CS228 Logic for Computer Science 2020

Lecture 4: Formal proofs

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Topic 4.1

Formal proofs

Consequence to derivation

Let us suppose for a (in)finite set of formulas Σ and a formula F, we have $\Sigma \models F$.

Can we syntactically infer $\Sigma \models F$ without writing the truth tables, which may be impossible if the size of Σ is infinite?

We call the syntactic inference "derivation". We derive the following statements.

$$\Sigma \vdash F$$

Example: derivation

Example 4.1

Let us consider the following simple example.

$$\underbrace{\Sigma \cup \{F\}}_{\textit{Left hand side(lhs)}} \vdash F$$

If F occurs in lhs, then F is clearly consequence of the lhs.

Therefore, we should be able to derive the above.

Proof rules

A proof rule allows us means to derive new statements from the old statements.

Called premises

$$\textsc{RuleName}\ \frac{\text{Stuff-already-there}}{\text{Stuff-to-be-added}}\ \text{Conditions to be met}$$

Called conclusion

A derivation proceeds by applying the proof rules.

What rules do we need for the propositional logic?

Proof rules - Basic

$$\textsc{Assumption}\frac{}{\Sigma \vdash F} F \in \Sigma$$

$$\textsc{Monotonic}\frac{\Sigma \vdash F}{\Sigma' \vdash F} \Sigma \subseteq \Sigma'$$

Derivation

Definition 4.1

A derivation is a list of statements that are derived from the earlier statements.

Example 4.2

A derivation due to the previous rules

Since assumption does not depend on any other statement, no need to refer.

1. $\{p \lor q, \neg\neg q\} \vdash \neg\neg q$ — Assumption
2. $\{p \lor q, \neg\neg q, r\} \vdash \neg\neg q$ — Monotonic applied to 1

We need to point at an earlier statement.

Proof rules for Negation

$$\textsc{DoubleNeg}\frac{\Sigma \vdash F}{\Sigma \vdash \neg\neg F}$$

Example 4.3

The following is a derivation

1. $\{p \vee q, r\} \vdash r$ *Assumption*
2. $\{p \vee q, \neg\neg q, r\} \vdash r$ *Monotonic applied to 1*
3. $\{p \vee q, \neg\neg q, r\} \vdash \neg\neg r$ *DoubleNeg applied to 2*

Proof rules for $\wedge$

$$\wedge-\textsc{intro}\frac{\Sigma \vdash F \quad \Sigma \vdash G}{\Sigma \vdash F \wedge G} \qquad \wedge-\textsc{Elim}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash F}$$

$$\wedge-\textsc{Symm}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash G \wedge F}$$

Example 4.4

The following is a derivation

1. $\{p \wedge q, \neg\neg q, r\} \vdash p \wedge q$ *Assumption*
2. $\{p \wedge q, \neg\neg q, r\} \vdash p$ *$\wedge$-Elim applied to 1*
3. $\{p \wedge q, \neg\neg q, r\} \vdash q \wedge p$ *$\wedge$-Symm applied to 1*

Proof rules for $\vee$

$$\vee - \text{INTRO}\frac{\Sigma \vdash F}{\Sigma \vdash F \vee G} \qquad \vee - \text{SYMM}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash G \vee F}$$

$$\vee - \text{DEF}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash \neg(\neg F \wedge \neg G)} \qquad \vee - \text{DEF}\frac{\Sigma \vdash \neg(\neg F \wedge \neg G)}{\Sigma \vdash F \vee G}$$

$$\vee - \text{ELIM}\frac{\Sigma \vdash F \vee G \qquad \Sigma \cup \{F\} \vdash H \qquad \Sigma \cup \{G\} \vdash H}{\Sigma \vdash H}$$

Commentary: We will use the same rule name if a rule can be applied in both the directions. For example, $\vee - \text{DEF}$.

