CS310 : Automata Theory 2019

Lecture 17: PDA == CFG

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Recognizing “by empty stack” to “by final state”

Theorem 17.1

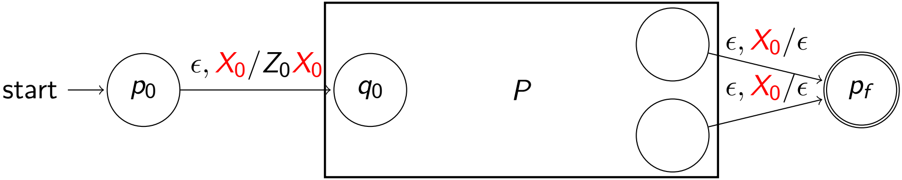
Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. There is a PDA P' such that $L(P') = L^{\epsilon}(P)$.

Proof. Let us construct the following PDA

$$P' = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta' \cup \delta, p_0, X_0, \{p_f\})$$

- new initial state p_0, accepting state p_f, and initial stack symbol X_0
- Extra transitions δ' are defined as follows

$$\delta'(q, \epsilon, X_0) \triangleq \begin{cases} \{(q_0, Z_0X_0)\} & q = p_0 \\ \{(p_f, \epsilon)\} & q \in Q \end{cases}$$



Recognizing "by empty stack" to "by final state" II

Proof(contd.).

claim: $L^{\epsilon}(P) \subseteq L(P')$

Let us assume $w \in L^{\epsilon}(P)$, i.e., w is accepting by P by empty stack.

Therefore $(q_0, w, Z_0) \vdash^*_{P} (q, \epsilon, \epsilon)$.

Since all transitions of P are in P', $(q_0, w, Z_0) \vdash^*_{P'} (q, \epsilon, \epsilon)$.

Due to properties of $\vdash^*$, $(q_0, w, Z_0X_0) \vdash^*_{P'} (q, \epsilon, X_0)$.

Due to the definition of δ',

$$(p_0, w, X_0) \vdash_{P'} (q_0, w, Z_0X_0) \vdash^*_{P'} (q, \epsilon, X_0) \vdash_{P'} (p_f, \epsilon, \epsilon)$$

Therefore, $w \in L(P')$. ...

Recognizing "by empty stack" to "by final state" III

Proof(contd.).

claim: $L(P') \subseteq L^\epsilon(P)$

Let us assume $w \in L(P')$, i.e., w is accepting by P by final state.

Therefore, $(p_0, w, X_0) \vdash^*_{P'} (p_f, \epsilon, \gamma)$.

Due to the definition of P', the first transition jumps to q_0 and pushes Z_0.
Let $q \in Q$ be the last state from where P' jumped to p_f.(why exists?)

Only the last transition can pop X_0 and no old transitions of P can push or pop X_0.

$$(p_0, w, X_0) \vdash_{P'} \underbrace{(q_0, w, Z_0X_0) \vdash^*_{P'} (q, \epsilon, X_0)}_{X_0 \text{ is always on stack}} \vdash_{P'} (p_f, \epsilon, \epsilon)$$

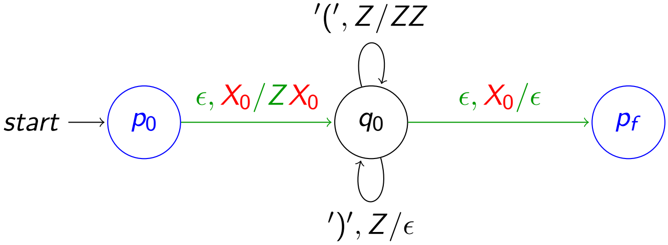
Therefore, $(q_0, w, Z_0) \vdash^*_{P} (q, \epsilon, \epsilon)$. Therefore, $w \in L^\epsilon(P)$. □

Example: recognizing "by empty stack" to "by final state"

Example 17.1

Consider the following PDA.

$$P = (\{q_0\}, \{'(', ')'\}, \{Z\}, \delta, q_0, Z, \{\})$$



$$P' = (\{q_0, p_0, p_f\}, \{'(', ')'\}, \{Z, X_0\}, \delta' \cup \delta, p_0, X_0, \{\})$$

By construction, $L^{\epsilon}(P) = L(P')$.

