# Automated Reasoning 2020

## Lecture 18: Difference and Octagonal logic

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Topic 18.1

# Difference logic

# Logic vs. theory

- theory = FOL + axioms
- logic = theory+syntactic restrictions

### Example 18.1

*LRA is a theory*
*QF_LRA is a logic that has only quantifier free LRA formulas*

# Difference Logic

**Difference Logic over reals(QF_RDL):** Boolean combinations of atoms of the form $x - y \leq b$, where $x$ and $y$ are real variables and $b$ is a real constant.

**Difference Logic over integers(QF_IDL):** Boolean combinations of atoms of the form $x - y \leq b$, where $x$ and $y$ are integer variables and $b$ is an integer constant.

Widely used in analysis of timed systems for comparing clocks.

**Commentary:** Lecture is based on: The octagon abstract domain.Antoine Miné. In Higher-Order and Symbolic Computation (HOSC), 19(1), 31-100, 2006. Springer.

# Difference Graph

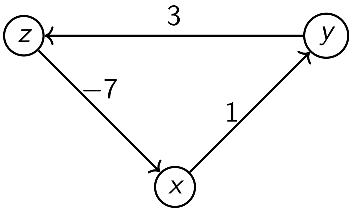
We may view $x - y \leq b$ as a weighted directed edge between nodes $x$ and $y$ with weight $b$ in a directed graph, which is called difference graph.

## Theorem 18.1

*A conjunction of difference inequalities is unsatisfiable iff the corresponding difference graph has negative cycles.*

## Example 18.2

$x - y \leq 1 \wedge y - z \leq 3 \wedge z - x \leq -7$



# Difference bound matrix

Another view of difference graph.

### Definition 18.1

*Let $F$ be conjunction of difference inequalities over rational variables $\{x_1, \ldots, x_n\}$. The **difference bound matrix(DBM)** $A$ is defined as follows.*

$$A_{ij} = \begin{cases} 0 & i = j \\ b & x_i - x_j \leq b \in F \\ \infty & otherwise \end{cases}$$

*Let* $F[A] \triangleq \bigwedge_{i,j \in 1..n} x_i - x_j \leq A_{ij}$.

Let $A_{i_0 \ldots i_m} \triangleq \sum_{k=1}^{m} A_{i_{k-1} i_k}$.

# Example: DBM

## Example 18.3

*Consider:* $x_2 - x_1 \leq 4 \wedge x_1 - x_2 \leq -1 \wedge x_3 - x_1 \leq 3 \wedge x_1 - x_3 \leq -1 \wedge x_2 - x_3 \leq 1$

*Constraints has three variables* $x_1$, $x_2$, *and* $x_3$.

*The corresponding DBM is*

$$\begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & \_\_\_ \\ 3 & \_\_\_ & 0 \end{bmatrix}$$

## Exercise 18.1

*Fill the blanks*

# Shortest path closure and satisfiability

## Definition 18.2

*The shortest path closure* $A^\bullet$ *of* $A$ *is defined as follows.*

$$(A^\bullet)_{ij} = \min_{i=i_0,i_1,\ldots,i_m=j \text{ and } m\leq n} A_{i_0\ldots i_m}$$

## Theorem 18.2

*$F$ is unsatisfiable iff* $\exists i \in 1..n.\ A^\bullet_{ii} < 0$

## Proof.

($\Leftarrow$) If RHS holds, trivially unsat.(why?)

($\Rightarrow$) if LHS holds,
due to Farkas lemma, there is a positive integral linear combination of difference inequalities that is $0 \leq -k$. ...

# Shortest path closure: there is a negative loop

## Proof(contd.)

**claim:** there is $A_{i_0,.....,i_m} < 0$ and $i_0 = i_m$.

Let $G = (\{x_1, ..., x_n\}, E)$ be a weighted directed graph such that

- $E = \{\underbrace{(x_i, b, x_j), .., (x_i, b, x_j)}_{\lambda \text{ times}} | x_i - x_j \leq b \text{ has } \lambda \text{ coefficient in the proof } \}$

Since each $x_i$ cancels out in the proof, $x_i$ has equal in and out degree in $G$.