Example : distributivity

Example 4.5

Let us show if we have $\Sigma \vdash (F \land G) \lor (F \land H)$, *we can derive* $\Sigma \vdash F \land (G \lor H)$.

1. $\Sigma \vdash (F \land G) \lor (F \land H)$ — Premise
2. $\Sigma \cup \{F \land G\} \vdash F \land G$ — Assumption
3. $\Sigma \cup \{F \land G\} \vdash F$ — $\land$-Elim applied to 2
4. $\Sigma \cup \{F \land G\} \vdash G \land F$ — $\land$-Symm applied to 2
5. $\Sigma \cup \{F \land G\} \vdash G$ — $\land$-Elim applied to 4
6. $\Sigma \cup \{F \land G\} \vdash G \lor H$ — $\lor$-Intro applied to 5
7. $\Sigma \cup \{F \land G\} \vdash F \land (G \lor H)$ — $\land$-Intro applied to 3 and 6

Example : distributivity (contd.)

8. $\Sigma \cup \{F \wedge H\} \vdash F \wedge H$ — Assumption
9. $\Sigma \cup \{F \wedge H\} \vdash F$ — $\wedge$-Elim applied to 8
10. $\Sigma \cup \{F \wedge H\} \vdash H \wedge F$ — $\wedge$-Symm applied to 8
11. $\Sigma \cup \{F \wedge H\} \vdash H$ — $\wedge$-Elim applied to 10
12. $\Sigma \cup \{F \wedge H\} \vdash H \vee G$ — $\vee$-Intro applied to 11
13. $\Sigma \cup \{F \wedge H\} \vdash G \vee H$ — $\vee$-Symm applied to 12
14. $\Sigma \cup \{F \wedge H\} \vdash F \wedge (G \vee H)$ — $\wedge$-Intro applied to 9 and 13

15. $\Sigma \vdash F \wedge (G \vee H)$ — $\vee$-elim applied to 1, 7, and 14

Proof rules for $\Rightarrow$

$$\Rightarrow-\textsc{Intro}\frac{\Sigma \cup \{F\} \vdash G}{\Sigma \vdash F \Rightarrow G}$$

$$\Rightarrow-\textsc{Elim}\frac{\Sigma \vdash F \Rightarrow G \quad \Sigma \vdash F}{\Sigma \vdash G}$$

$$\Rightarrow-\textsc{Def}\frac{\Sigma \vdash F \Rightarrow G}{\Sigma \vdash \neg F \lor G} \qquad \Rightarrow-\textsc{Def}\frac{\Sigma \vdash \neg F \lor G}{\Sigma \vdash F \Rightarrow G}$$

Example: central role of implication

Example 4.6

Let us prove $\{\neg p \vee q, p\} \vdash q$.

1. $\{\neg p \vee q, p\} \vdash p$ *Assumption*
2. $\{\neg p \vee q, p\} \vdash \neg p \vee q$ *Assumption*
3. $\{\neg p \vee q, p\} \vdash p \Rightarrow q$ $\Rightarrow$*-Def applied to 2*
4. $\{\neg p \vee q, p\} \vdash q$ $\Rightarrow$*-Elim applied to 1 and 3*

Attendance quiz: all the rules

$$\textsc{Assumption}\frac{}{\Sigma \vdash F} F \in \Sigma \quad \textsc{Monotonic}\frac{\Sigma \vdash F}{\Sigma' \vdash F} \Sigma \subseteq \Sigma' \quad \textsc{DoubleNeg}\frac{\Sigma \vdash F}{\Sigma \vdash \neg\neg F}$$

$$\wedge-\textsc{intro}\frac{\Sigma \vdash F \quad \Sigma \vdash G}{\Sigma \vdash F \wedge G} \quad \wedge-\textsc{Elim}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash F} \quad \wedge-\textsc{Symm}\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash G \wedge F}$$