Recognizing "by final state" to "by empty stack"

Theorem 17.2

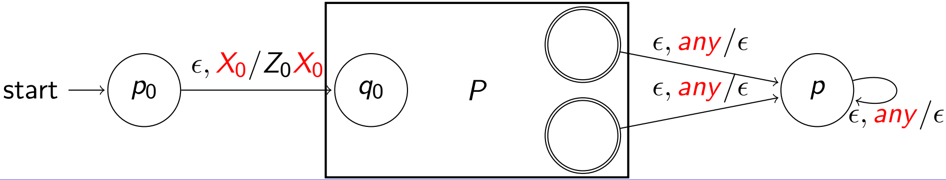
Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. There is a PDA P' such that $L^{\epsilon}(P') = L(P)$.

Proof. Let us construct the following PDA

$$P' = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta' \cup \delta, p_0, X_0, \{\})$$

- new initial state p_0, state p, and initial stack symbol X_0
- Extra transitions δ' are defined as follows

$$\delta'(q, \epsilon, Z) \triangleq \begin{cases} \{(q_0, Z_0X_0)\} & q = p_0 \wedge Z = X_0 \\ \{(p, \epsilon)\} & q \in F \cup \{p\} \wedge Z = \Gamma \cup \{X_0\} \end{cases}$$



Recognizing "by final state" to "by empty stack" II

Proof(contd.).

claim: $L(P) \subseteq L^{\epsilon}(P')$

Let us assume $w \in L(P)$, i.e., w is accepting by P by final state.

Therefore $(q_0, w, Z_0) \vdash_P^* (q, \epsilon, \gamma)$, where $q \in F$.

Since all transitions of P are in P', $(q_0, w, Z_0) \vdash_{P'}^* (q, \epsilon, \gamma)$.

Due to properties of $\vdash^*$, $(q_0, w, Z_0X_0) \vdash_{P'}^* (q, \epsilon, \gamma X_0)$.

Due to the definition of δ',

$$(p_0, w, X_0) \vdash_{P'} (q_0, w, Z_0X_0) \vdash_{P'}^* \underbrace{(q, \epsilon, \gamma X_0) \vdash_{P'}^* (p, \epsilon, \epsilon)}_{\text{loop at p will flush the stack}}$$

Therefore, $w \in L(P')$. ...

Recognizing "by final state" to "by empty stack" III

Proof(contd.).

claim: $L^{\epsilon}(P') \subseteq L(P)$

Let us assume $w \in L^{\epsilon}(P')$, i.e., w is accepting by P by empty stack.

Since the P transitions cannot pop X_0, the empty stack is reached only at p.

Therefore, $(p_0, w, X_0) \vdash^*_{P'} (p, \epsilon, \epsilon)$.

Due to the definition of P', the first transition jumps to q_0 and pushes Z_0.

Let $q_f \in F$ be the last state from where P' jumped to p.(why exists?)

$$(p_0, w, X_0) \vdash_{P'} \underbrace{(q_0, w, Z_0X_0) \vdash^*_{P'} (q_f, \epsilon, X_0\gamma)}_{X_0 \text{ is always on stack}} \vdash^*_{P'} (p, \epsilon, \epsilon)$$

Therefore, $(q_0, w, Z_0) \vdash^*_{P} (q, \epsilon, \gamma)$. Therefore, $w \in L(P)$. □

Topic 17.1

CFGs and PDAs

CFGs and PDAs

We have defined
context-free grammars (CFGs) and pushdown automata (PDAs).