Therefore, each SCC of $G$ has a Eulerian cycle (full traversal without repeating an edge). (why?)

The sum along one of the cycles must be negative.(why?) ...

## Exercise 18.2

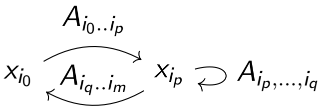
*Prove that if a directed graph is a strongly connected component(scc), and each node has equal in and out degree, there is a Eulerian cycle in the graph.*

# Shortest path closure(contd.)

## Proof.

**claim:** Shortest loop with negative sum has no repeated node

For $0 < p < q < m$, lets suppose $i_0 = i_m$ and $i_p = i_q$.




Since $A_{i_0..i_m} = \underbrace{A_{i_p..i_q}}_{\text{loop}} + \underbrace{(A_{i_q..i_m} + A_{i_m..i_p})}_{\text{loop}}$, one of the two sub-loops is negative.

Therefore, shorter loops exist with negative sum. Therefore, there is a negative simple loop. □

## Exercise 18.3

*If $F$ is sat, $A^\bullet_{ij} \leq A^\bullet_{ikj}$.*

# Floyd-Warshall Algorithm for shortest closure

We can compute $A^{\bullet}$ using the following iterations generating $A^0, \dots, A^n$.

$$A^0 = A$$
$$A^k_{ij} = \min(A^{k-1}_{ij}, A^{k-1}_{ikj})$$

## Theorem 18.3

$A^{\bullet} = A^n$

## Exercise 18.4

*a. Prove Theorem 18.3.* *Hint: Inductively show each loop-free path is considered*

*b. Extend the above algorithm to support strict inequalities*

*c. Does the above algorithm also work for $\mathbb{Z}$?*

# Example: DBM

**Example 18.4**

*Consider DBM:*
$$A^0 = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & \infty & 0 \end{bmatrix}$$

*First iteration:*
$$A^1 = min(A^0, \begin{bmatrix} A^0_{111} & A^0_{112} & A^0_{113} \\ A^0_{211} & A^0_{212} & A^0_{213} \\ A^0_{311} & A^0_{312} & A^0_{313} \end{bmatrix}) = min(A^0, \begin{bmatrix} 0 & -1 & -1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

*Second iteration:*
$$A^2 = min(A^1, \begin{bmatrix} A^1_{121} & A^1_{122} & A^1_{123} \\ A^1_{221} & A^1_{222} & A^1_{223} \\ A^1_{321} & A^1_{322} & A^1_{323} \end{bmatrix}) = min(A^1, \begin{bmatrix} 3 & -1 & 0 \\ 4 & 0 & 1 \\ 6 & 2 & 2 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

*Third iteration:*
$$A^3 = min(A^2, \begin{bmatrix} A^1_{131} & A^1_{132} & A^1_{133} \\ A^1_{231} & A^1_{232} & A^1_{233} \\ A^1_{331} & A^1_{332} & A^1_{333} \end{bmatrix}) = min(A^2, \begin{bmatrix} 2 & 1 & -1 \\ 4 & 3 & 1 \\ 3 & 2 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$$

# Incremental difference logic for SMT solvers

DBMs are not good for SMT solvers, where we need pop and unsat core.

SMT solvers implements difference logic constraints using difference graph.
Maintains a current assignment.

- push($x_1 - x_2 \leq b$):
  1. Adds corresponding edge from the graph
  2. If current assignment is feasible with new atom, exit
  3. If not, adjust assignments until it saturates z3:src/smt/diff_logic.h:make_feasible

- Pop($x_1 - x_2 \leq b$):
  - Remove the corresponding edge without worry

- Unsat core
  - If assignment fails to adjust, we can find the set of edges that required the adjustment
  - the edges form negative cycle, and reported as unsat core

Topic 18.2

# Difference logic : canonical representation

# Canonical representation

Sometimes a class for formulas have canonical representation.

