Lecture 6: Maps

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Topic 6.1

Maps
Morphisms

Definition 6.1
For sets $X$ and $Y$, $f : X \rightarrow Y$ is a morphism relative to functions $g : X \rightarrow X$ and $g' : Y \rightarrow Y$ if

$$\forall x. \ f(g(x)) = g'(f(x))$$

A morphism may be defined relative to relations

Definition 6.2
For sets $X$ and $Y$, $f : X \rightarrow Y$ is a morphism relative to relations $R : X \times X$ and $R' : Y \times Y$ such that

$$\forall x, y. \ (x, y) \in R \Rightarrow (f(x), f(y)) \in R'$$
Example: morphism

Example 6.1

Consider sets \( X = \{x, y, z\} \) and \( Y = \{a, b\} \)

Let \( R = \{(x, y), (x, z)\} \) and \( R' = \{(a, b)\} \)

The following is a morphism relative to \( R \) and \( R' \).

\[
f = \{x \rightarrow a, y \rightarrow b, z \rightarrow b\}
\]

Exercise 6.1

Is the following a morphism relative to \( R^{-1} \) and \( R' \)?

- \( f = \{x \rightarrow a, y \rightarrow a, z \rightarrow b\} \)
- \( f = \{x \rightarrow a, y \rightarrow a, z \rightarrow a\} \)
- \( f = \{x \rightarrow b, y \rightarrow a, z \rightarrow a\} \)
Two posets

We will be handling two posets. We need two notations.

One of them will be written as usual.

\((Y, \sqsubseteq, \sqcup, \sqcap)\)

And the other will be written as follows.

\((X, \leq, \lor, \land)\)
Monotone maps

Definition 6.3
For posets \((X, \leq)\) and \((Y, \sqsubseteq)\), \(f : X \to Y\) is a monotone map if

\[
\forall x, y \in X. \quad x \leq y \Rightarrow f(x) \sqsubseteq f(y)
\]

\(f\) is a morphism relative to \(\leq\) and \(\sqsubseteq\).

Example 6.2
The following are posets and a monotonic map between them (dashed lines).
Join preserving

Theorem 6.1
For posets \((X, \leq)\) and \((Y, \sqsubseteq)\), let \(f : X \to Y\) be monotone map. For each \(S \subseteq X\), if \(\forall S\) and \(\sqcup f(S)\) exist, then

\[ \sqcup f(S) \sqsubseteq f(\forall S) \]

Proof.
1. By def., all in \(S \leq \forall S\).
2. Due to 1 and \(f\) is monotone map, all in \(f(S) \subseteq f(\forall S)\).
3. Therefore, \(f(\forall S) \in ub(f(S))\).
4. Therefore, \(\sqcup f(S) \subseteq f(\forall S)\).

Exercise 6.2
Give a counterexample to show that the reverse of above theorem is not true?
Example: Join

Example 6.3

The following are posets and a monotonic map between them (dashed lines).

Consider $S = \{a, b\}$.

$f(\lor S) = c$

$\sqcup f(S) = d$

Clearly, $\sqcup f(S) \sqsubseteq f(\lor S)$
Exercise: reverse

Exercise 6.3

For join-semi lattices \((X, \leq, \lor)\) and \((Y, \sqsubseteq, \sqcup)\), a function \(f : X \rightarrow Y\) is monotone map if the following holds.

For each \(S \subseteq X\), if \(\lor S\) and \(\sqcup f(S)\) exist, then

\[
\sqcup f(S) \sqsubseteq f(\lor S)
\]
Order embedding

Definition 6.4
For posets \((X, \leq)\) and \((Y, \sqsubseteq)\), \(f : X \to Y\) is an order embedding if

\[
\forall x, y \in X. \ x \leq y \iff f(x) \sqsubseteq f(y)
\]

Example 6.4
\(X\) and \(Y\) are posets and dash-lines are an order embedding.

Theorem 6.2
For posets \((X, \leq)\) and \((Y, \sqsubseteq)\), let \(f : X \to Y\) be an order embedding, then \(f\) is injective.

Exercise 6.4
Prove 6.2
Order isomorphism

Definition 6.5
For posets \((X, \leq)\) and \((Y, \sqsubseteq)\),
f : X \rightarrow Y is an order isomorphism if f is order embedding and onto.

