CS615 2019

Lecture 6: Encoding into SAT problem

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Topic 6.1

Encoding in SAT

SAT encoding

Since SAT is a NP-complete problem, therefore any NP-hard problem can be encoded into SAT in polynomial size.

Therefore, we can solve hard problems using SAT solvers.

We will look into a few interesting examples.

Objective of an encoding.

- Compact encoding (linear if possible)
- Redundant clauses may help the solver
- Encoding should be "compatible" with CDCL

Encoding into CNF

CNF is the form of choice

- Most problems specify collection of restrictions on solutions
- Each restriction is usually of the form

$$\text{if-this} \Rightarrow \text{then-this}$$

The above constraints are naturally in CNF.

"Even if the system has hundreds and thousands of formulas, it can be put into CNF piece by piece without any multiplying out"

– Martin Davis and Hilary Putnam

Exercise 6.1

Which of the following two encodings of $ite(p, q, r)$ is in CNF?

1. $(p \wedge q) \vee (\neg p \wedge r)$
2. $(p \Rightarrow q) \wedge (\neg p \Rightarrow r)$

Coloring graph

Problem:

color a graph$(\{v_1, \ldots, v_n\}, E)$ with at most d colors such that if $(v_i, v_j) \in E$ then color of v_i is different from v_j.

SAT encoding

Variables: p_{ij} for $i \in 1..n$ and $j \in 1..d$. p_{ij} is true iff v_i is assigned jth color.
Clauses:

- Each vertex has at least one color

$$\text{for each } i \in 1..n \qquad (p_{i1} \vee \cdots \vee p_{id})$$

- if $(v_i, v_j) \in E$ then color of v_1 is different from v_2.

$$(\neg p_{ik} \vee \neg p_{jk}) \qquad \text{for each } k \in 1..d, \quad (v_i, v_j) \in 1..n$$

Exercise 6.2

a. Encode: "every vertex has at most one color."

b. Do we need this constraint to solve the problem?

Pigeon hole principle

Prove:
if we place $n+1$ pigeons in n holes then there is a hole with at least 2 pigeons

The theorem holds true for any n, but we can prove it for a fixed n.

SAT encoding
Variables: p_{ij} for $i \in 0..n$ and $j \in 1..n$. p_{ij} is true iff pigeon i sits in hole j.
Clauses:

- Each pigeon sits in at least one hole

$$\text{for each } i \in 0..n \qquad (p_{i1} \vee \cdots \vee p_{in})$$

- There is at most one pigeon in each hole.

$$(\neg p_{ik} \vee \neg p_{jk}) \qquad \text{for each } k \in 1..n, \quad i < j \in 1..n$$

Topic 6.2

Cardinality constraints

Cardinality constraints

$$p_1 + ... + p_n \bowtie k$$

where $\bowtie \in \{<, >, \leq, \geq, =, \neq\}$

Encoding $p_1 + ... + p_n = 1$

- At least one of p_i is true

$$(p_1 \vee .... \vee p_n)$$

- Not more than one p_is are true

$$(\neg p_i \vee \neg p_j) \qquad i,j \in \{1,..,n\}$$

Exercise 6.3

a. What is the complexity of at least one constraints?

b. What is the complexity of at most one constraints?

Sequential encoding of $p_1 + .. + p_n \leq 1$

The earlier encoding of at most one is quadratic. We can do better by introducing auxiliary (fresh) variables.

Let s_i be a fresh variable to indicate that the count has reached 1 by i.

The following constraints encode $p_1 + .. + p_n \leq 1$.

$$
\begin{array}{rcl}
 & (p_1 \Rightarrow s_1) & \wedge \\
\bigwedge_{1<i<n}(& ((p_i \vee s_{i-1}) \Rightarrow s_i) & \wedge \quad (s_{i-1} \Rightarrow \neg p_i) \quad) \\
 & & \wedge \quad (s_{n-1} \Rightarrow \neg p_n)
\end{array}
$$

If $p_i = 1$, for each $j \geq i$, $s_j = 1$.

If already seen a one, no more ones.

Exercise 6.4

a. Give a satisfying assignment when $p_3 = 1$ *and all other ps are 0.*

b. Give a satisfying assignments of s_i*s when all ps are* 0.

c. Convert the constraints into CNF

Bitwise encoding of $p_1 + ... + p_n \leq 1$

Let $m = \lceil \ln n \rceil$.