It is no coincidence that

PDAs and CFGs recognize same class of languages.

Let us see the proof.

Topic 17.2

CFGs to PDAs

CFGs to PDAs

Theorem 17.3

Let $G = (N, T, P, S)$ be a CFG grammar. The PDA

$$A = (\{q\}, T, N \cup T, \delta, q, S, \{\})$$

Single state — Input symbols — Stack symbols — Initial state — Start Stack symbols — No final state(why?)

recognizes $L(G)$ by empty stack, where δ is defined as follows.

$$\delta(q, \epsilon, B) \triangleq \{(q, \beta) | B \rightarrow \beta \in P\}$$

$$\delta(q, a, a) \triangleq \{(q, \epsilon)\}.$$

Push anticipated productions

Verify the anticipated guess while popping

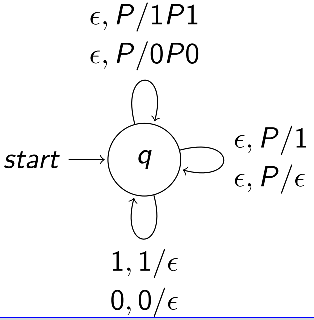
Example: CFGs to PDAs

Example 17.2

Consider CFG

1. $P \rightarrow 0P0$
2. $P \rightarrow 1P1$
3. $P \rightarrow \epsilon$
4. $P \rightarrow 1$

$$A = (\{q\}, \{0,1\}, \{0,1,P\}, \delta, q, P, \{\})$$



Proof : CFGs to PDAs

Proof.

Let $w \in L(G)$.

Consider the following left most derivation for w,

$$S \overset{lm}{\Longrightarrow} \gamma_1 \overset{lm}{\Longrightarrow} \dots \overset{lm}{\Longrightarrow} \gamma_n = w.$$

We will show by induction that

$$(q, w, S) \vdash_A^* (q, y_i, \alpha_i)$$

x_i has been matched and y_i is to be matched by α_i

such that

- there is an x_i such that $w = x_i y_i$ and $\gamma_i = x_i \alpha_i$ and
- if $\alpha_i \neq \epsilon$, then α_i begins with nonterminal .

x_i y_i

$\boxed{\alpha_i}$

Proof : CFGs to PDAs II

Proof(contd.)

base case:

For $i = 1$, $\gamma_1 = S$. We have

$$(q, w, S) \vdash_A^* (q, w, S),$$

i.e. nothing is processed. Therefore, $\alpha_i = S$, $y_1 = w$, and $x_1 = \epsilon$.

induction step:

By induction hypothesis, $(q, w, S) \vdash_A^* (q, y_i, \alpha_i)$ such that $w = x_i y_i$ and $\gamma_i = x_i \alpha_i$.

case $\alpha_i = \epsilon$: we are done.(why?)

...

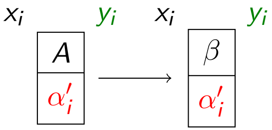
Proof : CFGs to PDAs III

Proof(contd.)

case $\alpha_i = A\alpha'_i$, where $A \in N$:

Therefore, $\gamma_i = x_i A\alpha'_i$ and $\gamma_{i+1} = x_i\beta\alpha'_i$ for some rule $A \to \beta \in P$.

PDA matches the move.



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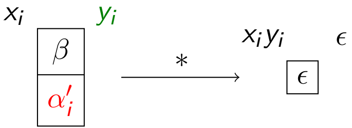
Proof : CFGs to PDAs IV

Proof(contd.)

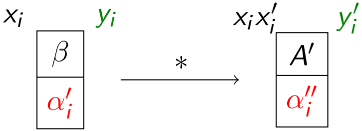
Two sub-cases

case $\beta\alpha'_i$ has no nonterminal.
Therefore, $i+1=n$. (why?)
Therefore, $\beta\alpha'_i = y_i$.
δ matches and pops all terminals.