### Definition 18.3

*A set of objects $R$ canonically represents a class of formulas $\Sigma$ if for each $F, F' \in \Sigma$ if $F \equiv F'$ and $o \in R$ represents $F$ then $o$ represents $F'$.*

# Tightness

## Definition 18.4

*A is tight if for all i and j*

- *if* $A_{ij} < \infty$, $\exists v \models F[A].\ v_i - v_j = A_{ij}$
- *if* $A_{ij} = \infty$, $\forall m < \infty.\ \exists v \models F[A].\ v_i - v_j > m$

## Theorem 18.4

*If F is sat,* $A^{\bullet}$ *is tight.*

## Proof.

Suppose there is a better bound $b < A^{\bullet}_{ij}$ exists such that $F[A^{\bullet}] \Rightarrow x_i - x_j \leq b$.

Like the last proof, there is a path $i_0..i_m$ such that $A_{i_0..i_m} \leq b$, $i_0 = i$ and $i_m = j$.(why?)

If $i_0..i_m$ has a loop then the sum along the loop must be positive.
Therefore, there must be a shorter path from $i$ to $j$ with smaller sum.(why?)
Therefore, a loopfree path from $i$ to $j$ exists with sum less than $b$. Therefore, $A^{\bullet}$ is tight □

# Implication checking and canonical form

### Theorem 18.5

*The set of shortest path closed DBMs canonically represents difference logic formulas.*

### Exercise 18.5

*Give an efficient method of checking equisatisfiablity and implication using DBMs.*

Topic 18.3

# Octagonal constraints

# Octagonal constraints

Definition 18.5

*Octagonal constraints are boolean combinations of inequalities of the form* $\pm x \pm y \leq b$ *or* $\pm x \leq b$ *where* $x$ *and* $y$ *are* $\mathbb{Z}/\mathbb{Q}$ *variables and* $b$ *is an* $\mathbb{Z}/\mathbb{Q}$ *constant.*

We can always translate octagonal constraints into equisatisfiable difference constraints.

# Octagon to difference logic encoding (contd.)

Consider conjunction of octagonal atoms $F$ over variables $V = \{x_1, \ldots, x_n\}$.

We construct a difference logic formula $F'$ over variables $V' = \{x'_1, \ldots, x'_{2n}\}$.

In the encoding, $x'_{2i-1}$ represents $x_i$ and $x'_{2i}$ represents $-x_i$.

# Octagon to difference logic encoding

$F'$ is constructed as follows

$$
\begin{array}{rrcll}
F \ni & x_i \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2i} \leq 2b & \in F' \\
F \ni & -x_i \leq b & \rightsquigarrow & x'_{2i} - x'_{2i-1} \leq 2b & \in F' \\
F \ni & x_i - x_j \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2j-1} \leq b,\ x'_{2j} - x'_{2i} \leq b & \in F' \\
F \ni & x_i + x_j \leq b & \rightsquigarrow & x'_{2i-1} - x'_{2j} \leq b, \quad x'_{2j-1} - x'_{2i} \leq b & \in F' \\
F \ni & -x_i - x_j \leq b & \rightsquigarrow & x'_{2i} - x'_{2j-1} \leq b, \quad x'_{2j} - x'_{2i-1} \leq b & \in F'
\end{array}
$$

## Theorem 18.6

*If $F$ is over $\mathbb{Q}$ then*

- *If $(v_1, \ldots, v_n) \models F$ then $(v_1, -v_1, \ldots, v_n, -v_n) \models F'$*
- *If $(v_1, v_2, \ldots, v_{2n-1}, v_{2n}) \models F'$ then $(\frac{(v_1 - v_2)}{2}, \ldots, \frac{(v_{2n-1} - v_{2n})}{2}) \models F$*

## Exercise 18.6

*a. Prove the above.* *b. Give an example over $\mathbb{Z}$ when Theorem 18.6 fails*