- Consider bits $r_1, ..., r_m$
- For each $i \in 1...n$, let $b_1, ..., b_m$ be the binary encoding of $(i-1)$. We add the following constraints for p_i to be 1.

$$(p_i \Rightarrow (r_1 = b_1 \wedge ... \wedge r_m = b_m))$$

Example 6.1

Consider $p_1 + p_2 + p_3 \leq 1$.
$m = \lceil \ln n \rceil = 2$.

We get the following constraints.		*Simplified*
$(p_1 \Rightarrow (r_1 = 0 \wedge r_2 = 0))$		$(p_1 \Rightarrow (\neg r_1 \wedge \neg r_2))$
$(p_2 \Rightarrow (r_1 = 0 \wedge r_2 = 1))$	$\rightsquigarrow$	$(p_2 \Rightarrow (\neg r_1 \wedge r_2))$
$(p_3 \Rightarrow (r_1 = 1 \wedge r_2 = 0))$		$(p_3 \Rightarrow (r_1 \wedge \neg r_2))$

Exercise 6.5

What are the variable and clause size complexities?

Encoding $p_1 + ... + p_n \leq k$

There are several encodings

- Generalized pairwise
- Sequential counter
- Operational encoding
- Sorting networks
- Cardinality networks

Exercise 6.6

Given the above encodings, how to encode $p_1 + ... + p_n \geq k$*?*

Generalized pairwise encoding for $p_1 + ... + p_n \leq k$

No $k+1$ variables must be true at the same time.

For each $i_1, ..., i_{k+1} \in 1..n$, we add the following clause

$$(\neg p_{i_1} \vee \cdots \vee \neg p_{i_{k+1}})$$

Exercise 6.7

How many clauses are added for the encoding?

Sequential counter encoding for $p_1 + ... + p_n \leq k$

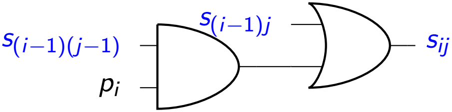
Let variable s_{ij} encode that the sum upto p_i has reached to j or not.

- Constraints for first variable p_1

$$(p_1 \Rightarrow s_{11}) \wedge \bigwedge_{j \in [2,k]} \neg s_{1j}$$

- Constraints for p_i, where $i > 1$

$$((p_i \vee s_{(i-1)1}) \Rightarrow s_{i1}) \wedge \bigwedge_{j \in [2,k]} ((\underbrace{p_i \wedge s_{(i-1)(j-1)}}_{add\ +1} \vee s_{(i-1)j}) \Rightarrow s_{ij})$$



Sequential counter encoding for $p_1 + ... + p_n \leq k$ (II)

- If the sum has reached to k at $i - 1$, no more ones

$$(s_{(i-1)k} \Rightarrow \neg p_i)$$

Exercise 6.8

What is the variable/clause complexity?

Operational encoding for $p_1 + ... + p_n \leq k$

Sum the bits using full adders. Compare the resulting bits against k.

Produces $O(n)$ encoding, however the encoding is not considered good for sat solvers, since it is not arc consistent.

Arc-consistency

Let $C(Ps)$ be a problem with variables $Ps = p_1, ..., p_n$.

Let $E(Ps, Ts)$ be encoding of the problem, where variables $Ts = t_1, ..., t_k$ are introduced by the encoding.

Definition 6.1

We say $E(Ps, Ts)$ is arc-consistent if for any partial model m of E

1. *If $m|_{Ps}$ is inconsistent with C, then unit propagation in E causes conflict.*
2. *If $m|_{Ps}$ is extendable to m' by local reasoning in C, then unit propagation in E obtains m'' such that $m''|_{Ps} = m'$.*

Unit propagation == Local reasoning

Example: arc-consistency

Example 6.2

Consider problem $p_1 + ... + p_n \leq 1$

An encoding is arc-consistent if

1. *If at any time two* p_i*s are made true, unit propagation should trigger unsatisfiability*
2. *If at any time* p_i *is made true, unit propagation should make all other* p_j*s false*

Commentary: The unit propagation in the encoding must mimic local reasoning of the problem. Intuitively, the encoding must not make the life of solver harder. If local reasoning can deduce something, then unit propagation must also deduce it. For more discussion, look in `http://minisat.se/downloads/MiniSat+.pdf` page5

Example: non arc-consistent encoding

Example 6.3

Consider problem $p_1 + p_2 + p_3 \leq 0$

Let us use full adder encoding

$$\begin{aligned} sum =& (p_1 \oplus p_2 \oplus p_3) \\ carry =& (p_1 \wedge p_2) \vee (p_2 \wedge p_3) \vee (p_1 \wedge p_3) \\ & \neg sum \wedge \neg carry \end{aligned}$$

Clearly p_1, p_2, p_3 *are* 0.

