# CS228 Logic for Computer Science 2020

## Lecture 15: First-order logic - Semantics

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Topic 15.1

# FOL - semantics

# Semantics : structures

### Definition 15.1

*For signature* $\mathbf{S} = (\mathbf{F}, \mathbf{R})$, *a* **S***-structure* $m$ *is a*

$$(D_m; \{f_m : D_m^n \rightarrow D_m | f/n \in \mathbf{F}\}, \{P_m \subseteq D_m^n | P/n \in \mathbf{R}\}),$$

*where* $D_m$ *is a nonempty set. Let* **S***-Mods denotes the set of all* **S***-structures.*

Some terminology

- $D_m$ is called domain of $m$.
- $f_m$ assigns meaning to $f$ under structure $m$.
- Similarly, $P_m$ assigns meaning to $P$ under structure $m$.

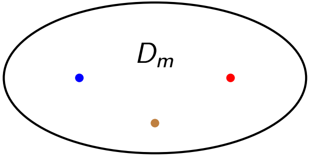
**Commentary:** Structures are also known as interpretations/models.

# Example: structure

### Example 15.1

*Consider* $\mathbf{S} = (\{c/0, f/1, g/2\}, \{H/1, M/2\})$.

*Let us suppose our structure m has domain* $D_m = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{brown}\bullet}\}$.




*We need to assign value to each function.*

- $c_m = {\color{blue}\bullet}$
- $f_m = \{{\color{blue}\bullet} \mapsto {\color{brown}\bullet}, {\color{brown}\bullet} \mapsto {\color{red}\bullet}, {\color{red}\bullet} \mapsto {\color{red}\bullet}\}$
- $$g_m = \{({\color{blue}\bullet}, {\color{blue}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{blue}\bullet}, {\color{brown}\bullet}) \mapsto {\color{blue}\bullet}, ({\color{blue}\bullet}, {\color{red}\bullet}) \mapsto {\color{red}\bullet},$$
  $$({\color{red}\bullet}, {\color{blue}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{red}\bullet}, {\color{brown}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{red}\bullet}, {\color{red}\bullet}) \mapsto {\color{brown}\bullet},$$
  $$({\color{brown}\bullet}, {\color{blue}\bullet}) \mapsto {\color{red}\bullet}, ({\color{brown}\bullet}, {\color{brown}\bullet}) \mapsto {\color{brown}\bullet}, ({\color{brown}\bullet}, {\color{red}\bullet}) \mapsto {\color{blue}\bullet}\}$$

*We also need to assign values to each predicate.*

- $H_m = \{{\color{blue}\bullet}, {\color{brown}\bullet}\}$ $\qquad M_m = \{({\color{blue}\bullet}, {\color{blue}\bullet}), ({\color{brown}\bullet}, {\color{red}\bullet})\}$

### Exercise 15.1

*a. How many structure are there for the signature with the above domain?*

*b. Suppose* $P/0 \in \mathbf{R}$, *give a value to* $P_m$.

# Example: structure

### Example 15.2

*Consider* $\mathbf{S} = (\{\cup/2\}, \{\in /2\})$.

$m = (\mathbb{N}; \cup_m = max, \in_m = \{(i,j) | i < j\})$ *is a* $\mathbf{S}$*-structure.*

# Semantics: assignments

Recall, We also have variables. Who will assign to the variables?

## Definition 15.2

*An assignment is a map* $\nu : \textbf{Vars} \rightarrow D_m$

## Example 15.3

*In our running example the domain is* $\mathbb{N}$. *We may have the following assignment.*

$$\nu = \{x \mapsto 2, y \mapsto 3\}$$

# Semantics: term value

## Definition 15.3

*For a structure m and assignment $\nu$, we define* $m^\nu : T_{\mathbf{S}} \to D_m$ *as follows.*

$$m^\nu(x) \triangleq \nu(x) \qquad\qquad x \in \mathbf{Vars}$$

$$m^\nu(f(t_1,\dots,t_n)) \triangleq f_m(m^\nu(t_1),\dots,m^\nu(t_n))$$

## Definition 15.4

*Let t be a closed term.* $m(t) \triangleq m^\nu(t)$ *for any* $\nu$.

## Example 15.4

*Consider assignment* $\nu = \{x \mapsto 2, y \mapsto 3\}$ *and term* $\cup(x,y)$.