# Example: octagonal DBM

## Definition 18.6

*The DBM corresponding to $F'$ are called octagonal DBMs(ODBMs).*

## Exercise 18.7

*Consider:* $x_1 + x_2 \leq 4 \wedge x_2 - x_1 \leq 5 \wedge x_1 - x_2 \leq 3 \wedge -x_1 - x_2 \leq 1 \wedge x_2 \leq 2 \wedge -x_2 \leq 7$

*Corresponding ODBM*

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

$x_1 + x_2 \leq 4 \rightsquigarrow x_1 - x_4 \leq 4, x_3 - x_2 \leq 4$

$x_2 - x_1 \leq 5 \rightsquigarrow x_3 - x_1 \leq 5, x_2 - x_4 \leq 5$

$x_1 - x_2 \leq 3 \rightsquigarrow x_1 - x_3 \leq 3, x_4 - x_2 \leq 3$

$-x_1 - x_2 \leq 1 \rightsquigarrow x_1 - x_4 \leq 1, x_3 - x_2 \leq 1$

$x_2 \leq 2 \rightsquigarrow x_3 - x_4 \leq 4$

$-x_2 \leq 7 \rightsquigarrow x_3 - x_4 \leq 14$

# Relating indices and coherence

Let $\overline{2k} \triangleq 2k - 1$ and $\overline{2k-1} \triangleq 2k$

### Example 18.5

$\bar{1}\bar{1} = 22 \quad \bar{2}\bar{1} = 12 \quad \bar{2}\bar{2} = 11$

### Exercise 18.8

- $\bar{3}\bar{1} =$
- $\bar{4}\bar{2} =$
- $\bar{3}\bar{2} =$
- $\bar{1}\bar{1} =$

# Relating indices and coherence II

Consider the following DBM due to 2 variable octagonal constraints.

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

Cells with matching colors are pairs $(ij, \overline{ji})$.

### Definition 18.7

*A DBM A is coherent if* $\forall i, j.\ A_{ij} = A_{\overline{ji}}$.

# Unsatisfiability

For $\mathbb{Q}$, any method of checking unsat of difference constraints will work on ODBMs.

Let $A$ be ODBM of $F$. $A^{\bullet}$ will let us know in $2n$ steps if $F$ is sat.

For $\mathbb{Z}$, we may need to interpret ODBMs differently.
We will cover this shortly.

Topic 18.4

# Octagonal constraints : canonical form

# Implication checking and canonical form

Floyd-Warshall Algorithm does not obtain canonical form for ODBMs.

$x'_k = -x'_{\bar{k}}$ is not needed for satisfiablity check. Consequently, $A^\bullet$ is not canonical over $\mathbb{Q}$.

We need to tighten the bounds that may be proven due to the above equalities.

### Exercise 18.9

*Give an example such that $A^\bullet$ is not tight for octagonal constraints.*

# Canonical closure for octagonal constraints

Let us define closure property for ODBMs.

**Definition 18.8**

*For a ODBM $A$, let $F[A]$ define the corresponding formula over original variables.*

**Definition 18.9**

*For both $\mathbb{Z}$ and $\mathbb{Q}$, an ODBM $A$ is tight if for all $i$ and $j$*

- *if $A_{ij} < \infty$ then $\exists v \models F[A].\ v'_i - v'_j = A_{ij}$ and*
- *if $A_{ij} = \infty$ then $\forall m < \infty.\ \exists v \models F[A].\ v'_i - v'_j > m$,*

*where $v'_{2k-1} \triangleq v_k$ and $v'_{2k} \triangleq -v_k$*

**Theorem 18.7**

*If $A$ is tight then $A$ is a canonical representation of $F[A]$*

# $\mathbb{Q}$ tightness condition

## Theorem 18.8

*Let us suppose $F[A]$ is sat.*
*If $\forall i, j, k,\ A_{ij} \leq A_{ikj}$ and $A_{ij} \leq (A_{i\bar{i}} + A_{j\bar{j}})/2$ then $A$ is tight*

