Automated Reasoning 2020

Lecture 1: Introduction and background

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Compile date: 2020-07-27

Topic 1.1

What is automated reasoning?

Automated reasoning (logic)

We will use reasoning and logic synonymously.

- Have you ever said to someone, “be reasonable”?
  - whatever your intuition was that is reasoning
- Why we care?

  Logic is the **calculus** of computer science

Commentary: ‘Automated reasoning’ consists of two words. Let us start with an explanation of the word ‘reasoning’. The question is like asking what is time? We can not explain time. We can only experience time. Similarly, if someone says something nonsensical. You may say, “be reasonable.” That is the experience of reasoning. We can not explain it. As computer scientists, we care about logic because it is the calculus of computer science. What do we mean by that? Recall from your high school studies, we used differential equations to analyzed electrical circuits. Differential equations are calculus of electrical engineering.

Example: applying logic

Logic is inferring conclusions from given premises

Example 1.1

1. *Humans are mortal*
2. *Socrates is a human*

✓

Socrates is mortal

1. *Apostles are twelve*
2. *Peter is an apostle*

✗

Peter is twelve

Invalid reasoning?

Commentary: Let us consider the examples of reasoning here. We think that the first reasoning is correct. However, the second reasoning is clearly faulty. What is wrong even though both the reasoning follow the same pattern? In the second reasoning, the first premise is talking about the size of the set of apostles instead of the properties of each of them. The English language made the matters ambiguous. We methods to separate good and bad reasonings.

Automated reasoning aims to

enable machines to

identify the valid reasoning!!

Commentary: Now let us focus on the word 'automated'. Like many other aspects of human life, we use machines to help us in identifying the bad reasoning. Automated reasoning is the field that consists of the formalism, algorithms, and tools that enables the identification. In the course, we will study them.

Automated reasoning is a backbone technology!!

Applications in verification, synthesis, solving NP-hard problems, and so on.

Automated reasoning for verification tools are like engines for the cars.

Commentary: Why do we need automated reasoning tools? There are many practical problems, where we need automated reasoning tools as an oracle to solve sub-problems. For example, software verification where we want to check if the software has indeed implemented the intended specification. The check boils down to automated reasoning problem.

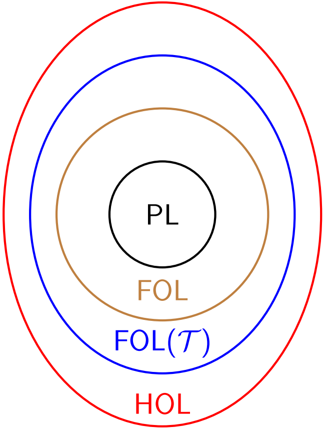
Topic 1.2

Spectra of logic

Spectra of logic

Logic has been divided into increasing complexity of classes.

1. Propositional logic (PL)
2. First-order logic (FOL)
3. First-order logic with theories (FOL($\mathcal{T}$))
4. Higher-order logic (HOL)



Commentary: Logic is a broad subject. We will study logic in the four classes of reasoning with increasing complexity. This structure allows us to study logic in convenient way. Each class introduces an additional level of complexity.

Propositional logic (PL)

Propositional logic

- deals with propositions,
- only infers from the structure over the propositions, and
- does not look inside the propositions.

Commentary: Propositional logic is also called Boolean logic. It is the most basic form of reasoning. One needs this class to solve the puzzles such as Sudoku.

Example : propositional logic

Example 1.2

Is the following argument valid?

If the seed catalog is correct then if seeds are planted in April then the flowers bloom in July. The flowers do not bloom in July. Therefore, if seeds are planted in April then the seed catalog is not correct.

Let us symbolize our problem

If c then if s then f. not f. Therefore, if s then not c.

- c = *the seed catalogue is correct*
- s = *seeds are planted in April*
- f = *the flowers bloom in July*

PL reasons over propositional symbols and logical connectives

Commentary: The symbolic form of an argument is obtained after replacing propositions by their symbols. Writing an argument in symbolic form allows us to focus on the reasoning involved in the argument. One can easily see that the argument holds.

First-order logic (FOL)

First-order logic

- looks inside the propositions,
- much more expressive,
- includes parameterized propositions and quantifiers over individuals, and
- can express lots of interesting math.

Example 1.3

Is the following argument valid?
Humans are mortal. Socrates is a human. Therefore, Socrates is mortal.