But, $\underbrace{\textit{the unit propagation without any decisions}}_{\textit{Local reasoning}}$ *does not give the model.*

Exercise 6.9

Does Tseitin encoding preserve the arc-consistency?

Cardinality constraints via sorted variables $O(n \ln^2 n)$

Let us suppose we have a circuit that produces sorted bits in decreasing order.

$$([y_1, .., y_n], Cs) := sort(p_1, ..p_n)$$

We can encode the cardinality constraints as follows

$$p_1 + .. + p_n \leq k \qquad \{y_{k+1} = 0\} \cup Cs$$

$$p_1 + .. + p_n \geq k \qquad \{y_k = 1\} \cup Cs$$

Exercise 6.10

a. How to encode $p_1 + .. + p_n < k$

b. How to encode $p_1 + .. + p_n > k$

c. How to encode $p_1 + .. + p_n = k$

For details : look at the extra slides at the end of the lecture.

Topic 6.3

Pseudo-Boolean constraints

Pseudo-Boolean constraints

Let p_1,....,p_n be Boolean variables.

The following is a pseudo-Boolean constraint.

$$c_1 p_1 + ... + c_n p_n \leq c,$$

where $c_1, .., c_n, c \in \mathbb{Z}$.

How should we solve them?

- Using Boolean reasoning
- Using arithmetic reasoning

Here we will see the Boolean encoding for the constraints.

Observations on pseudo-Boolean constraints

- Replacing negative coefficients to positive

$$t - c_i p_i \leq c \quad \rightsquigarrow \quad t + c_i(\neg p_i) \leq c + c_i$$

- Divide the whole constraints by $d := gcd(c_1, ..., c_n)$.

$$c_1 p_1 + ... + c_n p_n \leq c \quad \rightsquigarrow \quad (c_1/d)p_1 + .. + (c_n/d)p_n \leq \lfloor c/d \rfloor$$

- Trim large coefficients to $c + 1$. Let us suppose $c_i > c$.

$$t + c_i p_i \leq c \quad \rightsquigarrow \quad t + (c+1)p_i \leq c$$

- Trivially true are replaced by $\top$. If $c >= c_i + ... + c_n$

$$c_1 p_1 + ... + c_n p_n \leq c \quad \rightsquigarrow \quad \top$$

- Trivially false are replace by $\perp$. If $c < 0$

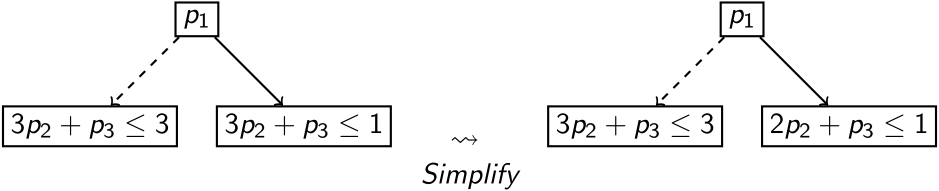
$$c_1 p_1 + ... + c_n p_n \leq c \quad \rightsquigarrow \quad \perp$$

Translating to decision diagrams

We choose a 0 and 1 for each variable to split cases and simplify.

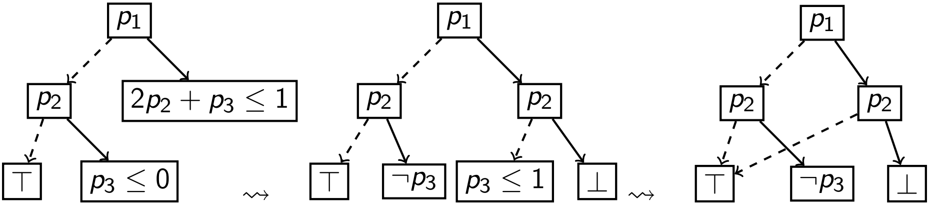
Example 6.4

Consider $2p_1 + 3p_2 + p_3 \leq 3$

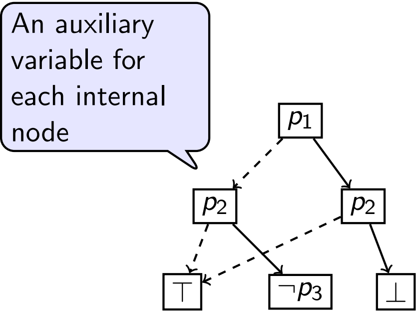


Example: translating to decision diagrams

We can split node left node $3p_2 + p_3 \leq 3$ further on p_2.



