CS228 Logic for Computer Science 2020

Lecture 5: Formal proofs - derived rules

Instructor: Ashutosh Gupta

IITB, India

Compile date: 2020-01-24

Topic 5.1

Derived rules

Derived rules

In logical thinking, we have many deductions that are not listed in our rules.

The deductions are consequence of our rules. We call them derived rules.

Let us look at a few.

Derived rules : Modus ponens

Theorem 5.1

If we he have $\Sigma \vdash \neg F \lor G$ and $\Sigma \vdash F$, we can derive $\Sigma \vdash G$.

Proof.

1. $\Sigma \vdash \neg F \lor G$ — Premise
2. $\Sigma \vdash F$ — Premise
3. $\Sigma \vdash F \Rightarrow G$ — $\Rightarrow$-Def applied to 1
4. $\Sigma \vdash G$ — $\Rightarrow$-Elim applied to 2 and 3

$\square$

We can use the above derivation as a sub-procedure and introduce the following proof rule.

$$\lor\text{-}\textsc{ModusPonens}\ \frac{\Sigma \vdash \neg F \lor G \qquad \Sigma \vdash F}{\Sigma \vdash G}$$

Example: implication

Example 5.1

Let us prove $\{(\neg p \vee r), (p \vee \neg q)\} \vdash (q \Rightarrow p \wedge r)$.

1. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash q$ *Assumption*
2. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash (p \vee \neg q)$ *Assumption*
3. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash (\neg q \vee p)$ *$\vee$-Symm applied to 2*
4. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash p$ *$\vee$-ModusPonens applied to 1 and 3*
5. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash (\neg p \vee r)$ *Assumption*
6. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash r$ *$\vee$-ModusPonens applied to 4 and 5*
7. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash p \wedge r$ *$\wedge$-Intro applied to 4 and 6*
8. $\{(\neg p \vee r), (p \vee \neg q)\} \vdash (q \Rightarrow p \wedge r)$ *$\Rightarrow$-Intro applied to 7*

Tautology

I drink water when it rains or when it does not.

A convoluted way of saying something is always true.

Derived rules: tautology rule

Theorem 5.2

For any F and a set Σ of formulas, we can always derive $\Sigma \vdash \neg F \vee F$.

Proof.

1. $\Sigma \cup \{F\} \vdash F$ — Assumption
2. $\Sigma \vdash F \Rightarrow F$ — $\Rightarrow$-Intro applied to 1
3. $\Sigma \vdash \neg F \vee F$ — $\Rightarrow$-Def applied to 2

$\square$

Again, we can introduce the following proof rule.

$$\textsc{Tautology}\,\frac{}{\Sigma \vdash \neg F \vee F}$$

Contradiction

If I eat a cake and not eat it, then sun is cold.

In logic, once we introduce an absurdity (formally contradiction), there are no limits in absurdity.

Commentary: To explain the importance of logic. Once Bertrand Russell made the following argument, 1. 2+2 = 5 2. 4=5 3. 4-3 = 5-3 4. 1=2 5. I and Pope are two. 6. I and Pope are one. 6. I am Pope.

Derived rules: contradiction rule

Theorem 5.3

If we have $\Sigma \vdash F \land \neg F$, we can always derive $\Sigma \vdash G$.

Proof.

1. $\Sigma \vdash F \land \neg F$ Premise
2. $\Sigma \vdash \neg F \land F$ $\land$-Symm applied to 1
3. $\Sigma \vdash \neg F$ $\land$-Elim applied to 2
4. $\Sigma \vdash \neg F \lor G$ $\lor$-Intro applied to 3
5. $\Sigma \vdash F$ $\land$-Elim applied to 1
6. $\Sigma \vdash G$ $\land$-Modus Ponens applied to 4 and 5

$\square$

Therefore, we may declare the following derived proof rule

$$\textsc{Contra}\frac{\Sigma \vdash \neg F \land F}{\Sigma \vdash G}$$

Contrapositive

I think, therefore I am. -Descartes

$$\Leftrightarrow$$

I am not, therefore I do not think.