## Proof.

Consider cell $ij$ in $A$ s.t. $i \neq j$.(otherwise trivial)
Suppose $A_{ij}$ is finite.
Let $A' = A[ji \mapsto -A_{ij}, \overline{ij} \mapsto -A_{ij}]$

**claim:** $v \models F[A]$ and $v_i' - v_j' = A_{ij}$ iff $v \models F[A']$
Forward direction easily holds.(why?)
Since $A$ has no negative cycles, $A_{ij} + A_{ji} \geq 0$. So, $A_{ji} \geq -A_{ij}$. So, $A_{ji} \geq A'_{ji}$. Therefore, $A$ is pointwise greater than $A'$. Therefore, $F[A'] \Rightarrow F[A]$.
Since $A'_{ij} = -A'_{ji}$, if $v \models F[A']$ then $v_i' - v_j' = A_{ij}$. Backward direction holds. ...

# $\mathbb{Q}$ tightness condition(contd.)

## Proof(contd.)

Now we are only left to show the following.

**claim:** $F[A']$ is sat, which is there are no negative cycles in $A'$

$A'$ can have negative cycles only if $ji$ or $\overline{ij}$ occur in the cycle.(why?)

Wlog, we assume only $ji$ occurs in a negative cycle $i = i_0..i_m = j$

Therefore, $A'_{ji} + \sum_{l \in 1..m} A'_{i_{(l-1)} i_l} < 0$. Therefore, $-A_{ij} + \sum_{l \in 1..m} A_{i_{(l-1)} i_l} < 0$.

Therefore, $\sum_{l \in 1..m} A_{i_{(l-1)} i_l} < A_{ij}$. Contradiction.

One more case to consider

Assume both $ji$ and $\overline{ij}$ occur in a negative cycle $i = i_0..i_m i'_0..i_{m'} = j$, where $i_m = \bar{i}$ and $\bar{j} = i'_0$.

Therefore, $A'_{ji} + A'_{\overline{ij}} + \sum_{l \in 1..m} A'_{i_{l-1} i_l} + \sum_{l \in 1..m'} A'_{i'_{l-1} i'_l} < 0$.

Therefore, $-2A_{ij} + \sum_{l \in 1..m} A'_{i_{l-1} i_l} + \sum_{l \in 1..m'} A'_{i'_{l-1} i'_l} < 0$.

Therefore, $-2A_{ij} + A_{i\bar{i}} + A_{j\bar{j}} < 0$. Contradiction. □

## Exercise 18.10

*a. Prove the $A_{ij} = \infty$ case.*     *b. Does converse of the theorem hold?*

# Computing canonical closure for octagonal constraints

Due to the previous theorem and desire of efficient computation, let us redefine $A^\bullet$ for ODBMs.

### Definition 18.10

*We compute $A^\bullet$ using the following iterations generating $A^0, \dots, A^{2n} = A^\bullet$. Let $o = 2k - 1$ for some $k \in 1..n$.*

$$
\begin{aligned}
A^0 &= A \\
(A^{o+1})_{ij} &= \min(A^o_{ij}, \frac{A^o_{i\bar{i}} + A^o_{\bar{j}j}}{2}) && \textit{(odd rule)} \\
(A^o)_{ij} &= \min(A^{o-1}_{ij}, A^{o-1}_{ioj}, A^{o-1}_{i\bar{o}j}, A^{o-1}_{io\bar{o}j}, A^{o-1}_{i\bar{o}oj}) && \textit{(even rule)}
\end{aligned}
$$

In the even rule, three new paths are considered to exploit the structure of ODBMs.

We will prove that $A^\bullet$ is tight in post lecture slides.

# Even rule intuition

In octagon formulas, $x_k$ may insert itself between $x_{\lceil i/2 \rceil}$ and $x_{\lceil j/2 \rceil}$ in the following four ways.