$\alpha_{i+1} = y_{i+1} = \epsilon$

case Let $\beta\alpha'_i = x'_i A' \alpha''_i$.
Therefore, x'_i is prefix of y_i.(why?)
Let $y_i = x'_i y'_i$.
δ will match and pop top terminals.



$\alpha_{i+1} = A'\alpha''_i$ and $y_{i+1} = y'_i$

...

Proof : CFGs to PDAs V

Proof(contd.)

Proving the other direction

Assume $w \in L^{\epsilon}(A)$. So we have $(q, w, S) \vdash_A^* (q, \epsilon, \epsilon)$

For any $B \in N$ and $x \in T^*$, we will prove

$$\text{if } (q, x, B) \vdash_A^* (q, \epsilon, \epsilon) \text{ then } B \overset{*}{\Rightarrow} x$$

also says that any derivation of B does not look below B in the stack

We prove by induction on the length of derivation n.

...

Proof : CFGs to PDAs VI

Proof(contd.)

base case:

$n = 1$.

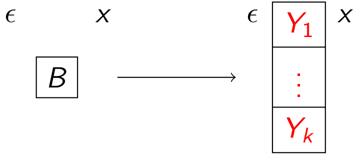
We can finish the run in a single transition only if $B \to \epsilon \in P$. Therefore

$$\text{if } (q, \epsilon, B) \underset{A}{\vdash} (q, \epsilon, \epsilon) \text{ then } B \Rightarrow \epsilon$$

induction step:

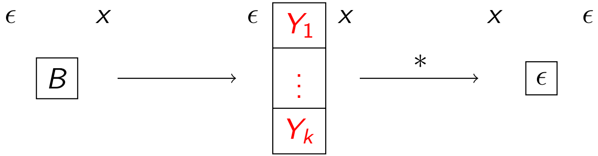
$n > 1$

Let $B \to Y_1 ... Y_k \in P$ is used in the first transition of A, which pops B and stack is filled with $Y_1, ..., Y_k$ without consuming anything.

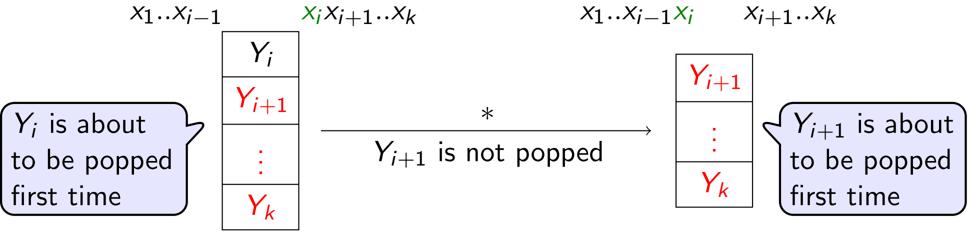


Proof : CFGs to PDAs VII

Proof(contd.)



Y_i has to be eventually popped.(why?) Let us divide $x = x_1x_2...x_k$ such that



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Proof : CFGs to PDAs VII

Proof(contd.)

Therefore, we have $(q, x_i, Y_i) \vdash_A^* (q, \epsilon, \epsilon)$ in less than n steps.

Due to induction hypothesis, $Y_i \stackrel{*}{\Rightarrow} x_i$.

Due to the properties of derivations,

$$x_1..x_{i-1} Y_i Y_{i+1}..Y_k \stackrel{*}{\Rightarrow} x_1..x_{i-1} x_i Y_{i+1}..Y_k$$

By stitching the derivations for each i, we will obtain

$$B \Rightarrow Y_1...Y_k \stackrel{*}{\Rightarrow} x_1...x_k.$$

Since we have $(q, w, S) \vdash_A^* (q, \epsilon, \epsilon)$, $S \stackrel{*}{\Rightarrow} w$.

□

End of Lecture 17