Example: decision diagrams to clauses



$$\rightsquigarrow \quad \begin{array}{l} (\neg p_1 \Rightarrow temp1) \land \\ (temp1 \land \neg p_2 \Rightarrow \top) \land \\ (temp1 \land p_2 \Rightarrow \neg p_3) \land \\ \\ (p_1 \Rightarrow temp2) \land \\ (temp2 \land \neg p_2 \Rightarrow \top) \land \\ (temp2 \land p_2 \Rightarrow \bot) \end{array}$$

Exercise 6.11

a. Simplify the clauses

b. Complexity of the translation from pseudo-Boolean constraints?

Exercise: Pseudo-Boolean constraints

Exercise 6.12

Let p_1, p_2, and p_3 be Boolean variables. Convert the following pseudo-Boolean inequalities into BDDs while applying simplifications eagerly, and thereafter into equivsatisfiable CNF clauses.

- $2p_1 + 6p_3 + p_2 \leq 3$
- $2p_1 + 6p_3 + p_2 \geq 3$
- $2p_1 + 3p_3 + 5p_2 \leq 6$

Exercise: equivalent ranges in pseudo-Boolean constraints

Exercise 6.13

Let p_1, p_2, and p_3 be Boolean variables. Let us consider pseudo-Boolean constraint $2p_1 + 3p_3 + 5p_2 \leq K$, for some non-negative integer K. For which of the following ranges of K, the constraint has same set of satisfying models?

- $[0, 2]$
- $[3, 4]$
- $[7, 7]$
- $[10, 12]$

Exponential sized BDDs for Pseudo-Boolean constraints

Consider the following pseudo-Boolean constraint

$$\sum_{i=1}^{2n} \sum_{j=1}^{2n} (2^{j-1} + 2^{2n+i-1}) p_{ij} \leq (2^{4n} - 1)n$$

Any BDD representing the above constraints have at least 2^n nodes.

Proof in : A New Look at BDDs for Pseudo-Boolean Constraints,

https://www.cs.upc.edu/~oliveras/espai/papers/JAIR-bdd.pdf

API for pseudo-Boolean constraints in Z3

```
from z3 import *
p = Bool("p")
q = Bool("q")  # declare a Boolean variable

c1 = PbLe([(p,1),(q,2)], 3) # encodes p+2q =< 3
c2 = PbGe([(p,1),(q,-1)], 4) # encodes p-q => 4

s = Solver()
s.add( And(c1,c2) )
s.check()
```

Topic 6.4

More problems

Solving Sudoku using SAT solvers

Example 6.5

4	2	6	5	7	1	3	9	8
8	5	7	2	9	3	1	4	6
1	3	9	4	6	8	2	7	5
9	7	1	3	8	5	6	2	4
5	4	3	7	2	6	8	1	9
6	8	2	1	4	9	7	5	3
7	9	4	6	3	2	5	8	1
2	6	5	8	1	4	9	3	7
3	1	8	9	5	7	4	6	2

Sudoku

- *Variables:* $v_{i,j,k} \in \mathcal{B}$ *and* $i, j, k \in \{1, ..., 9\}$
- *If* $v_{i,j,k} = 1$*, column* i *and row* j *contains* k*.*
- *Value in each cell is valid:*

$$\sum_{k=1}^{9} v_{i,j,k} = 1 \qquad i, j \in \{1, .., 9\}$$

- *Each value used exactly once in each row:*

$$\sum_{i=1}^{9} v_{i,j,k} = 1 \qquad j, k \in \{1, .., 9\}$$

- *Each value used exactly once in each column:*

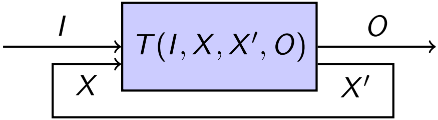
$$\sum_{j=1}^{9} v_{i,j,k} = 1 \qquad i, k \in \{1, .., 9\}$$

- *Each value used exactly once in each* 3×3 *grid*

$$\sum_{s=1}^{3} \sum_{r=1}^{3} v_{3i+r,j+s,k} = 1 \quad i, j \in \{0, 1, 2\}, k \in \{1, .., 9\}$$

Bounded model checking

Consider a Mealy machine



- I is a vector of variables representing input
- O is a vector of variables representing output
- X is a vector of variables representing current state
- X' is a vector of variables representing next state

Prove: After n steps, the machines always produces output O that satisfies some formula $F(O)$.