1. $\pm x_{\lceil i/2 \rceil} - x_k \leq A_{io}$ and $x_k \pm x_{\lceil j/2 \rceil} \leq A_{oj}$ Update using $A_{io} + A_{oj}$
2. $\pm x_{\lceil i/2 \rceil} + x_k \leq A_{i\bar{o}}$ and $-x_k \pm x_{\lceil j/2 \rceil} \leq A_{\bar{o}j}$ Update using $A_{i\bar{o}} + A_{\bar{o}j}$
3. $\pm x_{\lceil i/2 \rceil} + x_k \leq A_{i\bar{o}}$, $x_k \pm x_{\lceil j/2 \rceil} \leq A_{oj}$, and $-x_k \leq A_{\bar{o}o}/2$ Update using $A_{i\bar{o}} + A_{\bar{o}o} + A_{oj}$
4. $\pm x_{\lceil i/2 \rceil} - x_k \leq A_{io}$, $-x_k \pm x_{\lceil j/2 \rceil} \leq A_{\bar{o}j}$, and $x_k \leq A_{o\bar{o}}/2$ Update using $A_{io} + A_{o\bar{o}} + A_{\bar{o}j}$

The above cases are considered in the four paths in the definition 18.10.

# Example: canonical closure of ODBM

## Example 18.6

*Consider:*

$$\begin{bmatrix} 0 & \infty & 3 & 4 \\ \infty & 0 & 1 & 5 \\ 5 & 4 & 0 & 4 \\ 1 & 3 & 14 & 0 \end{bmatrix}$$

*First we apply the even rule $o = 1$:*

$$A^1_{ij} = A^1_{\bar{j}\bar{i}} = \min(A^0_{ij}, A^0_{i1j}, A^0_{i2j}, A^0_{i12j}, A^0_{i21j})$$
$$A^1_{12} = A^1_{21} = \min(A^0_{12}, A^0_{112}, A^0_{122}, A^0_{1122}, A^0_{1212}) = min(\infty, \infty, \infty, \infty, \infty) = \infty$$
$$A^1_{24} = A^1_{13} = min(A^0_{24}, A^0_{214}, A^0_{224}, A^0_{2124}, A^0_{2214}) = min(5, \infty, 5, \infty, \infty) = 5$$
$$A^1_{34} = A^1_{34} = \min(A^0_{34}, A^0_{314}, A^0_{324}, A^0_{3124}, A^0_{3214}) = min(4, 9, 9, \infty, \infty) = 4$$
$$A^1_{43} = A^1_{43} = \min(A^0_{43}, A^0_{413}, A^0_{423}, A^0_{4123}, A^0_{4213}) = min(14, 4, 4, \infty, \infty) = 4$$

## Exercise 18.11

*Find the tight ODBM for the following octagonal constraints:*

$2 \leq x + y \leq 7 \wedge x \leq 9 \wedge y - x \leq 1 \wedge -y \leq 1$

# Octagonal constraints over $\mathbb{Z}$

For $\mathbb{Z}$, we need a stronger property to ensure tightness.

**Theorem 18.9**

*Let $A$ be ODBM interpreted over $\mathbb{Z}$.*
*if $\forall i, j, k,\ A_{ij} \leq A_{ikj},\ A_{ij} \leq (A_{i\bar{i}} + A_{j\bar{j}})/2$, and $A_{i\bar{i}}$ is even then $A$ is tight.*

**Exercise 18.12**

*Prove the above theorem.*

# Computing canonical closure for octgonal DBMs over $\mathbb{Z}$

In this case, let us present an incremental version of the closure iterations.

Lets suppose $A$ is tight and we add another octagonal atom in $A$ that updates $A_{i_0 j_0}$ and $A_{\bar{j_0}\bar{i_0}}$.