$m^\nu(\cup(x,y)) = max(2,3) = 3$

# Example: satisfiability

### Example 15.5

*Consider* $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and term* $s(x) + y$

*Consider structure* $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

$$m^{\nu}(s(x) + y) = m^{\nu}(s(x)) +^{\mathbb{N}} m^{\nu}(y) = succ(m^{\nu}(x)) +^{\mathbb{N}} 2 = succ(3) +^{\mathbb{N}} 2 = 6$$

# Semantics: satisfaction relation

## Definition 15.5

*We define the satisfaction relation $\models$ among structures, assignments, and formulas as follows*

- $m, \nu \models \top$
- $m, \nu \models P(t_1, \dots, t_n)$    *if* $(m^\nu(t_1), \dots, m^\nu(t_n)) \in P_m$
- $m, \nu \models t_1 = t_2$    *if* $m^\nu(t_1) = m^\nu(t_n)$
- $m, \nu \models \neg F$    *if* $m, \nu \not\models F$
- $m, \nu \models F_1 \lor F_2$    *if* $m, \nu \models F_1$ *or* $m, \nu \models F_2$

  *skipping other boolean connectives*
- $m, \nu \models \exists x.(F)$    *if there is* $u \in D_m : m, \nu[x \mapsto u] \models F$
- $m, \nu \models \forall x.(F)$    *if for each* $u \in D_m : m, \nu[x \mapsto u] \models F$

# Example: satisfiability

## Example 15.6

*Consider* $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and formula* $\exists z.s(x) + y = s(z)$

*Consider structure* $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

*We have seen* $m^{\nu}(s(x) + y) = 6$.

$m^{\nu[z\mapsto 5]}(s(x) + y) = m^{\nu}(s(x) + y) = 6$. *//Since z does not occur in the term*

$m^{\nu[z\mapsto 5]}(s(z)) = 6$

*Therefore,* $m, \nu[z \mapsto 5] \models s(x) + y = s(z)$.

$m, \nu \models \exists z.s(x) + y = s(z)$.

# Exercise: satisfaction relation

### Exercise 15.2

*Consider sentence* $F = \exists x.\forall y.\ \neg y \in x$ *(what does it say to you!)*

*Consider* $m = (\mathbb{N}; \cup_m = max, \in_m = \{(i,j)|i < j\})$ *and* $\nu = \{x \mapsto 2, y \mapsto 3\}$.

*Does* $m, \nu \models F$?

# Exercise: structure

Consider $\mathbf{S} = (\{c/0, f/1\}, \{H/1, M/2\})$. Let us suppose structure $m$ has $D_m = \{\color{red}\bullet, \color{blue}\bullet, \color{brown}\bullet\}$ and the values of the symbols in $m$ are

- $c_m = \color{blue}\bullet$
- $f_m = \{\color{blue}\bullet \mapsto \color{brown}\bullet, \color{brown}\bullet \mapsto \color{red}\bullet, \color{red}\bullet \mapsto \color{red}\bullet\}$
- $H_m = \{\color{blue}\bullet, \color{brown}\bullet\}$
- $M_m = \{(\color{blue}\bullet, \color{blue}\bullet), (\color{brown}\bullet, \color{red}\bullet)\}$

## Exercise 15.3

*Is the following sentence satisfied by the above structure?*

- $\exists x. H(x)$
- $\exists x. H(f(x))$
- $\forall x. H(x)$
- $H(c)$

# Attendance quiz

Is following sentence satisfied by the structure on the slide?

$c = f(f(c))$
$\exists x.H(x) \wedge H(f(x))$
$\exists x.M(x,x)$
$\exists x.\neg M(x,x)$
$\exists x.M(x,f(x))$
$\exists x.\neg M(x,f(x))$
$\forall x.(H(x) \Rightarrow f(x) \neq c)$
$\forall x.(c = x \Rightarrow H(x))$
$\forall x.(H(x) \Rightarrow \exists y.M(x,y))$

$c = f(c)$
$\exists x.H(f(f(x)))$
$\forall x.M(x,x)$
$\forall x.\neg M(x,x)$
$\forall x.M(x,f(x))$
$\forall x.\neg M(x,f(x))$
$\exists x.(H(x) \wedge f(x) = c)$
$\exists x.(c = x \wedge \neg H(x))$

$\exists x.(H(x) \wedge \forall y.\neg M(x,y))$

# Why nonempty domain?

We are required to have nonempty domain in the structure. Why?

### Example 15.7

*Consider formula* $\forall x.(H(x) \land \neg H(x))$.

*Should any structure satisfy the formula?* *Noooooooo..*

*But, if we allow* $m = \{\emptyset; H_m = \emptyset\}$ *then*

$$m \models \forall x.(H(x) \land \neg H(x)).$$

*Due to this counterintuitive behavior, the empty domain is disallowed.*

# Example: non-standard structures

## Example 15.8

*Consider* $\mathbf{S} = (\{\mathbf{0}/0, s/1, +/2\}, \{\})$ *and formula* $\exists z.s(x) + y = s(z)$

**Unexpected structure:** *Let* $m = (\{a, b\}^*; \epsilon, append\_a, concat)$.

- *The domain of m is the set of all strings over alphabet* $\{a, b\}$.
- *append_a: appends a in the input and*
- *concat: joins two strings.*

*Let* $\nu = \{x \mapsto ab, y \mapsto ba\}$.
*Since* $m, \nu[z \mapsto abab] \models s(x) + y = s(z)$, *we have* $m, \nu \models \exists z.s(x) + y = s(z)$.

