Lecture 11: Fixed points

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Topic 11.1

Fixed point theory
Fixed points

Let $X$ be a set.
A fixed point of a function $f : X \to X$ is $x \in X$ such that $f(x) = x$.

Let $f$ be a function on poset $(X, \leq)$:

- $fp(f) \triangleq \{ x \mid f(x) = x \}$
- $prefp(f) \triangleq \{ x \mid x \leq f(x) \}$
- $postfp(f) \triangleq \{ x \mid f(x) \leq x \}$
- least fixed point $lfp(f) \triangleq \min \{ fp(f) \}$
- greatest fixed point $gfp(f) \triangleq \max \{ fp(f) \}$

$lfp$ and $gfp$ may not exist.

Exercise 11.1

$fp(f) = prefp(f) \cap postfp(f)$
Example: fixed points

Example 11.1

Consider the following poset and a function $f$ (dashed-lines)

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d} \\
\end{array}
\]

\[
\text{\textit{prefp}}(f) = \{c, d, b\}
\]

\[
\text{\textit{postfp}}(f) = \{a, b\}
\]

\[
\text{\textit{postfp}}(f) = \{b\}
\]
Knaster-Tarski fixed point theorem

Theorem 11.1
A monotonic map $f : X \rightarrow X$ on a complete lattice $(X, \sqsubseteq, \top, \bot, \cap, \cup)$ has a least fixed point, which is

$$lfp(f) = \sqcap postfp(f) = \sqcap \{ x | f(x) \sqsubseteq x \}$$

Proof.

Reasons: complete lattice $\Rightarrow \sqcap$ exist $\exists x \in postfp(f)$
$f$ is monotone $f(a)$ is lb and $a$ is glb
transitivity $f$ is monotone $f(a) \in postfp(f)$
def. $postfp(f)$ def. $\sqcap postfp(f)$

Result: $f(a) = a \in postfp(f)$

Since $fp(f) \subseteq postfp(f)$, $a$ is lfp. (why $postfp(f) \neq \emptyset$?)
Knaster-Tarski theorem for gfp

Theorem 11.2
A monotonic map $f : X \rightarrow X$ on a complete lattice $(X, \sqsubseteq, \top, \bot, \land, \lor)$ has a greatest fixed point, which is

$$gfp(f) = \sqcup \text{prefp}(f) = \sqcup \{x | x \sqsubseteq f(x)\}$$

The proof is symmetrical to the previous case.
**Lfp greater than a prefix point**

**Definition 11.1**

Let $f : X \to X$ be a monotonic map on a complete lattice $(X, \sqsubseteq, \top, \bot, \sqcap, \sqcup)$. Let $a \in X$. Let $\text{lfp}_a(f)$ be the least fixed point of $f$ greater than $a$, i.e.,

$$a \sqsubseteq \text{lfp}_a(f) \quad \text{lfp}_a(f) = f(\text{lfp}_a(f)) \quad \forall x. \ a \sqsubseteq x = f(x) \Rightarrow \text{lfp}_a(f) \sqsubseteq x$$

**Theorem 11.3**

If $a \in \text{prefp}(f)$ then $\text{lfp}_a(f)$ exists and $\text{lfp}_a(f) = \text{lfp}(\lambda x. a \sqcup f(x))$.

**Proof.**

1. Let $p = \text{lfp}(\lambda x. a \sqcup f(x))$. So, $p = a \sqcup f(p)$.
2. By def. of $\sqcup$, $a \sqsubseteq p$, the first condition satisfied
3. Due to monotonic $f$, $f(a) \sqsubseteq f(p)$
4. Due to $a \sqsubseteq f(a)$ and transitivity, $a \sqsubseteq f(p)$
5. Therefore, $f(p) = a \sqcup f(p)$ and $p = f(p)$, the second condition satisfied
6. Choose $q$ such that $a \sqsubseteq q$ and $q = f(q)$, then $a \sqcap f(q) = q$.
7. Therefore, $q \in \text{postfp}(\lambda x. a \sqcup f(x))$.
8. Kanaster-Tarski, $p \sqsubseteq q$, the third condition satisfied
Exercise: $lfp_a(f)$ may not exists

Exercise 11.2
Show if $a \notin pref_p(f)$ then $lfp_a(f)$ may not exists.
Theorem 11.4
Let $f : X \rightarrow X$ be a monotonic map on a complete lattice $(X, \sqsubseteq, \top, \bot, \sqcap, \sqcup)$. $fp(f)$ forms a complete lattice.

