

CS310 : Automata Theory 2019

Lecture 11: Applications of pumping lemma

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Contrapositive of pumping lemma

Recall

Theorem 11.1

Let L be a language. L is not regular if, *for each n , there is a word $w \in L$ such that $|w| \geq n$ and for each breakup of w into three words $w = xyz$ such that*

1. $y \neq \epsilon$ and
2. $|xy| \leq n$,

then there is a $k \geq 0$ such that $xy^kz \notin L$.

How to use the pumping lemma?

In the theorem, there are two **exists** quantifiers, namely w and k .

Proving non regularity boils down to the following two quantifier instantiations.

- ▶ Choose a word w for each n
- ▶ Find k for **each breakup** of **the w**

The instantiations are the creative steps!

Proving a language non regular

Consider language L

- ▶ For each n , we propose a word w in L longer than n
- ▶ We define parameterized word x and non-empty word y such that
 - ▶ $xyz = w$ for some z and
 - ▶ parameter space covers all x and y such that $|xy| \leq n$.
- ▶ For each split x and y , we choose a k such that $xy^kz \notin L$.

We have proven that L is not regular

Example 1: using pumping lemma

Example 11.1

Consider again $L_{eq} = \{0^n 1^n \mid n \geq 0\}$.

- ▶ For each n we need a word. Let it be $w = 0^n 1^n$.
- ▶ The first n characters of w are 0^n . The breaks x and y are to be from within 0^n .
- ▶ Let $x = 0^i$ and $y = 0^j$, where $i + j \leq n$ and $j \neq 0$.

$$w = \underbrace{0^i}_x \underbrace{0^j}_y \underbrace{0^{n-j-i} 1^n}_z$$

i and j are encoding all possible breaks of 0^n .

- ▶ Now we choose $k = 0$ for each i and j . The corresponding word is

$$0^i (0^j)^0 0^{n-j-i} 1^n = 0^{n-j} 1^n.$$

- ▶ Clearly, $0^{n-j} 1^n \notin L_{eq}$. Therefore, L_{eq} is not regular.

Example 2: using pumping lemma

Example 11.2

Consider again $L = \{w \mid w \text{ has equal number of 0 and 1}\}$.

- ▶ For each n we need a word. Let it be $w = 0^n 1^n$.
- ▶ The first n characters of w are 0^n . The breaks x and y are to be from within 0^n .
- ▶ Let $x = 0^i$ and $y = 0^j$, where $i + j \leq n$ and $j \neq 0$.

$$w = \underbrace{0^i}_x \underbrace{0^j}_y \underbrace{0^{n-j-i} 1^n}_z$$

i and j are encoding all possible breaks of 0^n .

- ▶ Now we choose $k = 0$ for each i and j . The corresponding word is

$$0^i (0^j)^0 0^{n-j-i} 1^n = 0^{n-j} 1^n.$$

- ▶ Clearly, $0^{n-j} 1^n \notin L$. Therefore, L is not regular.

Example 3: using pumping lemma

Let word w^R be reverse of word w .

Example 11.3

Consider $L_{rev} = \{ww^R \mid w \in \{0,1\}^*\}$.

- ▶ For each n , let $w = 0^n 110^n$
- ▶ The first n characters of w are 0^n
- ▶ Let $x = 0^i$ and $y = 0^j$, where $i + j \leq n$ and $j \neq 0$.

$$w = \underbrace{0^i}_x \underbrace{0^j}_y \underbrace{0^{n-j-i}110^n}_z$$

- ▶ Let $k = 2$ for each i and j .

$$0^i(0^j)^2 0^{n-j-i}110^n = 0^{n+j}110^n \notin L_{rev}$$

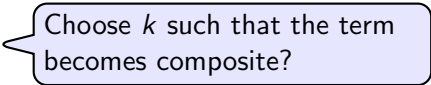
Example 4: using pumping lemma

Example 11.4

Consider $L_{prime} = \{1^p \mid p \text{ is a prime number.}\}$.

- ▶ For each n , let $w = 1^p$ such that $p > n + 2$
- ▶ The first n characters of w are 1^n .
- ▶ Let $x = 1^i$ and $y = 1^j$, where $i + j \leq n$ and $j \neq 0$.

$$w = \underbrace{1^i}_x \underbrace{1^j}_y \underbrace{1^{p-j-i}}_z$$

- ▶ So, $|xy^kz| = (p - j) + kj$. 
- ▶ Let $k = p - j$. Therefore, $|xy^kz| = (p - j)(1 + j)$.
- ▶ Since both $(p - j) > 1$ and $1 + j > 1$, $xy^kz \notin L_{prime}$.

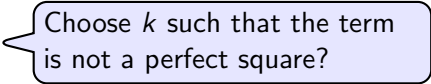
Example 5: using pumping lemma

Example 11.5

Consider $L = \{1^{p^2} \mid p \geq 0\}$.

- ▶ For each n , let $w = 1^{n^2}$
- ▶ The first n characters of w are 1^n .
- ▶ Let $x = 1^i$ and $y = 1^j$, where $i + j \leq n$ and $j \neq 0$.

$$w = \underbrace{1^i}_x \underbrace{1^j}_y \underbrace{1^{n^2-j-i}}_z$$

- ▶ So, $|xy^kz| = (n^2 - j) + kj$. 
- ▶ Let $k = 2$. Therefore, $|xy^kz| = n^2 + j$.
- ▶ Since $0 < j \leq n$ (why?), $n^2 < n^2 + j < n^2 + 2n + 1$ (why?).
Therefore, $xy^2z \notin L$.

Need for infinite memory

Feels like all non-regular languages needed to **remember infinite memory**.

Example 11.6

*In $\{0^n 1^n \mid n \geq 0\}$ we need to remember **the number** of seen 0s and count the 1s to match.*

*Finite number of states cannot count **unboundedly** increasing number.*

More generalized pumping lemma

We have been looking for evidence of **bad pumping** in the prefixes of the words.

We can look for such evidence for any subword of length greater than n .

Theorem 11.2

Let L be a language. L is not regular if,

- ▶ *for each n ,*
- ▶ *there are words u , and w such that $uw \in L$ and $|w| \geq n$*
- ▶ *for each breakup of w into three words $xyz = w$ such that $y \neq \epsilon$ and $|xy| \leq n$ then*
- ▶ *there is a $k \geq 0$ such that $uxy^kz \notin L$.*

In our earlier version of pumping lemma, $u = \epsilon$.

Converse does not hold!

Pumping lemma holds for the following language but is not regular.

$$L = \underbrace{\{ca^n b^n \mid n \geq 1\}}_{L_1} \cup \underbrace{\{c^n w \mid n \neq 1 \text{ and } w \in \{a, b\}^*\}}_{L_2}$$

Application of pumping lemma:

- ▶ $n = 1$
- ▶ Two cases
- ▶ **Case** take word $ca^j b^j \in L_1$
 - ▶ Let $x = \epsilon$, $y = c$, and $z = a^j b^j$.
 - ▶ For $k \neq 1$, $c^k a^j b^j \in L_2$, and for $k = 1$, $c^k a^j b^j \in L_1$
- ▶ **Case** take word $c^j w \in L_2$ for $j \neq 1$
 - ▶

Exercise 11.1

Complete the above application of pumping lemma

End of Lecture 11