In the symbolic form,
For all x if $H(x)$ then $M(x)$. $H(s)$. Therefore, $M(s)$.

- $H(x) =$ *x is a human*
- $M(x) =$ *x is mortal*
- $s =$ *Socrates*

FOL is not the most general logic.
Many arguments can not be expressed in FOL

FOL+logical theories

In a theory, we study validity of FOL arguments under specialized assumptions (called axioms).

Example 1.4

The number theory uses symbols $0, 1, .., <, +, \cdot$ *with specialized meanings*

The following sentence has no sense until we assign the meanings to $>$ *and* $\cdot$

$$\forall x \, \exists p \, (p > x \land (\forall v_1 \forall v2 \, (v_1 > 1 \land v_2 > 1 \Rightarrow p \neq v_1 \cdot v_2)))$$

Under the meanings *it says that there are arbitrarily large prime numbers.*

In the earlier example, we had no interpretation of predicate 'x is human'. Here we precisely know what is predicate '$x < y$'.

Commentary: The specialized meaning are defined using axioms. For example, the sentence $\forall x. \; 0 + x = x$ describes one of the properties of 0 and $+$.

Higher-order logic (HOL)

Higher-order logic

- includes quantifiers over “anything”,
- consists of hierarchy first order, second order, third order and so on,
- most expressive logic.

Example 1.5

$$\forall P \forall x.\ (P(x) \vee \neg P(x))$$

The quantifier is over proposition P. Therefore, the formula belongs to the second-order logic.

Commentary: The first quantifier is not allowed in the first-order logic. the first-order logic quantifies over individuals, the second-order logic quantifies over sets, the third-order logic quantifies over set of sets, and so on.

Topic 1.3

Satisfiability problem

Satisfiability for reasoning

How to covert a reasoning problem into a computational question?

Answer: satisfiability problem

Commentary: Computers can compute. We need to pose a computational question to the computers to solve logical problems. Therefore, we are changing track to understand the satisfiability problem. We will see that we can translate all logical problems into satisfiability problem, which is a computational question.

Example: satisfaction

Let x, y be rational variables.

Choose a value of x and y such that the following formula holds true.

$$x + y = 3$$

We say

$$\underbrace{\{x \mapsto 1, y \mapsto 2\}}_{\text{model}} \models \underbrace{x + y = 3}_{\text{formula}}.$$

Commentary: x and y are not rational numbers. They are variables, i.e, symbols that can hold numbers. A model is an assignment to the variables of the formula. $m \models F$ is read as 'm satisfies F'.

Evaluation

For a given model m and formula F,

$$m \models F?$$

Example 1.6

$$\{x \mapsto 1, y \mapsto 2\} \models x + y = 3.$$

Exercise 1.1

- $\{x \to 1\} \models x > 0$?
- $\{x \to 1, y \to 2\} \models x + y = 3 \land x > 0$?
- $\{x \to 1, y \to 2\} \models x + y = 3 \land x > 0 \land y > 10$?

Exercise 1.2

Can we say something more about the last formula?

Satisfiablity problem

Satisfiability == Is there any model?

$$\text{Is } F \text{ SAT?}$$

Harder problem!

Exercise 1.3

Are the following formulas sat?

- $x + y = 3 \land x > 0$
- $x + y = 3 \land x > 0 \land y > 10$
- $x > 0 \lor x < 1$

disjunction

Exercise 1.4

Can we say something more about the last formula?

Commentary: If one does not know that a formula has a model or not. Then one may ask if a given formula is satisfiable or not. A formula can be valid, i.e., all models satisfy the formula. A formula can be satisfiable, unsatisfiable, or valid.

Reasoning == Satisfiability problem

All reasoning problems can be reduced to satisfiability problems.

Often abbreviated to **SAT problem**

Exercise 1.5

a. How to convert checking a valid argument into a SAT problem?

b. Convert argument "if $x \geq 2$ then $x \geq 1$." into a SAT problem.

Commentary: A typical argument looks like $Premise_1 \wedge \ldots. \wedge Premise_n \Rightarrow Conclusion$. If we negate the logical statement, we obtain $Premise_1 \wedge \ldots \wedge Premise_2 \wedge \neg Conclusion$ after some simplifications. If we show that the negation is unsatisfiable, then only the argument is valid. Therefore, the validity check of an argument can be reduced to a satisfiability problem.

