CS615: Formal Specification and Verification of Programs 2019

Lecture 6: Maps

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Topic 6.1

Maps

Morphisms

Definition 6.1

For sets X and Y, $f : X \to Y$ is a morphism relative to functions $g : X \to X$ and $g' : Y \to Y$ if

$$\forall x.\ f(g(x)) = g'(f(x))$$

A morphism may be defined relative to relations

Definition 6.2

For sets X and Y, $f : X \to Y$ is a morphism relative to relations $R : X \times X$ and $R' : Y \times Y$ such that

$$\forall x, y.\ (x, y) \in R \Rightarrow (f(x), f(y)) \in R'$$

Example: morphism

Example 6.1

Consider sets $X = \{x, y, z\}$ *and* $Y = \{a, b\}$

Let $R = \{(x, y), (x, z)\}$ *and* $R' = \{(a, b)\}$

The following is a morphism relative to R *and* R'.

$$f = \{x \to a, y \to b, z \to b\}$$

Exercise 6.1

Is the following a morphism relative to R^{-1} *and* R'?

- $f = \{x \to a, y \to a, z \to b\}$
- $f = \{x \to a, y \to a, z \to a\}$
- $f = \{x \to b, y \to a, z \to a\}$

Two posets

We will be handling two posets. We need two notations.

One of them will be written as usual.

$$(Y, \sqsubseteq, \sqcup, \sqcap)$$

And the other will be written as follows.

$$(X, \leq, \vee, \wedge)$$

Monotone maps

Definition 6.3

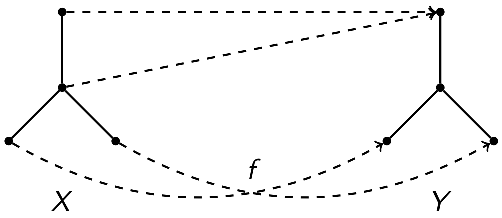
For posets $(X, \leq)$ and $(Y, \sqsubseteq)$, $f : X \rightarrow Y$ is a monotone map if

$$\forall x, y \in X.\ x \leq y \Rightarrow f(x) \sqsubseteq f(y)$$

f is a morphism relative to $\leq$ and $\sqsubseteq$.

Example 6.2

The following are posets and a monotonic map between them (dashed lines).



Join preserving

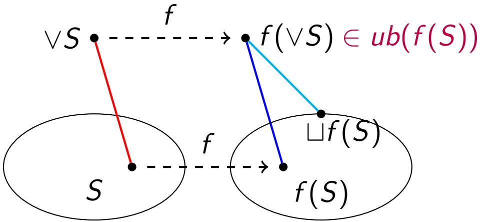
Theorem 6.1

For posets $(X, \leq)$ and $(Y, \sqsubseteq)$, let $f : X \to Y$ be monotone map. For each $S \subseteq X$, if $\vee S$ and $\sqcup f(S)$ exist, then

$$\sqcup f(S) \sqsubseteq f(\vee S)$$

Proof.

1. By def., all in $S \leq \vee S$.
2. Due to 1 and f is monotone map, all in $f(S) \sqsubseteq f(\vee S)$.
3. Therefore, $f(\vee S) \in ub(f(S))$.
4. Therefore, $\sqcup f(S) \sqsubseteq f(\vee S)$.



□

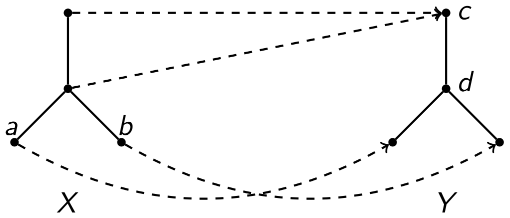
Exercise 6.2

Give a counterexample to show that the reverse of above theorem is not true?

Example: Join

Example 6.3

The following are posets and a monotonic map between them (dashed lines).



Consider $S = \{a, b\}$.

$f(\vee S) = c$

$\sqcup f(S) = d$

Clearly, $\sqcup f(S) \sqsubseteq f(\vee S)$

Exercise: reverse

Exercise 6.3

For join-semi lattices $(X, \leq, \vee)$ and $(Y, \sqsubseteq, \sqcup)$, a function $f : X \to Y$ is monotone map if the following holds.

For each $S \subseteq X$, if $\vee S$ and $\sqcup f(S)$ exist, then

$$\sqcup f(S) \sqsubseteq f(\vee S)$$

Order embedding

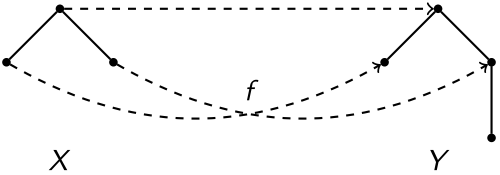
Definition 6.4

For posets $(X, \leq)$ and $(Y, \sqsubseteq)$, $f : X \to Y$ is an order embedding if

$$\forall x, y \in X.\ x \leq y \Leftrightarrow f(x) \sqsubseteq f(y)$$

Example 6.4

X and Y are posets and dash-lines are an order embedding.



Theorem 6.2

For posets $(X, \leq)$ and $(Y, \sqsubseteq)$, let $f : X \to Y$ be an order embedding, then f is injective.