Bounded model checking encoding

SAT encoding:

Variables:

- $I_0, \ldots, I_{n-1}$ representing input at every step
- $O_1, \ldots, O_n$ representing output at every step
- $X_0, \ldots, X_n$ representing internal state at every step

Clauses:

- Encoding system runs

$$T(I_0, X_0, X_1, O_1) \wedge \cdots \wedge T(I_{n-1}, X_{n-1}, X_n, O_n)$$

- Encoding property

$$\neg F(O_n)$$

If the encoding is unsat the property holds.

Topic 6.5

Input Format

DIMACS Input format

Example 6.6

Input CNF

```
c
c this is a comment
c
p cnf 4 6
-2   3  0
 1   3  0
-1   2  3 -4 0
-1  -2  0
 1  -2  0
 2  -3  0
```

Declares number of variables and clauses.

Each row is a clause ending with 0

Clause is $p_2 \vee \neg p_3$

Topic 6.6

Problems

SAT encoding: n queens

Exercise 6.14

Encode N-queens problem in a SAT problem.
N-queens problem: Place n queens in $n \times n$ chess such that none of the queens threaten each other.

SAT encoding: overlapping subsets

Exercise 6.15

For a set of size n, find a maximal collection of k sized sets such that any pair of the sets have exactly one common element.

SAT encoding: setting a question paper

Exercise 6.16

There is a datbase of questions with the following properties:

- *Hardness level* $\in$ {*Easy,Medium,Hard*}
- *Marks* $\in \mathbb{N}$
- *Topic* $\in \{T_1, ..., T_t\}$
- *LastAsked* $\in$ *Years*

Make a question paper with the following properties

- *It must contain x% easy, y% medium, and z% difficult marks.*
- *The total marks of the paper are given.*
- *The number of problems in the paper are given.*
- *All topics must be covered.*
- *No question that was asked in last five years must be asked.*

Write an encoding into SAT problem that finds such a solution. Test your encoding on reasonably sized input database. Devise a strategy to evaluate your tool and report plots to demonstrate the performance.

SAT encoding: finding a schedule

Exercise 6.17

An institute is offering m courses.

- *Each has a number of contact hours $==$ credits*

The institute has r rooms.

- *Each room has a maximum student capacity*

The institute has s weekly slots to conduct the courses.

- *Each slot has either 1 or 1.5 hour length*

There are n students.

- *Each student have to take minimum number of credits*
- *Each student has a set of preferred courses.*

Assign each course slots and a room such that all student can take courses from their preferred courses that meet their minimum credit criteria.
Write an encoding into SAT problem that finds such an assignment . Test your encoding on reasonably sized input. Devise a strategy to evaluate your tool and report plots to demonstrate the performance.

SAT encoding: synthesis by examples

Exercise 6.18

Consider an unknown function $f : \mathcal{B}^N \rightarrow \mathcal{B}$. *Let us suppose for inputs* $I_1, ..., I_m \in \mathcal{B}^N$, *we know the values of* $f(I_1), .., f(I_m)$.

a) Write a SAT encoding of finding a k-sat formula containing ℓ *clauses that represents the function.*

b) Write a SAT encoding of finding an NNF (negation normal form, i.e., $\neg$ *is only allowed on atoms) formula of height k and width* ℓ *that represents the function.(Let us not count negation in the height.)*

c) Write a SAT encoding of finding a binary decision diagram of height k and maximum width ℓ *that represents the function.*

Test your encoding on reasonably sized input. Devise a strategy to evaluate your tool and report plots to demonstrate the performance.

SAT encoding: Rubik's cube

Exercise 6.19

Write a Rubik's cube solver using a SAT solver

- *Input:*
  - *start state,*
  - *final state, and*
  - *number of operations k*
- *Output:*
  - *sequence of valid operations or*
  - *"impossible to solve within k operations"*

Test your encoding on reasonably many inputs. Devise a strategy to evaluate your tool and report plots to demonstrate the performance.

SAT encoding: square of squares

Exercise 6.20

Squaring the square problem: "Tiling an integral square using only other smaller integral squares such that all tiles have different sizes."

Consider a square of size $n \times n$, find a solution of above problem using a SAT solver using tiles less than k.