Exercise 11.3
For $S \subseteq fp(f)$, show that $\text{lfp}_S(f)$ exists and is lub of $X$ in poset $(fp(f), \sqsubseteq)$

*Hint: use theorem 11.3*
Fixed point compose

Theorem 11.5
Let \((X, \leq)\) and \((Y, \sqsubseteq)\) be complete lattices, and \(f : X \to Y\) and \(g : Y \to X\) are monotonic then
\[
g(lfp(f \circ g)) = lfp(g \circ f)
\]

Proof.
1. \((g \circ f)g(lfp(f \circ g)) = g(f \circ g(lfp(f \circ g))) = g(lfp(f \circ g))\)
2. Therefore, \(g(lfp(f \circ g))\) is a fixed point of \(g \circ f\)
3. Assume \(x = g \circ f(x)\)
4. \(\Rightarrow f(x) = f \circ g \circ f(x) \Rightarrow f(x) = f \circ g(f(x))\)
5. Therefore, by Kanaster-tarski, \(lfp(f \circ g) \sqsubseteq f(x)\)
6. Since \(g\) is monotone, \(g(lfp(f \circ g)) \leq g \circ f(x)\)
7. Due to 3, \(g(lfp(f \circ g)) \leq x\)
8. Therefore, \(g(lfp(f \circ g))\) is lfp of \(g \circ f\)
Theorem 11.6

Let $f, g : X \to X$ be monotonic maps on a complete lattice $(X, \sqsubseteq, \top, \bot, \cap, \cup)$ such that for all $x \in X$, $f(x) \sqsubseteq g(x)$ then

$$\mathrm{lfp}(f) \sqsubseteq \mathrm{lfp}(g)$$

Exercise 11.4

Prove the above theorem
Transfinite iterates

Let \( f : X \rightarrow X \) be a function on a poset \((X, \sqsubseteq, \sqcap, \sqcup)\).

**Definition 11.2**

For some ordinal number \( \lambda \), the *upward iterates* \((I^k, k \leq \lambda)\) of \( f \) from \( a \) is a sequence such that

- \( I^0 = a \)
- \( I^{k+1} = f(I^k) \)
- \( I^\lambda = \sqcup_{k<\lambda} I^k \)

**Definition 11.3**

For some ordinal number \( \lambda \), the *downward iterates* \((I^k, k \leq \lambda)\) of \( f \) from \( a \) is a sequence such that

- \( I^0 = a \)
- \( I^{k+1} = f(I^k) \)
- \( I^\lambda = \sqcap_{k<\lambda} I^k \)

In poset, \( \sqcap \) and \( \sqcup \) are partially defined. Consequently, iterates are partially defined. If \( X \) is a lattice or cpo then iterates are well-defined.
Recall: Complete partial order

Definition 11.4
A complete partial order (cpo) is a poset \((X, \sqsubseteq)\) such that every increasing chain in \(X\) has a lub in \(X\)
A condition for finite iterates converging to lfp

Theorem 11.7

If

- \((X, \subseteq, \cup, \cap)\) is poset,
- \(f : X \rightarrow X\) is a monotone function,
- \(a \in \text{prefp}(f)\),
- upward iterates \((I^k, k \leq \omega)\) of \(f\) from \(a\) exists, and
- \(I^\omega \in \text{fp}(f)\)

then \((I^k, k \leq \omega)\) is increasing chain and \(I^\omega = \text{lfp}_a(f)\).

Proof.
1. Since \(a \subseteq f(a)\), \(I^0 \subseteq I^1\)
2. Induction hyp, \(I^n \subseteq I^{n+1}\). Due to monotone \(f\), \(f(I^n) \subseteq f(I^{n+1}) \Rightarrow I^{n+1} \subseteq I^{n+2}\)
3. By induction, \(\forall n < \omega. I^n \subseteq I^{n+1}\)
4. Since \(I^\omega = \bigcup_{k<\omega} I^k\) and \(I^\omega\) exists, \(\forall n \leq m \leq \omega. I^n \subseteq I^m\) (proved increasing chain)
5. Since \(a = I^0\), \(a \subseteq I^\omega\)
6. Assume \(a = I^0 \subseteq x = f(x)\). Since \(f\) is monotone, \(I^n \subseteq x \Rightarrow I^{n+1} = f(I^n) \subseteq f(x) = x\)
7. By induction and def. of \(I^\omega\), \(\forall n \leq \omega. I^n \subseteq x\). Therefore \(I^\omega = \text{lfp}_a(f)\)
Kleene fixed point theorem

Theorem 11.8

If

- $(X, \sqsubseteq, \sqcup)$ is cpo,
- $f : X \rightarrow X$ is upper continuous,
- $a \in \text{prefp}(f)$, and
- $(I_k^k, k \leq \omega)$ be upward iterates of $f$ from $a$

then $I^\omega = \text{lfp}_a(f)$.

Proof.