## Exercise 15.4

- *Show* $m, \nu[y \mapsto bb] \not\models \exists z.s(x) + y = s(z)$
- *Give an assignment* $\nu$ *s.t.* $m, \nu \models x \neq 0 \Rightarrow \exists y.\ x = s(y)$.
  *Show* $m \not\models \forall x.\ x \neq 0 \Rightarrow \exists y.\ x = s(y)$.

# Satisfiable, true, valid, and unsatisfiable

We say

- $F$ is *satisfiable* if there are $m$ and $\nu$ such that $m, \nu \models F$
- Otherwise, $F$ is called unsatisfiable
- $F$ is *true* in $m$ ($m \models F$) if for all $\nu$ we have $m, \nu \models F$
- $F$ is *valid* ($\models F$) if for all $\nu$ and $m$ we have $m, \nu \models F$

If $F$ is a sentence, $\nu$ has no influence in the satisfaction relation.(why?)

For sentence $F$, we say

- $F$ is *true* in $m$ if $m \models F$
- Otherwise, $F$ is *false* in $m$.

# Extended satisfiability

We extend the usage of $\models$.

**Definition 15.6**
*Let $\Sigma$ be a (possibly infinite) set of formulas.*
*$m, \nu \models \Sigma$ if $m, \nu \models F$ for each $F \in \Sigma$.*

**Definition 15.7**
*Let $M$ be a (possibly infinite) set of structures.*
*$M \models F$ if for each $m \in M$, $m \models F$.*

# Implication and equivalence

## Definition 15.8

*Let $\Sigma$ be a (possibly infinite) set of formulas.*
$\Sigma \models F$ *if for each structure m and assignment $\nu$ if $m, \nu \models \Sigma$ then $m, \nu \models F$.*

$\Sigma \models F$ is read $\Sigma$ implies $F$. If $\{G\} \models F$ then we may write $G \models F$.

## Definition 15.9

*Let $F \equiv G$ if $G \models F$ and $F \models G$.*

# Equisatisfiable and equivalid

**Definition 15.10**
*Formulas F and G are equisatisfiable if*

$$F \text{ is sat} \quad \text{iff} \quad G \text{ is sat.}$$

**Definition 15.11**
*Formulas F and G are equivalid if*

$$\models F \quad \text{iff} \quad \models G.$$

# Topic 15.2

## Problems

# FOL to PL

### Exercise 15.5

*Give the restrictions on FOL such that it becomes the propositional logic. Give an example of FOL structure of a non-trivial propositional formula.*

# Valid formulas

## Exercise 15.6

*Prove/Disprove the following formulas are valid.*

- $\forall x.P(x) \Rightarrow P(c)$
- $\forall x.(P(x) \Rightarrow P(c))$
- $\exists x.(P(x) \Rightarrow \forall x.P(x))$
- $\exists y\forall x.R(x,y) \Rightarrow \forall x\exists y.R(x,y)$
- $\forall x\exists y.R(x,y) \Rightarrow \exists y\forall x.R(x,y)$

# Distributively

### Exercise 15.7

*Show the validity of the following formulas.*

1. $\neg\forall x.\ P(x) \Leftrightarrow \exists x.\ \neg P(x)$
2. $(\forall x.\ (P(x) \wedge Q(x))) \Leftrightarrow \forall x.\ P(x) \wedge \forall x.\ Q(x)$
3. $(\exists x.\ (P(x) \vee Q(x))) \Leftrightarrow \exists x.\ P(x) \vee \exists x.\ Q(x)$

### Exercise 15.8

*Show $\forall$ does not distribute over $\vee$.*

*Show $\exists$ does not distribute over $\wedge$.*

# Encode mod k

### Exercise 15.9

*Give an FOL sentence that encodes that there are $n$ elements in any satisfying structure, such that $n \mod k = 0$ for a given $k$.*

# Find structures

## Exercise 15.10

*For* each *of the following formula give a structure that satisfies the formula. If there is no structure that satisfies a formula, then report that the formula is unsatisfiable.*

1. $\forall x.\exists y R(x,y) \land \neg\exists x.\forall y R(x,y)$
2. $\neg\forall x.\exists y R(x,y) \land \exists x.\forall y R(x,y)$
3. $\neg\forall x.\exists y R(x,y) \land \neg\exists x.\forall y R(x,y)$
4. $\forall x.\exists y R(x,y) \land \exists x.\forall y R(x,y)$