Example: SAT problem(contd.)

Let x, y be rational variables.

Choose a value of x and y such that the following formula holds true.

$$x + y = 3 \wedge y > 10 \wedge x > 0$$

theory formulas

Easy

$$x + y = 3 \wedge y > 10 \wedge (x > 0 \vee x < -4)$$

quantifier-free formulas

Hard

$$\forall y.\ x + y = 3 \wedge y > 10 \wedge (x > 0 \vee x < -4)$$

quantified formulas

Very Hard

Commentary: The above are increasingly hard classes of satisfiability problems. The names used for hardness are informal to minimize jargon. The formal names of the classes are polynomial, NP-complete, and Undecidable.

Quantifier-free formulas

Quantifier-free formulas consists of

- theory atoms and
- Boolean structure

Example 1.7

$$x + y = 3 \land y > 10 \land (x > 0 \lor x < -4)$$

Theory atoms

Boolean Structure

Commentary: We typically assume $\land$(conjunction) occurs in any formula. We say a formula has Boolean structure if it has $\lor$(disjunction).

Propositional formulas

Propositional formulas are a special case, where the theory atoms are Boolean variables.

Example 1.8

Let p_1, p_2, p_3 be Boolean variables

$$p_1 \wedge \neg p_2 \wedge (p_3 \vee p_2)$$

A satisfying model:

$$\{p_1 \mapsto 1, p_2 \mapsto 0, p_3 \mapsto 1\} \models p_1 \wedge \neg p_2 \wedge (p_3 \vee p_2)$$

A bit of jargon

- Solvers for quantifier-free propositional formulas are called

  SAT solvers.

- Solvers for quantifier-free formulas with the other theories are called

  SMT solvers.

SMT $=$ satisfiability modulo theory

Theory solvers

SMT solvers are divided into two components.

- SAT solver: it solves the Boolean structure
- Theory solver: it solves the theory constraints

Example 1.9

Let x,y be rational variables.

$$x + y = 3 \land y > 10 \land x > 0$$

Since the formula has no $\lor$ (disjunction), a solver of linear rational arithmetic can find satisfiable model using simplex algorithm.

Commentary: Simplex can not directly handle the formulas that have disjunctions.

Quantified formulas

Quantified formulas also include quantifiers.

Example 1.10

The following formulas says: give x such that for each y the body holds true.

$$\forall y.\ \underbrace{(x + y = 3 \Rightarrow y > 10 \wedge (x > 0 \vee x < -4))}_{Body}$$

A satisfying model:

$$\{x \rightarrow -8\} \models \forall y.\ (x + y = 3 \Rightarrow y > 10 \wedge (x > 0 \vee x < -4))$$

Exercise 1.6

Can we eliminate y in the above formula to obtain constraints just over x?

Commentary: Since y is quantified, the model does not assign value to y.

Example: understanding quatified formula

Example 1.11

Consider the following formula and the model again.

$$\{x \to -8\} \models \forall y.\, (x + y = 3 \Rightarrow y > 10 \wedge (x > 0 \vee x < -4))$$

Substitute the value of x in the formula

$$\forall y.\, (-8 + y = 3 \Rightarrow y > 10 \wedge (-8 > 0 \vee -8 < -4))$$

After simplification, we obtain

$$\forall y.\, (y = 11 \Rightarrow y > 10).$$

The above is clearly true.(why?)

Exercise 1.7

Is $\forall y.\, (y = 11 \wedge y > 10)$ *true?*

Exercise

Exercise 1.8

Give satisfying assignments of the following formulas

- $\neg p_1 \wedge (p_1 \vee \neg p_2)$
- $x < 3 \wedge y < 1 \wedge (x + y > 5 \vee x - y < 3)$
- $\forall x\ (x > y \Rightarrow \exists z\ (2z = x))$

Topic 1.4

Derived problems in logic

Problems in logic

A logical problem may come in several forms.

- Validity
- Implication
- Quantifier elimination
- Induction
- Maximum satisfiability
- Interpolation
- Abduction

A typical solver needs to be able to handle the above.

Commentary: The last three logical problems will also be covered in the course. However, we will not discuss them in the lecture.

Validity

Is the formula true for all models?

$$\models F?$$

Is it harder problem than the SAT problem?

We can simply check satisfiability of $\neg F$.