1. $f$ is continuous $\Rightarrow$ $f$ is monotone $\Rightarrow (I_k^k, k \leq \omega)$ increasing chain
2. Since $X$ is cpo, $I^\omega$ exists.
3. $f(I^\omega) = f(\bigsqcup_{k<\omega} I^k)$
4. $= \bigsqcup_{k<\omega} f(I^k)$, since $f$ is continuous.
5. $= \bigsqcup_{0<k<\omega} I^k = a \sqcup \bigsqcup_{0<k<\omega} I^k = \bigsqcup_{k<\omega} I^k = I^\omega$
6. Due to the previous theorem, $I^\omega = \text{lfp}_a(f)$
Knaster-Tarski for CPOs

We can prove Knaster-Tarski Theorem like results on cpos.

**Theorem 11.9**

If

1. \((X, \sqsubseteq, \sqcup)\) is cpo,
2. \(f : X \rightarrow X\) is upper continuous, and
3. \(a \in \text{prefp}(f)\)

then \(lfp_a(f) = \sqcap\{x \in X | a \sqsubseteq x \land f(x) \sqsubseteq x\}\).

**Proof.**

Let \(P = \{x \in X | a \sqsubseteq x \land f(x) \sqsubseteq x\}\). Let \((I^k, k \leq \omega)\) are iterates of \(f\) from \(a\).

1. Due to previous theorem, \(lfp_a(f) = I^\omega\). And, \(I^\omega \in P\).
2. Choose \(x, a \sqsubseteq x \in P\)
3. Induction hyp, \(I^n \leq x \Rightarrow f(I^n) \leq f(x) \leq x \Rightarrow I^{n+1} \leq x\)
4. By induction, \(\forall n < \omega, I^n \leq x\).
5. By def. of \(I^\omega, I^\omega \leq x\)
6. Therefore, \(lfp_a(f) \sqsubseteq \sqcap P\)
Fixed point for monotone functions on cpos

Theorem 11.10

If

- \((X, \sqsubseteq, \sqcup)\) is cpo,
- \(f : X \to X\) is monotone function,
- \(a \in \text{prefp}(f)\), and
- for some ordinal \(\lambda\), \((I^k, k \leq \lambda)\) be upward iterates of \(f\) from \(a\)

then \((I^k, k \leq \lambda)\) is increasing chain, which is ultimately stationary and converges to \(\text{lfp}_a(f)\).

We will skip the proof. However, the length to the stationary point is bounded by the ordinal size of the cpo.

Monotone is a weaker condition than continuous. Therefore, we need larger ordinals.
Topic 11.2

Asynchronous iterations for fixed points
System of simultaneous fixed point equations

For $i \in 1..n$, $(X_i, \sqsubseteq_i, \perp_i, \top_i, \cup_i, \cap_i)$ be complete lattices.

Let complete lattice $(X, \sqsubseteq, \perp, \top, \cup, \cap)$ be

- $X = X_1 \times \cdots \times X_n$
- $x \sqsubseteq y = (\land_{i=1}^n x_i \sqsubseteq_i y_i)$

Let $f : X \times X$ and $f_i : X \times X_i$ be $f_i(X) = (f(X))_i$

The fixed point equation $x = f(x)$ can be written as the following simultaneous fixed point equation.

$$
\begin{align*}
  x_1 &= f_1(x_1, \ldots, x_n) \\
  \vdots \\
  x_n &= f_n(x_1, \ldots, x_n)
\end{align*}
$$
Asynchronous iterations
We need not update each component at each iteration. We only need to ensure that each component is updated fairly.

Definition 11.5 (Chaotic iterations)
Let \((J^k, k \in \mathbb{O})\) be a sequence of subsets of \([1, n]\), which is weakly fair, i.e.,

\[
\forall i \in 1..n \forall j \in \mathbb{O}. \exists k > j. \ i \in J^k
\]

The iterates \((I^k, k < \lambda)\) starting from \(a \in X\) for \(F\) defined by \((J^k, k \in \mathbb{O})\) is

\[
\begin{align*}
I^0 &= a \\
I_i^k &= f_i(I^{k-1}) \quad \text{if } i \in J^k \\
I_i^k &= I^{k-1} \quad \text{if } i \not\in J^k \\
I^\lambda &= \bigsqcup_{k<\lambda} I^k
\end{align*}
\]

Theorem 11.11
\((I^k, k < \lambda)\) is increasing chain, ultimately stationary, and limit is \(\text{lfp}_a(f)\)
Example: asynchronous iterations

- Jacobi iterations: $J^k = [1, n]$
  - update every component in each step
- Gauss-Seidel iterations: $J^k = \{ k \ mod \ n \}$
  - update only one component in each step
End of Lecture 11