After removing $\epsilon$ transition and applying $\epsilon$ closure, we obtain

Exercise 5.2

Is the above construction correct?
Correct construction for eliminating $\epsilon$ transitions

Earlier construction was handling the base case incorrectly!

We need cases between initial visit of $q_0$ and later repeated visits of $q_0$.

Correct proof for theorem 5.1.

We construct $A' = (Q \cup \{q'_0\}, \Sigma, \delta', q'_0, F')$, where

\[
\delta'(q'_0, a) \triangleq ECLOSE(\delta(ECLOSE(\{q_0\}), a)),
\]

\[
\delta'(q, a) \triangleq ECLOSE(\delta(q, a)) \text{ for each } a \in \Sigma \text{ and } q \in Q,
\]

\[
F' \triangleq \begin{cases} 
F \cup \{q'_0\} & \text{if } ECLOSE(\{q_0\}) \cap F \neq \emptyset \\
F & \text{otherwise.}
\end{cases}
\]

Intuitively, this time the above construction works.

Commentary: In the book, the DFA construction is presented. Here we only translating to NFA, because we already know how to convert from NFA to DFA. Please be mindful of this difference.
Example II: eliminating $\epsilon$ transitions (corrected)

Example 5.8

Consider automaton

The corresponding NFA is

$$\delta'(q'_0, a) \triangleq \text{ECLOSE}(\delta(\text{ECLOSE}({q_0}), a))$$
$$= \text{ECLOSE}(\delta(\{q_0, q_1\}, a)) = \text{ECLOSE}(\{q_2, q_0\}) = \{q_0, q_1, q_2\}$$
$\epsilon$-NFAs

$\epsilon$-NFAs still recognize regular languages.

$\epsilon$-NFAs, why are they useful?

To be continued...
End of Lecture 5