Example 1.12

$x > 0 \lor x < 1$ is valid because $x \leq 0 \land x \geq 1$ is unsatisfiable.

Commentary: Any validity problem can be converted into a sat problem. Therefore, we do not need another solver for checking validity.

Implication

$$F \Rightarrow G?$$

We need to check $F \Rightarrow G$ is a valid formula.
We check if $\not\models \neg(F \Rightarrow G)$, which is equivalent to checking if $\not\models F \land \neg G$.

Example 1.13

Consider implication $(x = y + 1 \land y \geq z + 3) \Rightarrow x \geq z$*.*

After negating the implication, we obtain $x = y + 1 \land y \geq z + 3 \land x < z$*.*

After simplification, we obtain $x - z \geq 4 \land x - z < 0$*.*

Therefore, the negation is unsatisfiable and the implication is valid.

Commentary: Many solvers provide interfaces to check implications, which is an important API. Under the hood, the above procedure is followed.

Quantifier elimination

given F, find G such that

$$G(y) \equiv \exists x.\ F(x, y)$$

Is this a harder problem? yes(why?)

Commentary: Quantifier elimination asks us to remove a given variable from a formula. It is projecting an object in lower dimension space. In many calculations, we need to eliminate variables. There are algorithms for quantifier elimination for some classes for formulas.

Example: quantifier elimination

Example 1.14

Consider formula $\exists x.\ x > 0 \wedge x' = x + 1$

After substituting x *by* $x' - 1$, $\exists x.\ x' - 1 > 0$.

Since x *is not in the formula, we drop the quantifier and obtain* $x' > 1$.

Exercise 1.9

a. *Eliminate quantifiers:* $\exists x, y.\ x > 2 \wedge y > 3 \wedge y' = x + y$

b. *What do we do when* $\vee$ *is in the formula?*

c. *How to eliminate universal quantifiers?*

Induction principle

$$F(0) \wedge \forall n : F(n) \Rightarrow F(n+1)$$
$$\Rightarrow$$
$$\forall n : F(n)$$

Very difficult to automate.

Because, we need to prove $\forall n : G(n)$, but the induction proof technique fails on G.

Then, we need to find $F(n)$ such that $\forall n : F(n) \Rightarrow G(n)$ and induction works on $F(n)$.

Commentary: $G(n)$s are often poorly posed questions. We need to make a guess to find $F(n)$ such that the induction method can be applied. A machine does not know how to make an informed guess.

Example : Induction principle

Example 1.15

We prove $F(n) = (\sum_{i=0}^{n} i = n(n+1)/2)$ by induction principle as follows

- $F(0) = (\sum_{i=0}^{0} i = 0(0+1)/2)$
- *We show that implication $F(n) \Rightarrow F(n+1)$ is valid, which is*

$$(\sum_{i=0}^{n} i = n(n+1)/2) \Rightarrow (\sum_{i=0}^{n+1} i = (n+1)(n+2)/2).$$

Exercise 1.10

Show the above implication holds using a satisfiability checker.

Topic 1.5

Course contents and logistics

Content of the course

We will study the following topics

- Background: propositional and first-order logic (FOL) basics
- SAT solvers: satisfiability solvers for propositional logic
- SMT solvers: satisfiability modulo theory solvers
- Decision procedures: algorithms for solving theory constraints
- Solvers for quantifiers
  - SMT+quantifier
  - Saturation solvers: FOL solvers
- Interactive theorem prover for higher order logics

Evaluation and website

Evaluation

- Programming assignments: 35%
- Quizzes : 25% (2% each)
- Midterm : 10%
- Presentation: 10% (15 min) [topics will be floated]
- Final: 20%

For the further information

`https://www.cse.iitb.ac.in/~akg/courses/2020-ar/`

All the assignments and slides will be posted on the website.

Topic 1.6

Problems

SMT solvers

Which of the following are true about SMT solvers?

1. SMT solvers can make tea for us
2. SMT solvers can natively handle quantified formulas
3. Z3 is an SMT solver
4. The theory solvers in SMT solvers can only check satisfiability of conjunction of literals

Progating equality

In which of the following conditions we can propagate equality $x = t$ in F.

1. $\forall x.(x = t \Rightarrow F)$
2. $\exists x.(x = t \Rightarrow F)$
3. $\forall x.(x = t \wedge F)$
4. $\exists x.(x = t \wedge F)$

End of Lecture 1