Test your encoding on reasonably sized n and k. Devise an strategy to evaluate your tool and report plots to demonstrate the performance.

SAT encoding: Mondrian art

Exercise 6.21

Mondiran art problem: "Divide an integer square into non-congruent rectangles. If all the sides are integers, what is the smallest possible difference in area between the largest and smallest rectangles?"

Consider a square of size $n \times n$, find a Mondrian solution above k using a SAT solver.

Pseudo-Boolean constraints

Exercise 6.22

Let a, b, and n be positive integers such that $\Sigma_{i=1}^{n} b^i < a$. *Let* $w_i = a + b^i$ *for each* $i \in 1..n$. *Show that the following pseudo-Boolean constraints are equivalent.*

$$w_1 p_1 + ... + w_n p_n \leq (an/2)$$

and

$$p_1 + ... + p_n \leq (n/2) - 1$$

Example : make mastermind player

Exercise 6.23

Mastermind is a two player game. There are n colors. Let $k < n$ be a positive number.

1. *Player one chooses a hidden sequence of k colors (colors may repeat)*
2. *The game proceeds iteratively as follows until player two has guessed the sequence correctly.*
   - *Player two makes a guess of sequence of k colors*
   - *Player one gives feedback to player two by giving*
     - *the number of correct colors in the correct positions, and*
     - *the number of correct colors in the wrong positions.*

One can play the game here
http://www.webgamesonline.com/mastermind/

Create player two using a SAT solver

Removing edges to be acyclic graph

Exercise 6.24

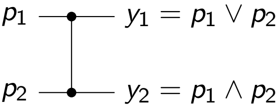
Give a SAT encoding for removing minimum number of edges in a (un)directed graphs such that the graph becomes acyclic.

Topic 6.7

Extra section : cardinality constraints via merge sort

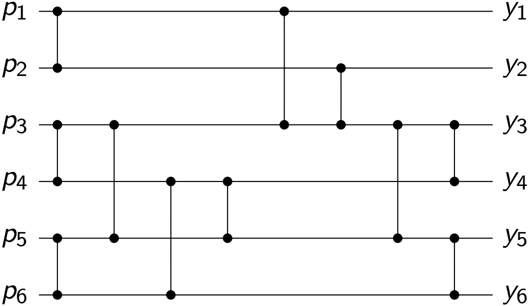
Sorting networks

The following circuit sorts two bits p_1 and p_2.



We can sort any number of bits by composing the circuit according to a sorting algorithm.

Example 6.7 *Sorting* 6 *bits using merge sort.*



Formal definition of sorting networks

base case:
$n = 1$

$$sort(p_1, p_2) \triangleq merge([p_1], [p_2]);$$

induction step:
$2n > 2$
Let,

sort/merge returns a vector of signals and a set of clauses.

$$([p'_1, ..., p'_n], Cs_1) := sort(p_1, .., p_n)$$
$$([p'_{n+1}, ..., p'_{2n}], Cs_2) := sort(p_{n+1}, .., p_{2n})$$
$$([y_1, ..., y_{2n}], Cs_M) := merge([p'_1, ..., p'_n], [p'_{n+1}, ..., p'_{2n}])$$

Then,

$$sort(p_1, .., p_{2n}) \triangleq ([y_1, .., y_{2n}], Cs_1 \cup Cs_2 \cup Cs_M)$$

Formally merge: odd-even merging network

Merge assumes that the input vectors are sorted.

base case:

$$merge([p_1],[p_2]) \triangleq ([y_1, y_2], \{y_1 \Leftrightarrow p_1 \wedge p_2, y_2 \Leftrightarrow p_1 \vee p_2\});$$

induction step:

Let

$$([z_1, .., z_n], Cs_1) := merge([p_1, p_3 ..., p_{n-1}], [y_1, y_3, ..., y_{n-1}])$$
$$([z'_1, .., z'_n], Cs_2) := merge([p_2, p_4 ..., p_n], [y_2, y_4, ..., y_n])$$
$$([c_{2i}, c_{2i+1}], CS^i_M) := merge([z_{i+1}], [z'_i]) \qquad \text{for each } i \in [1, n-1]$$

Then,

$$merge([p_1, ..., p_n], [y_1, ..., y_n]) \triangleq ([z_1, c_1, .., c_{2n-1}, z'_n], Cs_1 \cup Cs_2 \cup \bigcup_i CS^i_M)$$

Cardinality Networks: a theoretical and empirical study, 2011, Constraints

End of Lecture 6