In an argument, negation of the conclusion implies negation of premises.

Derived rules: contrapositive rule

Theorem 5.4

If we have $\Sigma \cup \{F\} \vdash G$*, we can always derive* $\Sigma \cup \{\neg G\} \vdash \neg F$*.*

Proof.

1. $\Sigma \cup \{F\} \vdash G$ — Premise
2. $\Sigma \cup \{F\} \vdash \neg\neg G$ — DoubleNeg applied to 1
3. $\Sigma \vdash F \Rightarrow \neg\neg G$ — $\Rightarrow$-Intro applied to 2
4. $\Sigma \vdash \neg F \vee \neg\neg G$ — $\Rightarrow$-Def applied to 3
5. $\Sigma \vdash \neg\neg G \vee \neg F$ — $\vee$-Symm applied to 4
6. $\Sigma \vdash (\neg G \Rightarrow \neg F)$ — $\Rightarrow$-Def applied to 5
7. $\Sigma \cup \{\neg G\} \vdash (\neg G \Rightarrow \neg F)$ — Monotonic applied to 6
8. $\Sigma \cup \{\neg G\} \vdash \neg G$ — Assumption
9. $\Sigma \cup \{\neg G\} \vdash \neg F$ — $\Rightarrow$-Elim applied to 7 and 8 □

Therefore, we may declare the following derived proof rule

$$\textsc{Contrapositive}\frac{\Sigma \cup \{F\} \vdash G}{\Sigma \cup \{\neg G\} \vdash \neg F}$$

Proof by cases and contradiction

We must have seen proof by cases and contradiction.

- Proof by cases

  If I have money, I run.
  If I do not have money, I run.
  Therefore, I run.

- Proof by contradiction

  Assume, I ate a dinosaur.
  My tummy is far smaller than a dianour. Contradiction.
  Therefore, I did not eat dinosaur.

Derived rules: proof by cases

Theorem 5.5

If we have $\Sigma \cup \{F\} \vdash G$ and $\Sigma \cup \{\neg F\} \vdash G$, we can always derive $\Sigma \vdash G$.

Proof.

1. $\Sigma \cup \{F\} \vdash G$ — Premise
2. $\Sigma \cup \{\neg F\} \vdash G$ — Premise
3. $\Sigma \vdash F \vee \neg F$ — Tautology
4. $\Sigma \vdash G$ — $\vee$-Elim applied to 1,2, and 3

$\square$

Therefore, we may declare the following derived proof rule

$$\textsc{ByCases}\frac{\Sigma \cup \{F\} \vdash G \qquad \Sigma \cup \{\neg F\} \vdash G}{\Sigma \vdash G}$$

Commentary: Sorry for the confusion in the class! There was a minor error.

Derived rules: proof by contradiction

Theorem 5.6

If we have $\Sigma \cup \{F\} \vdash G$ *and* $\Sigma \cup \{F\} \vdash \neg G$*, we can always derive* $\Sigma \vdash \neg F$*.*

Proof.

1. $\Sigma \cup \{F\} \vdash G$ Premise
2. $\Sigma \cup \{F\} \vdash \neg G$ Premise
3. $\Sigma \cup \{\neg G\} \vdash \neg F$ Contrapositive applied to 1
4. $\Sigma \cup \{\neg\neg G\} \vdash \neg F$ Contrapositive applied to 2
5. $\Sigma \vdash \neg F$ ByCases 3 and 4

$\square$

Therefore, we may declare the following derived proof rule

$$\textsc{ByContra}\frac{\Sigma \cup \{F\} \vdash G \qquad \Sigma \cup \{F\} \vdash \neg G}{\Sigma \vdash \neg F}$$

Reverse Double Negation

I do not dislike apples.

Therefore, I like apples.