(Observe: always updated together)

Let $A^0$ be the updated DBM.

$$(A^1)_{ij} = \min(A^0_{ij}, A^0_{ii_0j_0j}, A^0_{i\bar{j_0}\bar{i_0}j}) \qquad \text{if } i \neq \bar{j}$$

$$(A^1)_{i\bar{i}} = \min(A^0_{i\bar{i}}, A^0_{i\bar{j_0}\bar{i_0}i_0j_0\bar{i}}, A^0_{ii_0j_0\bar{j_0}\bar{i_0}\bar{i}}, 2\lfloor\frac{A^0_{ii_0j_0\bar{i}}}{2}\rfloor)$$

$$(A^2)_{ij} = \min(A^1_{ij}, \frac{A^1_{i\bar{i}} + A^1_{\bar{j}j}}{2})$$

### Theorem 18.10

*$A^2$ is tight*

Topic 18.5

# Problem

# Difference logic for integers

## Exercise 18.13

*Consider a difference logic formula with integer bounds. Show that it has an integer solution if it has a rational solution.*

# End of Lecture 18

## Topic 18.6

# Post lecture proofs

# Tightness of $A^\bullet$

## Theorem 18.11

*$A^\bullet$ (defined in 18.10) is tight.*

## Proof.

For each $i, j$, and $k$, we need to show $A^\bullet_{ij} \leq (A^\bullet_{i\bar{i}} + A^\bullet_{j\bar{j}})/2$ and $A^\bullet_{ij} \leq A^\bullet_{ikj}$.

**claim:** For $k > 0$, $A^{2k}_{ij} \leq (A^{2k}_{i\bar{i}} + A^{2k}_{j\bar{j}})/2$

Note $A^{2k}_{i\bar{i}} = A^{2k-1}_{i\bar{i}}$. (why?)

By def,

$$(A^{2k})_{ij} \leq \frac{A^{2k-1}_{i\bar{i}} + A^{2k-1}_{j\bar{j}}}{2}.$$

Therefore,

$$(A^{2k})_{ij} \leq \frac{A^{2k}_{i\bar{i}} + A^{2k}_{j\bar{j}}}{2}.$$

...

# Tightness of $A^\bullet$(contd.)

## Proof(contd.)

We are yet to prove $\forall i, j.\ A^\bullet_{ij} \leq A^\bullet_{ikj}$.

Let $Fact(k, o) \triangleq \forall i, j.\ A^o_{ij} \leq A^o_{ikj} \wedge A^o_{ij} \leq A^o_{i\bar{k}j}$

So we need to prove $\forall k \in 1..n.\ Fact(2k, 2n)$.

the following three will prove the above by induction:(why?)

1. In odd rules ($o$ is odd), $Fact(k, o) \Rightarrow Fact(k, o + 1)$ (preserve)
2. In even rules ($o$ is even), $Fact(k, o) \Rightarrow Fact(k, o + 1)$ (preserve)
3. After even rules ($o$ is even), $Fact(o, o)$ (establish)

...

# Tightness of $A^\bullet$: odd rules preserve the facts

## Proof(contd.)

**claim:** odd rule, if $\forall i,j.\ A^o_{ij} \leq A^o_{ikj} \wedge A^o_{ij} \leq A^o_{\bar{i}\bar{k}j}$ then $\forall i,j.\ A^{o+1}_{ij} \leq A^{o+1}_{ikj}$.

We have four cases(why?) and denoted them by pairs.

(1,1) $A^{o+1}_{ik} = A^o_{ik}$, $A^{o+1}_{kj} = A^o_{kj}$: $\underbrace{A^{o+1}_{ij} \leq A^o_{ij}}_{\text{odd rule}} \underbrace{\leq A^o_{ikj}}_{\text{lhs}} \underbrace{= A^{o+1}_{ikj}}_{\text{case cond.}}$

(2,1) $A^{o+1}_{ik} = (A^o_{i\bar{i}} + A^o_{k\bar{k}})/2$, $A^{o+1}_{kj} = A^o_{kj}$:

$$\underbrace{A^o_{ij} \leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}j}}{2}}_{\text{odd rule}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}\bar{k}j}}{2}}_{\text{lhs}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{j}\bar{k}kj}}{2}}_{\text{lhs}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{\bar{k}k} + A^o_{\bar{j}\bar{k}} + A^o_{kj}}{2}}_{\text{rewrite}} \underbrace{\leq \frac{A^o_{i\bar{i}} + A^o_{k\bar{k}}}{2} + A^o_{kj}}_{\text{coherence}} \underbrace{= A^{o+1}_{ikj}}_{\text{case cond.}}$$