# Unique quantifier

### Exercise 15.11

*We could consider enriching the language by the addition of a new quantifier. The formula $\exists! x F$. (read "there exists a unique x such that F") is to be satisfied in structure m and assignment $\nu$ iff there is one and only one $d \in D_m$ such that $m, \nu[x \to d] \models F$. Show that this apparent enrichment does not add any expressive power of FOL.*

# Hierarchy of formulas

Exercise 15.12

# Topic 15.3

## Extra slides: some properties of models

# Homomorphisms of models

## Definition 15.12

*Consider* $\mathbf{S} = (\mathbf{F}, \mathbf{R})$. *Let* $m$ *and* $m'$ *be* $\mathbf{S}$*-models.*
*A function* $h : D_m \rightarrow D_{m'}$ *is a homomorphism of* $m$ *into* $m'$ *if the following holds.*

- *for each* $f/n \in \mathbf{F}$*, for each* $(d_1, .., d_n) \in D_m^n$

$$h(f_m(d_1, .., d_n)) = f_{m'}(h(d_1), .., h(d_n))$$

- *for each* $P/n \in \mathbf{R}$*, for each* $(d_1, .., d_n) \in D_m^n$

$$(d_1, .., d_n) \in P_m \quad \textit{iff} \quad (h(d_1), .., h(d_n)) \in P_{m'}$$

## Definition 15.13

*A homomorphism* $h$ *of* $m$ *into* $m'$ *is called isomorphism if* $h$ *is one-to-one.*
$m$ *and* $m'$ *are called isomorphic if an* $h$ *exists that is also onto.*

# Example : homomorphism

Example 15.9

*Consider* $\mathbf{S} = (\{+/2\}, \{\})$.

*Consider* $m = (\mathbb{N}, +^{\mathbb{N}})$ *and* $m = (\mathcal{B}, \oplus^{\mathcal{B}})$,

$h(n) = n \text{ mod } 2$ *is a homomorphism of* $m$ *into* $m'$.

# Homomorphism theorem for terms and quantifier-free formulas without $=$

## Theorem 15.1

*Let $h$ be a homomorphism of $m$ into $m'$. Let $\nu$ be an assignment.*

1. *For each term $t$, $h(m^{\nu}(t)) = m'^{(\nu \circ h)}(t)$*
2. *If formula $F$ is quantifier-free and has no symbol "$=$"*

$$m^{\nu} \models F \quad \textit{iff} \quad m'^{(\nu \circ h)} \models F$$

## Proof.

Simple structural induction. □

## Exercise 15.13

*For a quantifier-free formula $F$ that may have symbol "$=$", show*

$$\textit{if} \quad m^{\nu} \models F \quad \textit{then} \quad m'^{(\nu \circ h)} \models F$$

*Why the reverse direction does not work?*

# Homomorphism theorem with $=$

## Theorem 15.2

*Let $h$ be a homomorphism of $m$ into $m'$. Let $\nu$ be an assignment. If $h$ is isomorphism then the reverse implication also holds for formulas with "$=$".*

## Proof.

Let us suppose $m'^{(\nu \circ h)} \models s = t$.
Therefore, $m'^{(\nu \circ h)}(s) = m'^{(\nu \circ h)}(t)$.
Therefore, $h(m^{\nu}(s)) = h(m^{\nu}(t))$.
Due to the one-to-one condition of $h$, $m^{\nu}(s) = m^{\nu}(t)$.
Therefore, $m^{\nu} \models s = t$. □

## Exercise 15.14

*For a formula $F$ (with quantifiers) without symbol "$=$", show*

$$\text{if} \quad m'^{(\nu \circ h)} \models F \quad \text{then} \quad m^{\nu} \models F.$$

**Commentary:** Note that that implication direction is switched from the previous exercise.

# Homomorphism theorem with quantifiers

## Theorem 15.3

*Let h be a isomorphism of m into $m'$ and $\nu$ be an assignment.*
*If h is also onto, the reverse direction also holds for the quantified formulas.*

## Proof.

Let us assume, $m^{\nu} \models \forall x.F$.
Choose $d' \in D_{m'}$.
Since $h$ is onto, there is a $d$ such that $d = h(d')$.
Therefore, $m^{\nu[x \mapsto d]} \models F$.
Therefore, $m'^{\nu[x \mapsto d']} \models F$.
Therefore, $m'^{(\nu \circ h)} \models \forall x.\ F$. □

## Theorem 15.4

*If m and $m'$ are isomorphic then for all sentences F,*

$$m \models F \quad \textit{iff} \quad m' \models F.$$

**Commentary:** The reverse direction of the above theorem is not true.

# End of Lecture 15