Reverse Double Negation

Theorem 5.7

If we have $\Sigma \vdash \neg\neg F$, we can always derive $\Sigma \vdash F$.

Proof.

1. $\Sigma \vdash \neg\neg F$ — Premise
2. $\Sigma \cup \{\neg F\} \vdash \neg\neg F$ — Monotonic applied to 1
3. $\Sigma \cup \{\neg F\} \vdash \neg F$ — Assumption
4. $\Sigma \cup \{\neg F\} \vdash \neg F \land \neg\neg F$ — $\land$-Intro applied to 2 and 3
5. $\Sigma \cup \{\neg F\} \vdash F$ — Contradiction applied to 4
6. $\Sigma \cup \{F\} \vdash F$ — Assumption
7. $\Sigma \vdash F$ — Proof by cases applied to 5 and 6 $\square$

Therefore, we may declare the following derived proof rule

$$\textsc{RevDobuleNeg}\frac{\Sigma \vdash \neg\neg F}{\Sigma \vdash F}$$

Derived rules : Resolution

Theorem 5.8

If we he have $\Sigma \vdash \neg F \vee G$ and $\Sigma \vdash F \vee H$, we can derive $\Sigma \vdash G \vee H$.

Proof.

1. $\Sigma \vdash \neg F \vee G$ — Premise
2. $\Sigma \cup \{F\} \vdash \neg F \vee G$ — Monotonic applied to 1
3. $\Sigma \cup \{F\} \vdash F$ — Assumption
4. $\Sigma \cup \{F\} \vdash G$ — Modes Ponens applied to 2 and 3
5. $\Sigma \cup \{F\} \vdash G \vee H$ — $\vee$-Intro applied to 4

6. $\Sigma \vdash F \vee H$ — Premise
7. $\Sigma \cup \{F\} \vdash \neg\neg F$ — DoubleNeg applied to 3
8. $\Sigma \cup \{F\} \vdash \neg\neg F \vee H$ — $\vee$-Intro applied to 7
9. $\Sigma \cup \{H\} \vdash H$ — Assumption

...

Derived rules : Resolution (contd.)

10. $\Sigma \cup \{H\} \vdash H \lor \neg\neg F$ — $\lor$-Intro applied to 9
11. $\Sigma \cup \{H\} \vdash \neg\neg F \lor H$ — $\lor$-Symm applied to 10
12. $\Sigma \vdash \neg\neg F \lor H$ — $\lor$-Elim applied to 6,8, and 11

13. $\Sigma \cup \{\neg F\} \vdash \neg\neg F \lor H$ — Monotonic applied to 12
14. $\Sigma \cup \{\neg F\} \vdash \neg F$ — Assumption
15. $\Sigma \cup \{\neg F\} \vdash H$ — Modes Ponens applied to 13 and 14
16. $\Sigma \cup \{\neg F\} \vdash H \lor G$ — $\lor$-Intro applied to 15
17. $\Sigma \cup \{\neg F\} \vdash G \lor H$ — $\lor$-Symm applied to 16
18. $\Sigma \vdash G \lor H$ — Proof by cases applied to 5 and 17

Therefore, we may declare the following derived proof rule

$$\textsc{Resoluton}\frac{\Sigma \vdash F \lor G \qquad \Sigma \vdash \neg F \lor H}{\Sigma \vdash G \lor H}$$

Soundness

We need to show that

Theorem 5.9

if

proof rules derive a statement $\Sigma \vdash F$

then

$\Sigma \models F$.

Proof.

We will make an inductive argument. We will assume that the theorem holds for the premises of the rules and show that it is also true for the conclusions.

…

Proving soundness

Proof(contd.)

Consider the following rule

$$\wedge - \textsc{Elim} \frac{\Sigma \vdash F \wedge G}{\Sigma \vdash F}$$

Consider assignment $m \models \Sigma$. By the induction hypothesis, $m \models F \wedge G$.