Exercise 6.4

Prove 6.2

Order isomorphism

Definition 6.5

For posets $(X, \leq)$ *and* $(Y, \sqsubseteq)$,
$f : X \rightarrow Y$ *is an order isomorphism if* f *is order embedding and onto.*

Definition 6.6

$(X, \leq)$ *and* $(Y, \sqsubseteq)$ *are order isomorphic if there is an order isomorphism between them.*

Characterize order isomorphism

Theorem 6.3

Posets $(X, \leq)$ and $(Y, \sqsubseteq)$ are order-isomorphic iff there exists $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that

- $f \circ g = \Delta_X$,
- $g \circ f = \Delta_Y$, *and*
- f *and* g *are monotone.*

Exercise 6.5

Prove theorem 6.3

Continuous maps

Definition 6.7

For posets $(X, \leq)$ and $(Y, \sqsubseteq)$, $f : X \to Y$ is continuous (upper-continuous) if for all chains $C \subseteq X$ such that $\vee C$ exists then $\sqcup f(C)$ exists and

$$f(\vee C) = \sqcup f(C)$$

Symmetrically, lower-continuous is defined.

Theorem 6.4

Continuous maps are monotonic

Exercise 6.6

Show if $x \sqsubseteq y$ and f is continuous then $f(y) = f(x) \sqcup f(y)$

Exercise 6.7

Give an example where a monotone map is not continuous.

Exercise: monotone and ACC $\Rightarrow$ continuous

Exercise 6.8

Let poset $(X, \leq)$ *satisfies ascending chain condition(ACC). Let* $(Y, \sqsubseteq)$ *be a post. Let* $f : X \rightarrow Y$ *be a monotone function. Prove* f *is continuous.*

Topic 6.2

Closure operators

Closure operators

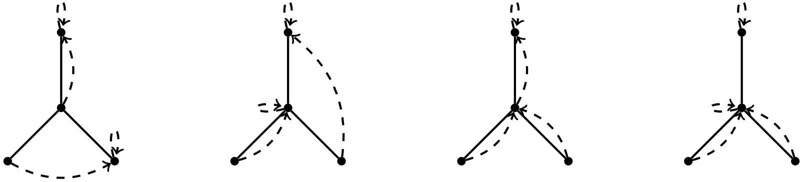
Definition 6.8

On a poset $(X, \leq)$, f is an *upper closure operator* *if for all $x, y \in X$*

- $x \leq f(x)$ *(extensive)*
- $x \leq y \Rightarrow f(x) \leq f(y)$, *and* *(monotone)*
- $f(f(x)) = f(x)$. *(idempotent)*

Example 6.5

Consider the following posets and functions (dashed edges).
Are the functions upper closure operator?



Characterize upper-closure operators

Theorem 6.5

f is upper closure operator iff $x \leq f(y) \Leftrightarrow f(x) \leq f(y)$

Proof.

Assume f is upper closure

claim: $x \leq f(y) \Rightarrow f(x) \leq f(y)$

1. Consider $x \leq f(y)$
2. Since f is monotone, $f(x) \leq f(f(y))$
3. Sine f is idempotent, $f(x) \leq f(y)$

claim: $f(x) \leq f(y) \Rightarrow x \leq f(y)$

1. $f(x) \leq f(y)$
2. $x \leq f(x) \leq f(y)$
3. $x \leq f(y)$

Forward direction is proven. …

Characterize upper-closure operators II

Proof (contd.)

Assume $x \leq f(y) \Leftrightarrow f(x) \leq f(y)$

We need to prove that f is extensive, monotone, and idempotent.

claim: f is extensive

1. $f(x) \leq f(x)$ always holds
2. Therefore, $f \leq f(x)$

claim: f is monotone

1. consider $x \leq y$
2. Use previous result, $x \leq y \leq f(y)$
3. Therefore, $x \leq f(y)$
4. Apply the assumption, $f(x) \leq f(y)$

...

Characterize upper-closure operators II

Proof (contd.)

Recall assumption $x \leq f(y) \Leftrightarrow f(x) \leq f(y)$

claim: f is idempotent

1. Clearly, $f(x) \leq f(x)$
2. Apply forward of the assumption, $f(f(x)) \leq f(x)$
3. Again clearly, $f(f(x)) \leq f(f(x))$
4. Apply backward of the assumption, $f(x) \leq f(f(x))$
5. Due to 2 and 4, $f(x) = f(f(x))$

□

Exercise 6.9

Use Z3 to prove the above theorem.

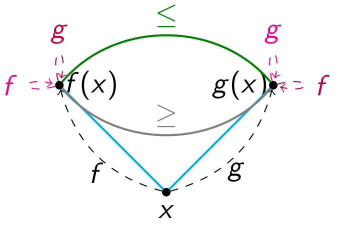
Fixed points of closure operators

Theorem 6.6

An upper closure operator is uniquely defined by its fixed points

Proof.

Let f and g be two upper closure operators that have same fixed points.
But, differ at point x, i.e., $f(x) \neq g(x)$.



due to idempotence of f and g

due to shared fixed points

due to extensive property

due to monotone g

due to monotone f

Contradiction: $f(x) = g(x)$

□

Exercise 6.10

a. *Show $f(x) \sqcup f(y)$ is a fixed point*

b. *Show image of complete lattice by closure operator is complete lattice*

End of Lecture 6