Definition 6.6
\((X, \leq)\) and \((Y, \sqsubseteq)\) are order isomorphic if there is an order isomorphism between them.
Characterize order isomorphism

Theorem 6.3
Posets \((X, \leq)\) and \((Y, \sqsubseteq)\) are order-isomorphic iff there exists \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\) such that

- \(f \circ g = \Delta_X\),
- \(g \circ f = \Delta_Y\), and
- \(f\) and \(g\) are monotone.

Exercise 6.5
Prove theorem 6.3
Continuous maps

Definition 6.7
For posets \((X, \leq)\) and \((Y, \sqsubseteq)\), \(f : X \to Y\) is continuous (upper-continuous) if for all chains \(C \subseteq X\) such that \(\vee C\) exists then \(\sqcup f(C)\) exists and

\[ f(\vee C) = \sqcup f(C) \]

Symmetrically, lower-continuous is defined.

Theorem 6.4
Continuous maps are monotonic

Exercise 6.6
Show if \(x \sqsubseteq y\) and \(f\) is continuous then \(f(y) = f(x) \sqcup f(y)\)

Exercise 6.7
Give an example where a monotone map is not continuous.
Exercise: monotone and ACC $\Rightarrow$ continuous

Exercise 6.8
Let poset $(X, \leq)$ satisfies ascending chain condition (ACC). Let $(Y, \sqsubseteq)$ be a post. Let $f : X \rightarrow Y$ be a monotone function. Prove $f$ is continuous.
Topic 6.2

Closure operators
Closure operators

Definition 6.8
On a poset \((X, \leq)\), \(f\) is an upper closure operator if for all \(x, y \in X\)

- \(x \leq f(x)\)  
  (extensive)
- \(x \leq y \Rightarrow f(x) \leq f(y)\), and  
  (monotone)
- \(f(f(x)) = f(x)\).  
  (idempotent)

Example 6.5
Consider the following posets and functions (dashed edges). Are the functions upper closure operator?
Characterize upper-closure operators

Theorem 6.5

$f$ is upper closure operator iff $x \leq f(y) \iff f(x) \leq f(y)$

Proof.
Assume $f$ is upper closure

**claim:** $x \leq f(y) \Rightarrow f(x) \leq f(y)$

1. Consider $x \leq f(y)$
2. Since $f$ is monotone, $f(x) \leq f(f(y))$
3. Since $f$ is idempotent, $f(x) \leq f(y)$

**claim:** $f(x) \leq f(y) \Rightarrow x \leq f(y)$

1. $f(x) \leq f(y)$
2. $x \leq f(x) \leq f(y)$
3. $x \leq f(y)$

Forward direction is proven.
Characterize upper-closure operators II

Proof (contd.)
Assume $x \leq f(y) \iff f(x) \leq f(y)$

We need to prove that $f$ is extensive, monotone, and idempotent.

claim: $f$ is extensive
1. $f(x) \leq f(x)$ always holds
2. Therefore, $f \leq f(x)$

claim: $f$ is monotone
1. consider $x \leq y$
2. Use previous result, $x \leq y \leq f(y)$
3. Therefore, $x \leq f(y)$
4. Apply the assumption, $f(x) \leq f(y)$ ...
Characterize upper-closure operators II

Proof (contd.)
Recall assumption $x \leq f(y) \Leftrightarrow f(x) \leq f(y)$

claim: $f$ is idempotent

1. Clearly, $f(x) \leq f(x)$
2. Apply forward of the assumption, $f(f(x)) \leq f(x)$
3. Again clearly, $f(f(x)) \leq f(f(x))$
4. Apply backward of the assumption, $f(x) \leq f(f(x))$
5. Due to 2 and 4, $f(x) = f(f(x))$

Exercise 6.9
Use Z3 to prove the above theorem.
Fixed points of closure operators

Theorem 6.6
An upper closure operator is uniquely defined by its fixed points

Proof.
Let $f$ and $g$ be two upper closure operators that have same fixed points. But, differ at point $x$, i.e., $f(x) \neq g(x)$.

Contradiction: $f(x) = g(x)$

Exercise 6.10
a. Show $f(x) \sqcup f(y)$ is a fixed point
b. Show image of complete lattice by closure operator is complete lattice
End of Lecture 6