(2,1) $A^{2k}_{ik} = A^o_{ik}$, $A^{o+1}_{kj} = (A^o_{k\bar{k}} + A^o_{\bar{j}j})/2$ (Symmetric to the last case)

(2,2) $A^{o+1}_{ik} = (A^o_{i\bar{i}} + A^o_{k\bar{k}})/2$ and $A^{o+1}_{kj} = (A^o_{k\bar{k}} + A^o_{\bar{j}j})/2$: (left for exercise) ...

## Exercise 18.14

*Prove the last case.*

# Tightness of $A^\bullet$ : even rules preserve the facts

## Proof(contd.)

**claim:** even rule, if $\forall i,j.\ A^{o-1}_{ij} \leq A^{o-1}_{ikj} \wedge A^{o-1}_{ij} \leq A^{o-1}_{i\bar{k}j}$ then $\forall i,j.\ A^{o}_{ij} \leq A^{o}_{ikj}$.

Here, we have 25 cases(why?) and denoted them by pairs:

(1,1) $A^o_{ik} = A^{o-1}_{ik}, A^o_{kj} = A^{o-1}_{kj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ij}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{ikj}}_{\text{lhs}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$

(2,1) $A^o_{ik} = A^{o-1}_{iok}, A^o_{kj} = A^{o-1}_{kj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{iokj}}_{\text{lhs}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$ .

(4,5) $A^o_{ik} = A^{o-1}_{io\bar{o}k}, A^o_{kj} = A^{o-1}_{k\bar{o}oj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{ioj} + A^{o-1}_{o\bar{o}o} + A^{o-1}_{\bar{o}k\bar{o}}}_{\text{no negative loops}} \underbrace{\leq A^{o-1}_{io\bar{o}k} + A^{o-1}_{k\bar{o}oj}}_{\text{rewrite}} \underbrace{= A^o_{ikj}}_{\text{case cond.}}$

...

## Exercise 18.15

*Prove cases (1,4), (2,3) and (3,3).*

*Hint: key proof technique: introduce cycles, introduce k*

# Tightness of $A^\bullet$ : even rule establishes the fact

## Proof(contd.)

**claim:** even rule, $\forall i,j.\ A^o_{ij} \leq A^o_{ioj} \land A^o_{ij} \leq A^o_{i\bar{o}j}$
We only prove $A^o_{ij} \leq A^o_{ioj}$, the other inequality is symmetric.
Again, we have 25 cases.(why?)
Since there are no negative cycles and $A^o_{oo} = 0$,
$A_{io} = A_{ioo} \leq A_{io\bar{o}o}$ and $i\bar{o}o \leq i\bar{o}oo$.
Therefore, only four cases left to consider.(why?)

(1,1) $A^o_{io} = A^{o-1}_{io}, A^o_{oj} = A^{o-1}_{oj}$: $\underbrace{A^o_{ij} \leq A^{o-1}_{ioj}}_{\text{even rule}} \underbrace{= A^o_{ioj}}_{\text{case cond.}}$

(2,2) $A^o_{io} = A^{o-1}_{i\bar{o}o}, A^o_{oj} = A^{o-1}_{o\bar{o}j}$:
$$\underbrace{A^o_{ij} \leq A^{o-1}_{i\bar{o}j}}_{\text{even rule}} \underbrace{\leq A^{o-1}_{i\bar{o}j} + A^{o-1}_{o\bar{o}o}}_{\text{no negative cycles}} \underbrace{\leq A^{o-1}_{i\bar{o}o} + A^{o-1}_{o\bar{o}j}}_{\text{rewrite}} \underbrace{= A^o_{ioj}}_{\text{case cond.}}$$
□

## Exercise 18.16

*Prove case (1,2).*