$$\vee-\textsc{intro}\frac{\Sigma \vdash F}{\Sigma \vdash F \vee G} \quad \vee-\textsc{Symm}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash G \vee F} \quad \vee-\textsc{def}\frac{\Sigma \vdash F \vee G}{\Sigma \vdash \neg(\neg F \wedge \neg G)}*$$

$$\vee-\textsc{Elim}\frac{\Sigma \vdash F \vee G \qquad \Sigma \cup \{F\} \vdash H \qquad \Sigma \cup \{G\} \vdash H}{\Sigma \vdash H}$$

$$\Rightarrow-\textsc{Intro}\frac{\Sigma \cup \{F\} \vdash G}{\Sigma \vdash F \Rightarrow G} \quad \Rightarrow-\textsc{Elim}\frac{\Sigma \vdash F \Rightarrow G \quad \Sigma \vdash F}{\Sigma \vdash G} \quad \Rightarrow-\textsc{def}\frac{\Sigma \vdash F \Rightarrow G}{\Sigma \vdash \neg F \vee G}*$$

* Works in both directions

Example: another proof

Example 4.7

Let us prove $\emptyset \vdash (p \Rightarrow q) \vee p$.

1.	$\{\neg p\} \vdash \neg p$	*Assumption*	Case 1
2.	$\{\neg p\} \vdash \neg p \vee q$	$\vee$*-Intro applied to 1*	
3.	$\{\neg p\} \vdash (p \Rightarrow q)$	$\Rightarrow$*-Def applied to 2*	
4.	$\{\neg p\} \vdash (p \Rightarrow q) \vee p$	$\vee$*-Intro applied to 3*	
5.	$\{p\} \vdash p$	*Assumption*	Case 2
6.	$\{p\} \vdash p \vee (p \Rightarrow q)$	$\vee$*-Intro applied to 5*	
7.	$\{p\} \vdash p \vee (p \Rightarrow q)$	$\wedge$*-Symm applied to*	
8.	$\{\} \vdash (p \Rightarrow p)$	$\Rightarrow$*-Intro applied 5*	Only two cases
9.	$\{\} \vdash (\neg p \vee p)$	$\Rightarrow$*-Def applied 8*	
10.	$\{\} \vdash p \vee (p \Rightarrow q)$	$\vee$*-Elim applied 9*	

Proof rules for punctuation

$$() - \textsc{Intro}\frac{\Sigma \vdash F}{\Sigma \vdash (F)} \qquad () - \textsc{Elim}\frac{\Sigma \vdash (F)}{\Sigma \vdash F}$$

$$\wedge - \textsc{Paren}\frac{\Sigma \vdash (F \wedge G) \wedge H}{\Sigma \vdash F \wedge G \wedge H} \qquad \vee - \textsc{Paren}\frac{\Sigma \vdash (F \vee G) \vee H}{\Sigma \vdash F \vee G \vee H}$$

Proof rules for $\Leftrightarrow$

$$\Leftrightarrow -\mathrm{DEF}\frac{\Sigma \vdash F \Leftrightarrow G}{\Sigma \vdash G \Rightarrow F} \qquad \Leftrightarrow -\mathrm{DEF}\frac{\Sigma \vdash F \Leftrightarrow G}{\Sigma \vdash F \Rightarrow G}$$

$$\Leftrightarrow -\mathrm{DEF}\frac{\Sigma \vdash G \Rightarrow F \qquad \Sigma \vdash F \Rightarrow G}{\Sigma \vdash G \Leftrightarrow F}$$

Exercise 4.1

Define rules for $\oplus$

Commentary: this set of proof rules does not cover $\oplus$. We will cover them in greater detail.

Topic 4.2

Problems

Exercise: the other direction of distributivity

Exercise 4.2

Show if we have $\Sigma \vdash F \wedge (G \vee H)$*, we can derive* $\Sigma \vdash (F \wedge G) \vee (F \wedge H)$*.*

Hint: Case split on G and $\neg G$.

End of Lecture 4