Using the the truth table, we can show that if $m \models F \wedge G$ then $m \models F$.

$m(F)$	$m(G)$	$m(F \wedge G)$
0	0	0
0	1	0
1	0	0
1	1	1

Therefore, $\Sigma \models F$. ...

Proof

Proof.

Consider one more rule

$$\Rightarrow -\textsc{Intro}\frac{\Sigma \cup \{F\} \vdash G}{\Sigma \vdash F \Rightarrow G}$$

Consider assignment $m \models \Sigma$. There are two possibilities.

- **case** $m \models F$:
  Therefore, $m \models \Sigma \cup \{F\}$. By the induction hypothesis, $m \models G$.
  Therefore, $m \models (F \Rightarrow G)$.
- **case** $m \not\models F$: Therefore, $m \models (F \Rightarrow G)$.

Therefore, $\Sigma \vdash F \Rightarrow G$.

Similarly, we need to draw truth table or case analysis for each of the rules to check the soundness. □

Mathematics vs. computer science

So far we see rules of reasoning.

We have seen that the rules are correct and will see in a few lectures that they are also sufficient, i.e., all true statements are derivable.

Our inner mathematician is happy!!

However, our inner computer scientist is unhappy

- Too many rules - dozens of rules
- no instructions (or algorithm) for applying them on a given problem

We will embark upon simplifying and automating the reasoning process.

Topic 5.2

Problems

Formal proofs

Exercise 5.1

Prove that the following statements

1. $\{(p \Rightarrow q), (p \vee q)\} \vdash q$
2. $\{(p \Rightarrow q), (q \Rightarrow r)\} \vdash \neg(\neg r \wedge p)$
3. $\{(q \vee (r \wedge s)), (q \Rightarrow t), (t \Rightarrow s)\} \vdash s$
4. $\{(p \vee q), (r \vee s)\} \vdash ((p \wedge r) \vee q \vee s)$
5. $\{(((p \Rightarrow q) \Rightarrow q) \Rightarrow q)\} \vdash (p \Rightarrow q)$
6. $\emptyset \vdash (p \Rightarrow (q \vee r)) \vee (r \Rightarrow \neg p)$
7. $\{p\} \vdash (q \Rightarrow p)$
8. $\{(p \Rightarrow (q \Rightarrow r))\} \vdash ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
9. $\{(\neg p \Rightarrow \neg q)\} \vdash (q \Rightarrow p)$

Attendence quiz

Which of the following is a derived proof rule? Let / be the derivation line.

$\Sigma \cup \{F\} \vdash G, \Sigma \cup \{F\} \vdash G / \Sigma \vdash G$
$\Sigma \cup \{F\} \vdash G, \Sigma \cup \{F\} \vdash \neg G / \Sigma \vdash F$
$\Sigma \vdash \neg F / \Sigma \vdash \neg F$
$\Sigma \cup \{F\} \vdash G / \Sigma \cup \{G\} \vdash F$
$\Sigma \vdash \neg F \lor F / \Sigma \vdash G$
$/ \Sigma \vdash \neg F \land F$
$\Sigma \vdash F \lor \neg G, \Sigma \vdash F / \Sigma \vdash G$

$\Sigma \cup \{F\} \vdash G, \Sigma \cup \{\neg F\} \vdash G / \Sigma \vdash G$
$\Sigma \cup \{F\} \vdash G, \Sigma \cup \{F\} \vdash \neg G / \Sigma \vdash \neg F$
$\Sigma \vdash \neg\neg F / \Sigma \vdash F$
$\Sigma \cup \{F\} \vdash G / \Sigma \cup \{\neg G\} \vdash \neg F$
$\Sigma \vdash \neg F \land F / \Sigma \vdash G$
$/ \Sigma \vdash \neg F \lor F$

$\Sigma \vdash \neg F \lor G, \Sigma \vdash F / \Sigma \vdash G$

End of Lecture 5